Interaction between internal and surface waves in a two-layers fluid

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joint work with Walter Craig, Philippe Guyenne
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I. Physical context

- Fluid domain : Two layers of immiscible fluid of different densities separated by a interface. Idealization of a fluid with a sharp variation in temperature or salinity.
- Large amplitude - long wavelength nonlinear waves can be generated at the interface and they can propagate over large distances.

  Internal waves are generated for example when tides cause water to move over submerged mountains on the ocean floor. Cold water from the bottom gets pushed up over the ridge and sets up a disturbance.
- Early measurements [Perry-Schimke (1965)] in the Andaman sea : Groups of internal waves up to 80m high, 2000 m long on the main thermocline at 500 m in water 1500 m deep.

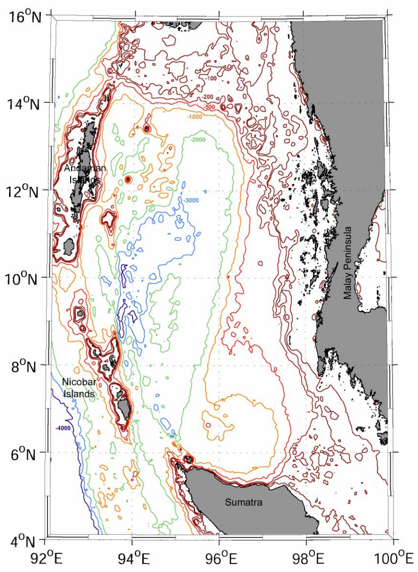

Figure: Bathymetry map of the Andaman sea

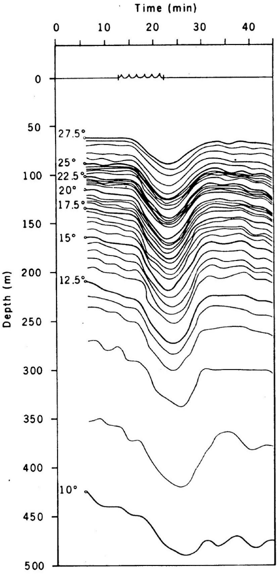

Figure: Isotherm contours on Oct. 25, 1976; Internal wave is materialized by isotherms. Osborne-Burch, ‘Internal Solitons in the Andaman Sea’, Science, **208** (1980), 451.

- Combination of in situ and remote sensing observations + progress in detection technology over the last 40 years, have shown that internal soliton-like waves are common features of costal oceans.
- On the practical side, internal waves can influence measurements of currents, undersea navigation, submerged engineering construction ... They play an important role in mixing different layers of water in the ocean.
- They are not directly visible to the observer, but they may sometimes produce small scale patterns at the surface that appears as a strip of rough waters, propagating over the internal wave.
  These distinct features on the surface are a signature of the presence of an internal wave.

Next 2 slides :

Sequence of photographs of the Andaman sea surface taken from an observation vessel [Oct. 27, 1976] as *a rip of rough waters approaches* from the west at a speed of 2.2m/sec .
(Osborne-Burch 1980).

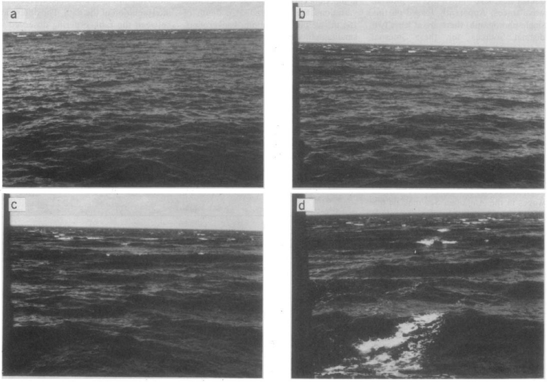

Figure: (a) The rip is seen in the distance, stretching from one horizon to the other as a *well-defined line of breaking waves*. The background sea state preceding the *'rip'* was ~ 0.6 m. (b) continues to approach; (c) the rip has just arrived at the vessel with wave heights 1.8 m. (d) The vessel is tossed about in the 1.8m waves.

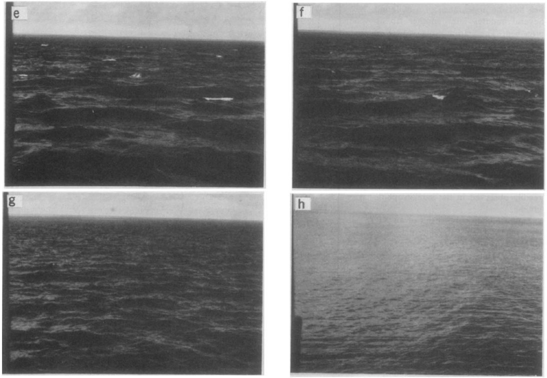

Figure: (e) The rear-ward edge of the rip is visible in 1.8 m waves; (f) The rear-ward edge of the rip has receded as the wave dropped to 1.3 m; (g) The wave amplitudes have dropped to 0.6 m; (h) The rip has completely passed as waves dropped to ripples of 0.1 m. : *'the mill pond effect'*.

Photographs taken from the space shuttle : The ripples induced by the internal waves have been imaged under the highly incident light of late afternoon. Under oblique lighting, their presence gives rise to a differential reflectancy property.

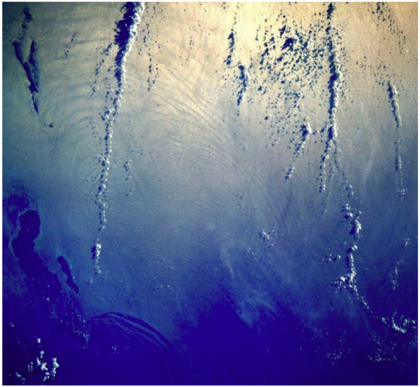

Figure: Photograph taken on May 5, 1985. [*Atlas of Oceanic internal solitary waves, the Andaman sea; Office of Naval Research, 2002.*

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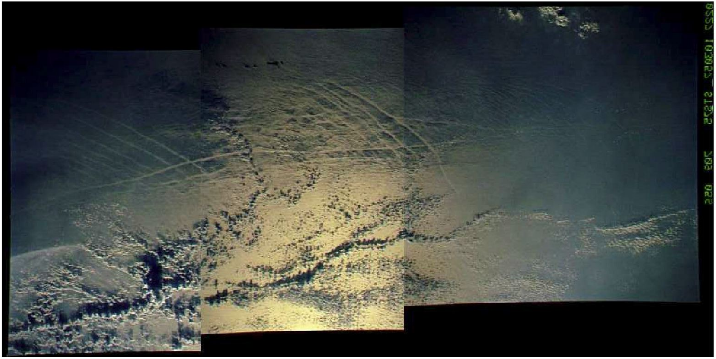


On a historical note, there is a description of such a phenomenon in a book by F.M. Maury :
'Physical Geography of the sea and its meteorology' (1885) (quoted in Osborne-Burch 1980)

In the entrance of the Malacca Straits, near the Nicobar and Acheen Islands, and between them and Junkseylon, there are often very strong ripplings, particularly in the southwest monsoon; these are alarming to persons unacquainted, for the broken water makes a great noise when the ship is passing through the ripplings in the night. In most places, ripplings are thought to be produced by strong currents, but here they are frequently seen when there is no perceptible current..... so as to produce an error in the course and distance sailed, yet the surface of the water is impelled foreward by some indiscovered cause.

The ripplings are seen in calm weather approaching from a distance, and in the night their noise is heard a considerable time before they come near. They beat against the sides of the ship with great violence, and pass on , the spray sometimes coming on deck; and a small boat could not always resist the turbulence of these remarkable ripplings.

Internal waves in the *Strait of Gibraltar*.

The two layers of fluid correspond to different salinity and the current is caused by the tides passing through the Strait.

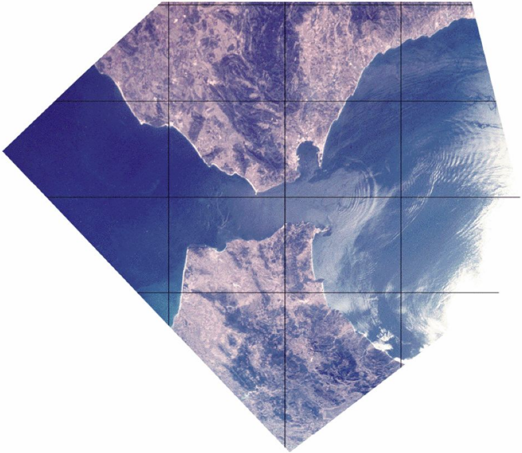


Figure: Strait of Gibraltar; from *the Atlas of Oceanic internal solitary waves, Office of Naval Research*

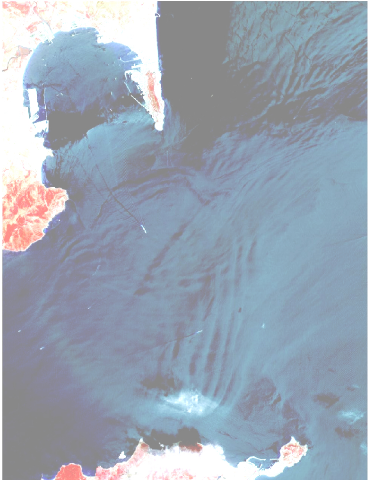
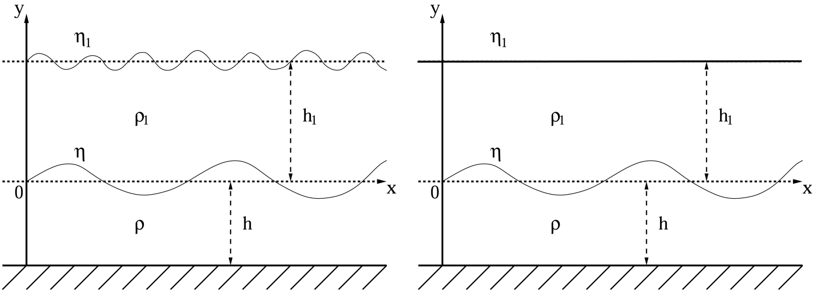


Figure: Strait of Gibraltar; [*the Atlas of Oceanic internal solitary waves, Office of Naval Research*]

- Extensive collection of measurements and images of various regions in the world can be found at *http://www.internalwaveatlas.com*.

II. Mathematical Models

Due to its importance in oceanography, large literature on internal waves in a variety of scaling regimes.
2 physical settings : (i) fixed lid; (ii) or internal/surface wave coupling,



- Stable configuration : $\rho > \rho_1$; ρ_1/ρ close to 1
- layer thickness ratio h_1/h plays important role.

- Fix lid : Large class of scaling regimes .
  Weakly nonlinear models for interface (Boussinesq, KdV, BO, ILW) ; Benjamin '67, Ono, '75, Camassa-Choi '96, '06, Nguyen-Dias '07
  Fully nonlinear models 1d: Matsuno (1993), and in the 2d case : Camassa and Choi (1999).
  Derivation + consistency analysis : Bona-Lannes-Saut '08
- coupling interface/free surface: Long wave/long wave Gear-Grimshaw '84, Parau-Dias, '01 Matsuno '93, Craig-Guyenne-Kalisch '05, Barros-Gravilyuk-Teshukov, '07
- Fix lid, internal wave propagation over periodic bottom topography Ruis de Zárate-Vigo-Alfaro-Nachbin-Choi 2009

Recent survey article by Helfrich and Melville (Ann. Rev.Fluid Mech 2006) with an overview of properties of internal solitary waves and vast bibliography.

Long wave/short wave interaction

- Regime that displays features of the pictures shown earlier: i.e. a free surface displaying small rough ripples created by the presence of a relatively large interface.
- In classical oceanographic models, the 'ray method' gives a qualitative description of the rip.
  A passing internal wave is represented by a near-surface current with a prescribed form. Surface waves are described statistically by a density function.
- Surface waves tend to shorten and steepen in regions of a converging current induced by a passing internal wave. (Gargett-Hughes 1972, Lewis-Lake-Ko 1974, Caponi et al. 1988, Bakhanov & Ostrovsky 2002).

- A classical explanation of the mill pond effect is that they are thus more likely to break, leaving calmer water after the internal wave passage.
- Direct numerical simulations by Donato, Peregrine and Stocker (1999) show the focusing and near-breaking of short surface waves due to a sinusoidal current.

III. Overview of results

From the Euler equations for stratified potential flow (restrict to 2d physical problem), we derive a mathematical model in the physically relevant regime:

- Long wave regime for the interface, interacting with ‘small’ modulated quasi-monochromatic surface waves.
- Resonant interaction: We identify the resonant surface wave numbers obtained when the group velocity of the free-surface coincides with the phase velocity of long internal waves.
- The resulting system : *KdV* for the evolution of the internal wave coupled to *a linear Schrödinger* equation for the modulation of the free surface. Solutions to the KdV equation appear as a potential in the Schrödinger equation.

$$\partial_\tau r + a_1 r \partial_X r + a_2 \partial_X^3 r = 0 \ ,$$
$$\partial_{\tau_1} v = i\Big[\frac{1}{2}\omega_1''(k_0)\partial_X^2 v + a_3 r(X, \varepsilon\tau_1) v\Big] \ ,$$

with $\tau = \varepsilon\tau_1$.

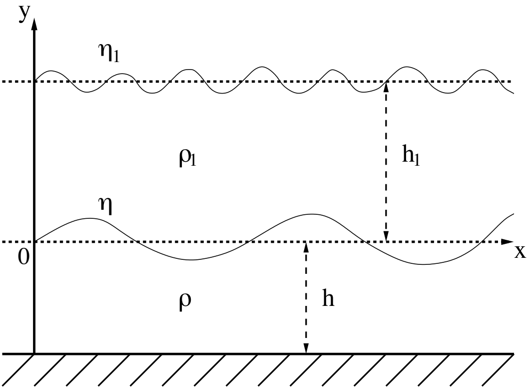
- The coefficients in the system only depend on the *physical parameters* h_1/h (mean depth ratios) and ρ_1/ρ (density ratio) . They are given by explicit formulas.
  $\rightarrow$ Detailed parametric analysis in realistic situations:
  $\rho_1/\rho \sim 1$, small h_1/h .
- This regime of parameters corresponds to *i.e. an internal soliton of depression* for KdV, (the potential for Schrödinger being a well) and a very small coefficient of dispersion for the Schrödinger equation (*semi-classical limit*).

We interpret the evolution of the free surface as a problem of radiative absorption and reflection:

- We relate the surface modulation of waves advected by fluid motion induced by internal waves to bound states of the linear Schrödinger equation.
  Wave energy from the sea surface is trapped by these bound states during the internal wave passage, which gives rise to the phenomenon of a region of rip.
- These bound states are computed numerically and found to be very localized in space, which is consistent with observations that the rip usually appears as narrow bands of rough water.

- We show that the reflection and transmission coefficients, $b(k)$ and $c(k)$ respectively, for the solution of the Schrödinger equation in the semi-classical regime, are asymptotic to $b = -1$ and $c = 0$ for a range of sideband wave numbers k near zero. In the reference frame of the internal soliton entering a background sea state, quasi-monochromatic surface waves are absorbed into the rip as bound states, or else they are effectively reflected in front of it.
- The fact that very little of the surface sea state is transmitted through the soliton region is an explanation of the mill pond effect.

IV.a The Euler Equations for stratified potential flow.



$$\Delta\varphi = 0\,, \quad \text{in the lower domain } S(t;\eta)$$
$$\Delta\varphi_1 = 0\,, \quad \text{in the upper domain } S_1(t;\eta,\eta_1)\,.$$

Boundary conditions

On the fixed bottom $\{y = -h\}$ of the lower fluid, the boundary condition is

$$\partial_y \varphi(x, -h) = 0 \ ,$$

On the interface $\{y = \eta(x,t)\}$, three boundary conditions - 2 kinematic , 1 dynamic (Bernouilli):

$$\begin{aligned}
&\partial_t \eta = \partial_y \varphi - \partial_x \eta \, \partial_x \varphi \\
&\partial_t \eta = \partial_y \varphi_1 - \partial_x \eta \, \partial_x \varphi_1 \\
&\rho(\partial_t \varphi + \frac{1}{2}|\nabla \varphi|^2 + g\eta) = \rho_1(\partial_t \varphi_1 + \frac{1}{2}|\nabla \varphi_1|^2 + g\eta) \ ,
\end{aligned}$$

On the top free surface $\{y = h_1 + \eta_1(x,t)\}$, 2 boundary conditions:

$$\begin{aligned}
&\partial_t \eta_1 = \partial_y \varphi_1 - \partial_x \eta_1 \, \partial_x \varphi_1 \\
&\partial_t \varphi_1 + \frac{1}{2}|\nabla \varphi_1|^2 + g\eta_1 = 0 \ .
\end{aligned}$$

The goal is to describe simultaneously the evolution of the free surface and free interface.

IV.b. Hamiltonian Formulation

It is possible to write the system in the form of a *Hamiltonian system* where the canonical variables are obtained in analogy with methods of *classical mechanics*.

- If the absence of internal wave (one fluid), the canonical variables are (η, ξ) where
  η is the free surface
  $\xi = \varphi(x, \eta(x))$ the trace of the velocity potential on the free surface (Zakharov 1968).
  The water wave problem takes the form

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \end{pmatrix} ,$$

- Hamiltonian = Total energy.

Dirichlet to Neuman operator : Central objet in the study of Water Waves.

Restrict to a single layer.
Let φ be harmonic in domain $\{(x, y), -h < y < \eta\}$, with Neumann boundary conditions on the bottom $(\partial_y \varphi(x, -h) = 0$. The Dirichlet–Neumann operator for the domain is defined by

$$G(\eta)\xi = \sqrt{1 + (\partial_x \eta)^2}\, \frac{\partial \varphi}{\partial n}\Big|_{y=\eta} .$$

where $\xi = \varphi(x, \eta(x))$ is the trace of the velocity potential on $y = \eta(x)$. The kinetic energy takes the form:

$$\int |\nabla \varphi|^2 dxdy = \int \xi \; G(\eta)\xi \; dx.$$

Choice of canonical variables follow *principles of classical mechanics*: Given a curve $\eta(\cdot, t)$ in configuration space, the Lagrangian given by

$$L := L(\eta, \dot{\eta}) = \text{kinetic energy} - \text{potential energy}$$

Rewrite the kinetic energy entirely in terms of $(\eta, \dot{\eta})$: Use the kinematic equation on free surface

$$\dot{\eta} = \partial_y \varphi - \partial_x \eta \partial_x \varphi = \sqrt{(1 + \eta_x^2)} \frac{\partial \varphi}{\partial n} = G(\eta)\xi$$

$$L(\eta, \dot{\eta}) = \int \frac{1}{2} \dot{\eta} G^{-1}(\eta) \dot{\eta} \, dx - \int \frac{g}{2} \eta^2(x) \, dx \ .$$

The Legendre transform will identify the coordinate canonically conjugate to η. Indeed,

$$\delta_{\dot{\eta}} L = G^{-1}(\eta) \dot{\eta}$$

which dictates that $\xi(x) = \varphi(x, \eta(x))$ is the appropriate choice.

- $G(\eta)$ has a Taylor expansion with respect to η (Coifman-Meyer '82); Terms can be calculated explicitly (recursion formulas). Useful to derive asymptotic models.
- Well-posedness theory (Lannes '05, Metivier-Alazard'09, Alazard-Burq-Zuily '11,'12...)
- Numerical simulations of water waves (Guyenne '06-08...)

- For stratified fluids, similar construction of canonical variables $(\eta, \xi, \eta_1, \xi_1)$.(Benjamin-Bridges, 1997).
  η is the interface ; $h_1 + \eta_1$ is the free surface
  $\xi = \rho\Phi - \rho_1\Phi_1, \quad \xi_1 = \rho_1\Phi_2$ where
  $\Phi = \varphi(x, \eta(x)),\ \Phi_1 = \varphi_1(x, \eta(x)),\ \Phi_2 = \varphi_1(x, h_1 + \eta_1(x))$.
- Hamiltonian = Kinetic energy + Potential energy.

The kinetic energy is the weighted sum of the Dirichlet integrals

$$K = \frac{1}{2}\int\int_{-h}^{\eta(x)} \rho|\nabla\varphi|^2\, dydx + \frac{1}{2}\int\int_{\eta(x)}^{h_1+\eta_1(x)} \rho_1|\nabla\varphi_1|^2\, dydx\ ,$$

and the potential energy is

$$V = \frac{1}{2}\int g(\rho - \rho_1)\eta^2(x)\, dx + \frac{1}{2}\int g\rho_1\left[(h_1 + \eta_1)^2(x) - h_1^2\right] dx\ .$$

$$\partial_t \begin{pmatrix} \eta \\ \xi \\ \eta_1 \\ \xi_1 \end{pmatrix} \equiv J\nabla H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \\ \delta_{\eta_1} H \\ \delta_{\xi_1} H \end{pmatrix}\ ,$$

Express the kinetic energy in terms of the canonical variables.

In the case of a 2 layer-fluids, we define

- The Dirichlet–Neumann operator for the lower domain is defined by

$$G(\eta)\varphi(x,\eta(x,t),t) = \sqrt{1+(\partial_x\eta)^2}\,\nabla\varphi\cdot n\Big|_{y=\eta}.$$

- For the upper fluid, the traces of φ_1 on η, and η_1 contribute to the exterior normal derivative of φ_1 on each boundary. The Dirichlet–Neumann operator is a matrix operator:

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} \varphi_1(x,\eta(x,t),t) \\ \varphi_1(x,h_1+\eta_1(x,t),t) \end{pmatrix} = \begin{pmatrix} -\sqrt{1+(\partial_x\eta)^2}\,\nabla\varphi_1\cdot n\Big|_{y=\eta} \\ \sqrt{1+(\partial_x\eta_1)^2}\,\nabla\varphi_1\cdot n_1\Big|_{y=h_1+\eta_1} \end{pmatrix}.$$

IV.c Linear analysis near fluid at rest

Linearized equations:

$$\begin{aligned} \partial_t \eta &= \delta_\xi H^{(2)}, \ \partial_t \xi &= -\delta_\eta H^{(2)} \\ \partial_t \eta_1 &= \delta_{\xi_1} H^{(2)}, \ \partial_t \xi_1 &= -\delta_{\eta_1} H^{(2)} \ . \end{aligned}$$

The quadratic part $H^{(2)}$ of the Hamiltonian is given by ($D = -i\partial_x$)

$$H^{(2)} = \frac{1}{2} \int (\xi, \xi_1) \mathcal{A}(D) \begin{pmatrix} \xi \\ \xi_1 \end{pmatrix} + g(\rho - \rho_1)\eta^2 + g\rho_1[(h_1 + \eta_1)^2 - h_1^2]$$

where $\mathcal{A}(D)$ is a 2×2 matrix of Fourier multipliers.
To simplify $H^{(2)}$, we make two canonical transformations. First, a rescaling of the dependent variables $(\eta, \xi, \eta_1, \xi_1)$. This replaces the two last terms of $H^{(2)}$ by $\eta^2 + \eta_1^2$. We then diagonalize the kinetic energy in $H^{(2)}$ by performing a rotation.
As a result, $H^{(2)}$ takes the simpler form

$$H^{(2)} = \frac{1}{2} \int \left[\xi \omega^2(D) \xi + \eta^2 + \xi_1 \omega_1^2(D) \xi_1 + \eta_1^2 \right] dx \ ,$$

(ξ, η) represent the 'internal' modes, while (ξ_1, η_1) are the 'free-surface' modes

$(\omega^2(k), \omega_1^2(k))$ are the two roots of the quadratic equation defining the dispersion relation :

$$
\begin{aligned}
\omega^4 \quad &- \quad g\rho k \frac{1 + \tanh(kh)\coth(kh_1)}{\rho\coth(kh_1) + \rho_1\tanh(kh)}\omega^2 \\
&+ \quad g^2(\rho - \rho_1)k^2 \frac{\tanh(kh)}{\rho\coth(kh_1) + \rho_1\tanh(kh)} = 0 \,.
\end{aligned}
$$

V.a Scaling regimes

h : typical depth of lower fluid
a : order of amplitude of interface
λ : order of wavelength of interface

- Long wave/small amplitude regime for interface (KdV)

$$\frac{h}{\lambda} = \varepsilon, \quad \frac{a}{h} = \varepsilon^2$$

h_1 : typical depth of upper fluid
a_1 : order of amplitude of free surface

- (very) small amplitude for surface: (modulational regime)

$$\frac{a_1}{h_1} = \varepsilon_1, \quad \varepsilon_1 = \varepsilon^{2+\alpha}, \quad \alpha \geq 0$$

V.b Scaling Ansatz and resonant condition

Roughly speaking...

- Long wave/small amplitude regime for interface (KdV)

$$\eta \sim \varepsilon^2 r(X,\tau);\ X = \varepsilon x,\ \tau = \varepsilon^3 t$$

- (very) small amplitude for surface: (modulational regime)

$$\eta_1 \sim \varepsilon_1 v(X,\tau_1) e^{i(k_0 x - \omega_1(k_0)t)} + \mathrm{c.c},\ \varepsilon_1 = \varepsilon^{2+\alpha},\ \tau_1 = \varepsilon^2 t$$

- Assume unidirectional motion for the interface at velocity c (where $c^2 = \omega^2(0)''/2$. The wavenumber k_0 will be chosen such that wave packets on the free surface (moving at group velocity $\omega_1'(k_0)$) move at same speed as interface:

$$\omega_1'(k_0) = c$$

'Linear resonant condition' between internal and surface waves.

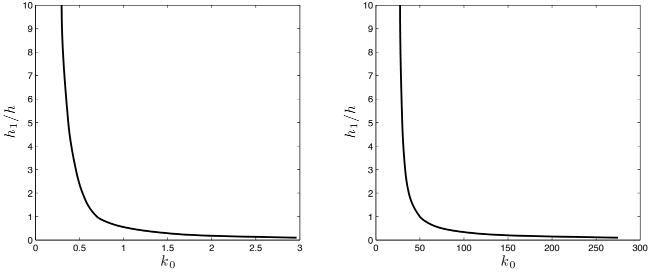


Figure: Depth ratio h_1/h vs. wavenumber k_0 corresponding to the linear resonance condition for $\rho_1/\rho = 0.1$(left); $\rho_1/\rho = 0.99$(right).

There is always a surface mode of wavenumber k_0 satisfying the resonance condition and thus traveling at the same linear speed as a long internal mode.

The smaller h_1/h, or the closer ρ_1/ρ to 1, the larger k_0 (hence the shorter the surface mode). Also, k_0 varies monotonically as a function of h_1/h.

V.c Long-wave scaling, modulational Ansatz

We assume that the 'internal' modes are long wave-small amplitude:

$$X = \varepsilon x \ , \quad \xi(x,t) = \varepsilon^2 \tilde{\xi}(X,t) \ , \quad \eta(x,t) = \varepsilon \tilde{\eta}(X,t) \ ,$$

The 'surface' modes are quasi-monochromatic waves obeying the modulational Ansatz, ($\varepsilon_1 = \varepsilon^{2+\alpha}$, $\alpha > 0$)

$$\eta_1(x,t) = \frac{\varepsilon_1}{\sqrt{2}} \omega^+(D)^{1/2} \Big(v(X,t) e^{ik_0 x} + \mathrm{c.c} \Big) + \varepsilon_1^2 \tilde{\eta}_1(X,t) \ ,$$

$$\xi_1(x,t) = \frac{\varepsilon_1}{\sqrt{2} i} \omega^+(D)^{-1/2} \Big(v(X,t) e^{ik_0 x} - \mathrm{c.c} \Big) + \frac{\varepsilon_1^2}{\varepsilon} \tilde{\xi}_1(X,t) \ ,$$

The function v represents the complex envelope of the free-surface modes, and $\widetilde{\eta}_1$ and $\widetilde{\xi}_1$ the associated mean fields

- Since both internal and surface wave propagate with their respective speeds, change the equations into a moving frame of reference. This is done by subtracting a multiple of the momentum I (Benjamin 1967)

$$I = \int \Big(\rho \int_{-h}^{\eta(x)} \partial_x \varphi \, dy + \rho_1 \int_{\eta(x)}^{h_1+\eta_1(x)} \partial_x \varphi_1 \, dy\Big) dx$$

from the Hamiltonian, $H \to H - cI$. (total momentum is also a conserved quantity)

The next step is to enter these scalings into the Hamiltonian and expand in powers of ε.....

Look at the dynamics of the system in a preferred direction of propagation by decomposing the interface into two components : $r(X, t)$ is the component that is principally right-moving, while $s(X, t)$ (of smaller amplitude $O(\varepsilon^2)$ is principally left-moving.

This leads to a reduced Hamiltonian and the corresponding equations of motion:

$$\partial_t U = J' \nabla H'$$

and J' a modified symplectic matrix, from which one derives the evolution equations.

VI.a Effective equations

In the regime of long-wave internal modes in resonance with the free surface, the motion is described by the coupled system

$$\partial_\tau r + a_1 r \partial_X r + a_2 \partial_X^3 r = \varepsilon^{2\alpha} \lambda_3 \partial_X |v|^2$$
$$\partial_{\tau_1} v = i\Big[\frac{1}{2}\omega_1''(k_0)\partial_X^2 v + a_3 r(X, \varepsilon\tau_1) v + \kappa_1 \varepsilon^{2+2\alpha}|v|^2 v\Big].$$

- Time scales for KdV and Schrödinger are different, $\tau = \varepsilon^3 t$ and $\tau_1 = \varepsilon^2 t$, (t = physical time) .
- The component r closely related to the internal wave ; v is the modulated amplitude related to the fast oscillations of the surface wave.
- Nonlinear terms in the Schrödinger negligible.
- Coupling with the KdV equation appears through a linear operator given by the potential.
- if $\alpha > 0$, KdV equation decouples from the Schrödinger equation; If $\alpha = 0$, another coupling appears as a forcing term in the KdV equation.

VI.b. Dependence of coefficients on ρ_1/ρ and h_1/h

Assume that the amplitude of the surface wave is much smaller than that of the internal wave : $\alpha > 0$.
Leading-order approximation ($\tau = \varepsilon\tau_1$):

$$\partial_\tau r + a_1 r \partial_X r + a_2 \partial_X^3 r = 0 \ ,$$
$$\partial_{\tau_1} v = i\left[\frac{1}{2}\omega_1''(k_0)\partial_X^2 v + a_3 r(X, \varepsilon\tau_1) v\right] \ ,$$

All coefficients explicitly expressed in terms of two free parameters: the relative density ρ_1/ρ and the relative depth h_1/h.

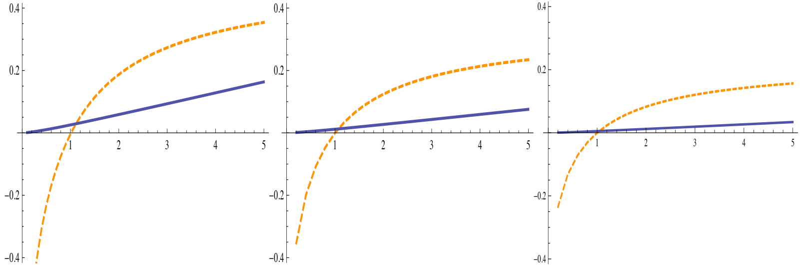


Figure: Parameters a_1 (dashed line) and a_2 (solid line) as functions of depth ratio h_1/h. The density ratio = $\rho_1/\rho = 0.95$ (left), 0.99 (middle) and 0.998 (right).

a_2 is always positive while a_1 changes sign when h_1/h passes a particular value of order 1. When h_1/h is small (realistic situations), the single KdV soliton

$$r(X,\tau) = \frac{3a_2 u_0}{a_1}\operatorname{sech}^2\left[\frac{\sqrt{u_0}}{2}(X - a_2 u_0 \tau)\right] ,$$

is a wave of *depression*.

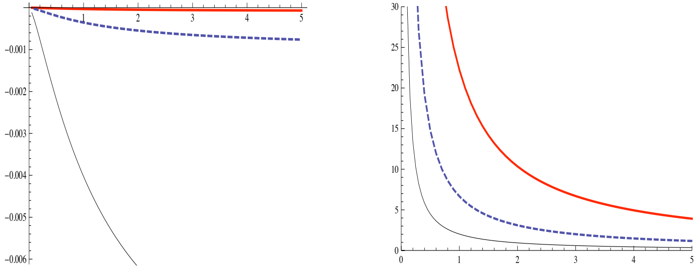


Figure: Dispersion coefficient $\omega_1''(k_0)$ (left) and parameter a_3 (right) as functions of h_1/h for $\rho_1/\rho = 0.95$ (thin line), 0.99 (dashed line) and 0.998 (thick line).

For ρ_1/ρ and h_1/h given, compute c_0; wave number k_0 uniquely determined from the resonance condition $c_0 = \omega_1'(k_0)$.
For ρ_1/ρ close to 1, $\omega_1''(k_0) < 0$ and $a_3 > 0$, always. As h_1/h decreases, $\omega_1''(k_0)$ tends to zero while $a_3 \sim O(1)$.

VII.a The case of a single KdV soliton (depression)

$$\partial_{\tau_1} v = i\left[\frac{1}{2}\omega_1''(k_0)\partial_X^2 v + a_3 r(X, \varepsilon\tau_1)v\right] ,$$

Change of variables

$$v(X,\tau_1) = e^{i(p_1 y + p_2 z)} w(y,z) ,$$

where $y = X - \varepsilon a_2 u_0 \tau_1$, $z = a_3\tau_1$ and

$$p_1 = \frac{\varepsilon a_2 u_0}{\omega_1''(k_0)} , \quad p_2 = \frac{(\varepsilon a_2 u_0)^2}{2a_3\omega_1''(k_0)} ,$$

and denoting

$$\frac{\omega_1''(k_0)}{2a_3} = -\delta^2 ,$$

the Schrödinger equation reduces to

$$-i\partial_z w = -\delta^2\partial_y^2 w + r(y)w .$$

$$-i\partial_z w = -\delta^2 \partial_y^2 w + r(y) w \ .$$

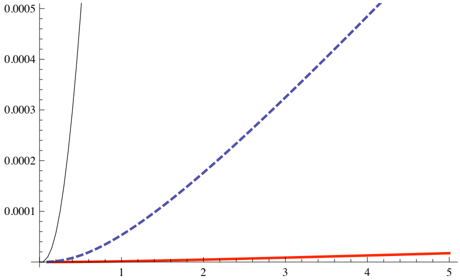


Figure: Parameter δ^2 as a function of h_1/h for $\rho_1/\rho = 0.95$ (thin line), $= 0.99$ (dashed line) and $= 0.998$ (thick line).

The general solution has a spectral decomposition in the form of a linear superposition of solutions

$$w(y, z) = \int e^{i\lambda z} u(y, \lambda) \, d\mu(\lambda) \ ,$$

where $u(y, \lambda)$ satisfies

$$-\delta^2 \partial_y^2 u + ru = \lambda u \ .$$

The spectrum of the operator $-\delta^2 \partial_y^2 + r$ is composed of a finite number of negative eigenvalues $\lambda_0 < \lambda_1 < \cdots < \lambda_N < 0$ along with the continuous spectrum $\lambda > 0$.

VII.b Bound states in the regime $h_1/h < 1$, $\rho_1/\rho \sim 1$.

Regime typical of coastal seas
$\delta \ll 1$ and the internal soliton plays the role of a potential well since $r < 0$.
This is analogous to the semi-classical limit in quantum mechanics. There is a large finite number of negative e.v $\{\lambda_j\}_{j=0,\ldots,N}$ with corresponding localized bound states $\{\psi_j(y)\}_{j=0,\ldots,N}$.

Numerical calculation for $\rho_1/\rho = 0.997$, $h_1/h = 0.2$ and $|r|_{L^\infty} = 0.5$

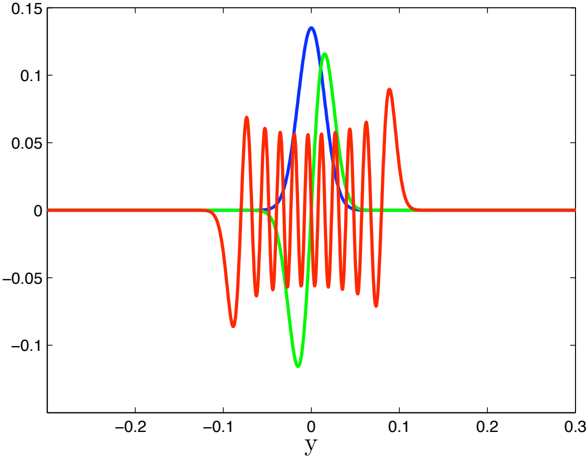


Figure: Superimposed image of three bound states ψ_j (ground state, $j = 2$ and $j = 20$) .

Oceanic conditions corresponding to the Andaman Sea :
$\rho_1/\rho = 0.997$, $h_1/h = 0.266$ and $\varepsilon^2 \sim a/h = 0.069$, as reported by Osborne & Burch (1980) .

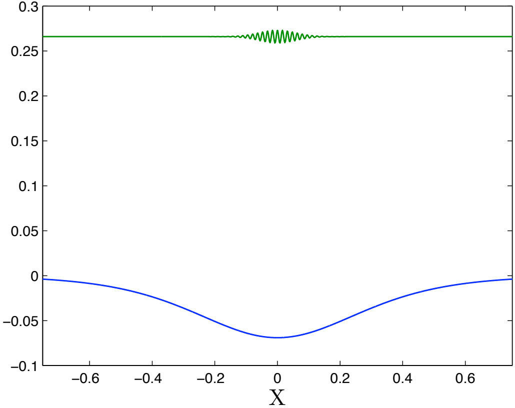


Figure: Internal and surface waves in the physical variables (η, η_1) for the choice of a ground bound state at the free surface. For clarity, the surface wave is magnified.

Off the Oregon coast:
Typical parameters are $\rho_1/\rho = 0.998$, $h_1/h = 0.035$ and $\varepsilon^2 \sim a/h = 0.192$, as given by Bakhanov & Ostrovsky (2002)

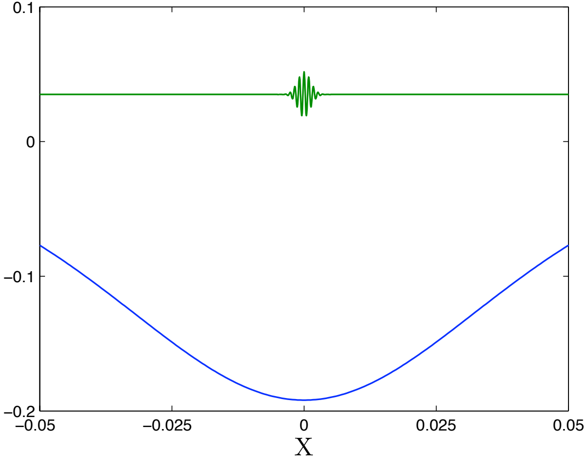


Figure: Internal and surface waves in the physical variables (η, η_1)

Both figures illustrate characteristic features of the rip phenomenon (Osborne & Burch (1980) and Alpers (1985)).

- the disparity in length scales between the internal and surface waves
- the localized shape of the surface modulation.

VII.c. Estimate the absorption of energy into the bound states

$w(y,0)$: initial state of the sea-surface modulation.
Projection of $w(y,0)$ onto the bound states :

$$\langle w(y,0), \psi_j(y)\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S_j(k)\widehat{w}(k,0)\,dk \ ,$$

where

$$S_j(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\psi_j(x)\,dx \ .$$

represents the Fourier absorption profile of the bound state ψ_j.

Proposition

In the limit $\delta \to 0$, for wave numbers k such that $0 < |k| < \delta\sqrt{|r(0)| - \lambda_j}$, there is a lower bound

$$C\,(\delta/j)^{1/4} \le S_j(k) \ ,$$

on the amplitude of the absorption spectral density.

VII.d. Reflection and transmission coefficients

We now turn to the continuous spectrum : semi-axis $\lambda > 0$.

- From the frame of reference of the soliton, the background sea state is incident on the potential well given by $r(y)$, and Fourier components of this sea state are absorbed, reflected or transmitted by it.
- In the semi-classical regime $\delta \ll 1$, the potential well acts as a barrier : We show that for an interval of wave numbers $k^2 = \lambda \in [0, \lambda(\delta)]$, the reflection coefficient is dominant for waves incident on the potential, and the transmission coefficient is small.

We look for a solution with $\lambda > 0$ in the continuous spectrum, in the form of plane waves at $\pm\infty$ with

$$\begin{aligned} u(y) &\sim e^{-i\sqrt{\lambda}y/\delta} + b\, e^{i\sqrt{\lambda}y/\delta}\,, \quad \text{as} \quad y \to -\infty\,, \\ u(y) &\sim c\, e^{-i\sqrt{\lambda}y/\delta}\,, \qquad\qquad\quad \text{as} \quad y \to +\infty\,. \end{aligned}$$

The coefficients b and c are respectively the reflection and transmission coefficients.

Proposition

In the limit δ and $\lambda \to 0$ with the condition $\sqrt{\lambda}/\delta \to 0$, the coefficients of reflection and transmission have the asymptotic values $b = -1 + O(\sqrt{\lambda})$ and $c = O(\sqrt{\lambda})$.

Interpretation of the ‘mill pond’ effect :

- From a stationary reference point on the sea surface, a passing internal soliton absorbs a significant component of the background sea state, while reflecting a further significant component of the wave numbers $k \in [-\sqrt{\lambda(\delta)}, \sqrt{\lambda(\delta)}]$, essentially sweeping this sea state in front of it, and permitting little radiation to be transmitted through the soliton to the sea surface behind it.
- Note that $k = \sqrt{\lambda}/\delta$ plays the role of sideband wave numbers since it is associated with solutions of the Schrödinger equation. Therefore, the limit $k \to 0$ for free-surface modes corresponds to the case of pure monochromatic waves of carrier wave number k_0.

Final remarks:

- This study is restricted to the physical water wave problem in two dimensions. To do: consider the 3d problem (a two-dimensional sea surface).
- The interpretation is restricted to internal wave in the form of a single KdV soliton. It allows a transformation to a stationary frame of reference.
- More general internal waves lead to the Schrödinger equation with a time-dependent potential. A possible approach would be through an adiabatic approximation of Schrödinger evolution with slowly varying potentials.
- Develop a rigorous consistency analysis to justify the asymptotic model.