Harmonic Analysis and Nonlinear Dispersive Equations

July. 17th/2011

→ Consider the Schrödinger Equation given by

$$\begin{cases} i\partial_t u + \Delta_x u = 0 \\ u(x,0) = \varphi(x) \end{cases}$$

Take the Fourier transform to get $0 = i\partial_t \hat{u} - 2\pi|\xi|^2 \hat{u}$. Thus, $\hat{u} = \exp\{-2\pi i|\xi|^2 t\}\,\hat{\varphi}(x)$. Thus, we get that

$$u(x,t) = \int_{\mathbb{R}^d} \varphi(x-y)\, C t^{-d/2} \exp\{ i|x|^2/4t \}\, dx$$

Properties of the linear Schrödinger flow:

(i) $\|e^{it\Delta}\varphi\|_{L^2} = \|\varphi\|_{L^2}$

(ii) $e^{it\Delta} \circ e^{is\Delta} = e^{i(t+s)\Delta}$

(iii) $e^{it\Delta} \circ \tau_{x_0} = \tau_{x_0} \circ e^{it\Delta}$ $[\tau_{x_0} f = f(x-x_0)]$

(iv) $e^{it\Delta} \circ \delta_\tau = \delta_\tau \circ e^{i(t/\tau^2)\Delta}$ $[\delta_\tau^t f = f(\tau f)]$,

where $e^{it\Delta}$ is given by $\widehat{e^{it\Delta}\varphi} = e^{-2\pi i t|\xi|^2}\hat{\varphi}$

(-1)

__Exercise__: (a) Prove (i) → (iv)

(b) Is it true that $\lim_{t\to 0} e^{it\Delta}\varphi = \varphi$? In what sense and for which φ?

→ Consider the Nonlinear Schrödinger Equation (NLS):

$$\begin{cases} (i\partial_t + \Delta_x)\mu = \chi \mu |\mu|^{p-1} \quad (\chi \in \mathbb{R}) \\ \mu(x,0) = \varphi(x) \end{cases}$$

__Note__: Rescale u by Cu and so we can make $\chi = \pm 1$.

Duhamel Formula:

$$u(t) = e^{it\Delta}\varphi - i\int_0^t \exp\{i\Delta(t-s)\}N(s)\,ds,$$

w/ $\quad N(t) = \chi\, u(t)\, |u(t)|^{p-1}$

Heuristically, start with the free solution and then modify it by how the non-linearity evolves the linear part.

- need some control to allow such a to scheme to converge like a contraction

Perturbative Iteration Scheme: Take

(i) $u_1(t) = e^{it\Delta}\varphi$

$\vdots$

(n+1) $u_{n+1}(t) = e^{it\Delta}\varphi - i\int_0^t e^{i(t-s)\Delta}\chi u_n(s)|u_n(s)|^{p-1}\,ds$

Exercise ①: Prove Formally that the conservation of mass and energy for solutions to NLS: if u is a nice solution on an interval I then the quantities $M(t) = \int |u(x,t)|^2\,dx$ and $E(t) = \frac{1}{2}\int \left(|\nabla_x u|^2 + \frac{\chi}{p-1}|u|^{p-1}\right)dx$ are conserved.

- To show conservation of M look at
  $\partial_t M = \int (\partial_t u\,\bar{u} + \partial_t \bar{u}\, u)$

$$= \int (\underbrace{\_\ \_\ \_}_{}) = 0$$

→ use Schrödinger Equation and integration by parts.

→ Assume $\varphi \in H^\sigma(\mathbb{R}^d)$. We need a space $X^\sigma = X^\sigma(I)$ such that $e^{it\Delta}\varphi \in X^\sigma(I)$ if $\varphi \in H^\sigma$ and $\int_0^t \exp\{i(t-s)\Delta\} f(s)|f(s)|^2\,ds \in X^\sigma(I)$ if $\cancel{f} \in X^\sigma(I)$

Exercise 1: Assume $d=3$, $p=2$. Show that the perturbative scheme produces a solution in $X^\sigma(I)=C(I,H^\sigma)$ provided that σ is sufficiently large and $|I|$ sufficiently small.

Strichartz Estimates: (a) if (q,r) are admissible (i.e. $2/q + d/r = d/2$ for $(q,r)\in(2,\infty]\times[2,\infty]$) then $\|e^{it\Delta}\varphi\|_{L^q_t L^r_x} \lesssim_q \|\varphi\|_{L^2}$

(b) if (q,r), $(\bar q,\bar r)$ are admissible (w/ $\frac{1}{q}+\frac{1}{q'}=1$ and $\frac{1}{r}+\frac{1}{r'}=1$) then

$$\Big\|\int_{-\infty}^{t} e^{i(t-s)\Delta}N(s)\Big\|_{L^q_t L^r_x} \lesssim_{q\bar q} \|N\|_{L^{\bar q'}_t L_x^{\bar r'}}$$

Notation: $L^q_t L^r_x = L^q(I,L^r)$

→ Another way of writing (b) is if (q,r) is an admissible pair then

$$\Big\|\int_{-\infty}^{t} e^{i(t-s)\Delta}N(s)\Big\|_{L^\infty_t L^2_x\cap L^q_t L^r_x} \lesssim_q \|N\|_{L^1_t L^2_x + L^{q'}_t L^{r'}_x}$$

Exercise: (a) Show that if $f:\mathbb{R}^d\times I\to\mathbb{C}$ is a measurable function $p,q\in[1,\infty]$ then

$$\|f\|_{L^p_t L^q_x} = \sup_{\|g\|_{L^{p'}_t L^{q'}_x}} \Big|\int_{\mathbb{R}^d\times I} fg\,dx\,dt\Big|$$

(b) Show that if (q,r), $(\bar q,\bar r)$ are admissible pairs $q\le\bar q$ then

$$\|f\|_{L^{\bar q}_t L^{\bar r}_x} \le \|f\|_{L^q_t L^r_x\cap L^\infty_t L^2_x}$$

$$\|f\|_{L^{\bar q'}_t L^{\bar r'}_x} \geq \|f\|_{L^{q'}_t L^{r'}_x + L^1_t L^2_x}$$

Hint: split f into $f = f 1_A + f 1_{\mathbb{R}\setminus A}$ with $A=\{|f|\le\lambda\}$

(5)

Note: $\|f\|_{A+B} = \inf_{f=f_1+f_2} [\|f_1\|_A + \|f_2\|_B]$

__Exercise 3:__ (Fractional Integration)

Hint: the maximal operator is bounded on L^p

$$\|f * |y|^{-\gamma}\|_{L^q} \leq C_{p,q} \|f\|_{L^p}$$

if $0<\gamma<d$, $1<p<q<\infty$, $\frac{1}{q} = 1/p - \frac{d-\gamma}{d}$

__Exercise 2:__ (Complex Interpolation) Let

Hint: Think Hadamard 3 line thm

$S = \{z \in \mathbb{C} \mid \mathrm{Re}\, z \in [0,1]\}$ and assume that $T_z : \mathcal{S}(\mathbb{R}^d) \to L^1_{loc}(\mathbb{R}^d)$, $z \in S$ is an analytic family of operators (i.e. the maps $z \to \int_{\mathbb{R}^d} T_z f \cdot g \, dz$ for $f, g \in \mathcal{S}(\mathbb{R}^d)$) is analytic & bounded in $\mathrm{int}(S)$ and continuous in S. Assume that $p_0, q_0, p_1, p_2 \in [1,\infty]$ and $\|T_{iy}\|_{L^{p_0} \to L^{q_0}} \leq M_0$, $\|T_{1+iy}\|_{L^{p_1} \to L^{q_1}} \leq M_1$ then for any $\theta \in [0,1]$ $\|T_\theta\|_{L^{p_\theta} \to L^{q_\theta}} \leq M_0^{1-\theta} M_1^{\theta}$ for $1/p_\theta = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $1/q_\theta = \frac{1-\theta}{q_0} + \theta/q_1$.

__Proof of the Strichartz Estimate:__

__Proof of (a)__ Game Plan: (i) Dispersive Estimates

$$\|e^{it\Delta}\|_{L^2_x \to L^2_x} \lesssim 1$$
$$\|e^{it\Delta}\|_{L^1_x \to L^\infty_x} \lesssim |t|^{-d/2}$$

(ii) Use Interpolation to get

$\frac{1}{r} + \frac{1}{r'} = 1 \longrightarrow$ $\|e^{it\Delta}\|_{L^{r'} \to L^r} \lesssim |t|^{-d(1/2 - 1/r)}$

Outline of Argument

$\rightsquigarrow$ The TT^* argument: we want

$$\|e^{it\Delta}\varphi\|_{L_t^q L_x^r} \lesssim_q \|\varphi\|_{L^2}$$

$$\overset{①}{\Longleftrightarrow} \int_{\mathbb{R}^d\times\mathbb{R}} (e^{it\Delta}\varphi)\, g(x,t)\, dx\, dt \lesssim_q \|\varphi\|_{L^2} \|g\|_{L_t^{q'} L_x^{r'}}$$

$$\overset{②}{\Longleftrightarrow} \Big\| \int_{\mathbb{R}^d\times\mathbb{R}} e^{-it|\xi|^2} e^{ix\cdot\xi} g(x,t)\, dx\, dt \Big\|_{L^2_\xi} \lesssim_q \|g\|_{L_t^{q'} L_x^{r'}}$$

$$\overset{③}{\Longleftrightarrow} \Big| \int_{\mathbb{R}^d\times\mathbb{R}} \int_{\mathbb{R}^d\times\mathbb{R}} g(x,t)\, \overline{g}(y,s)\, K(x-y, t-s)\, dx\, dt\, dy\, ds \Big| \lesssim_q \|g\|^2_{L_t^{q'} L_x^{r'}}$$

①// $\quad (e^{it\Delta}\varphi) = c\int e^{ix\cdot\xi} e^{-it|\xi|^2} \hat{\varphi}\, d\xi$

$$\int_{\mathbb{R}^d\times\mathbb{R}} (e^{it\Delta}\varphi)\, g = c\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}} e^{ix\xi} e^{-it|\xi|^2} \hat{\varphi}\, g(x,t)\, d\xi\, dx\, dt$$

Fubini or Tonelli $\Rightarrow = c\int_{\mathbb{R}^d} \hat{\varphi} \left(\int_{\mathbb{R}^d\times\mathbb{R}} e^{ix\xi} e^{-it|\xi|^2} g(x,t)\, dx\, dt \right) d\xi$

②// Notice that the middle part is indeed ②

③// This is what we need. To show this let

$$H(\xi) = \int_{\mathbb{R}^d\times\mathbb{R}} e^{ix\xi} e^{-it|\xi|^2} g(x,t)\, dx\, dt$$

then $\quad \|H(\xi)\|_{L^2} = \Big| \int_{\mathbb{R}^d\times\mathbb{R}} \int_{\mathbb{R}^d\times\mathbb{R}} g(x,t)\, \overline{g}(y,s)\, K(x-y, t-s)\, dx\, dt\, dy\, ds \Big|$

by just expanding. Could have $g(x,t)\overline{h(y,s)}$ to get $\lesssim \|g\|_{L_t^{q'} L_x^{r'}} \|h\|_{L_t^{q'} L_x^{r'}}$

The outlined gameplan is heuristically:

$T: L^2 \to X, \quad T^*: X^* \to L^{2*} \cong L^2, \quad TT^*: X^* \to X$

However, using the dispersive estimate

$$\left| \int_{\mathbb{R}^{2d}} g(x,t)\, h(y,s)\, K(x-y, t-s)\, dx\, dy \right| \lesssim G(t) G(s) |t-s|^{-d(1/2 - 1/r)}$$

where $G(t) = \left[\int_{\mathbb{R}^d} |g|^{r'} \right]^{1/r'}$. The desired bound follows from fractional integration. That is, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} G(t) G(s) |t-s|^{-d(1/2 - 1/r')}$$

$$= \int_{\mathbb{R}} G(t)\; G * |t|^{-d(1/2 - 1/r')}\, dt$$

$$\lesssim \|G(t)\|_{L^q} \, \| G * |t|^{-\gamma} \|_{L^{q'}}$$

exercise ③ tells us $\to$ $\lesssim \|G\|_{L^{q'}} \|G\|_{L^{q'}}$

$$= \|g\|_{L^{q'}_t L^{r'}_x}$$

$\rightsquigarrow$ Check that this γ is good. Note, the fractional integration argument fails for $q=2$ and so the argument doesn't hold for $q=2$.

$\boxed{\text{QED}}$

<u>Sketch of (b)</u>: $\left\| \int_{-\infty}^{t} e^{i(t-s)\Delta} N(s)\, ds \right\|_{L^2_x}$

$$\leq \int_{-\infty}^{t} \| e^{i(t-s)\Delta} N(s) \|_{L^2_x}\, ds \lesssim \int_{-\infty}^{t} \|N(s)\|_{L^2_x}\, ds$$

Exercise 4: Prove the remaining Strichartz Estimate
Hint: there are 3 other estimates required

$\boxed{QED(b)}$

$\boxed{QED}$

Fixed Point Theorem: Let B be a Banach Space. Given an ~~[illegible]~~ operator $T: B \to B$, if $\|Tu - Tv\| \le c\|u-v\|_B$ with $c<1$ then $\exists! \, u \in B \ni Tu = u$.

Proof: Take $u_0 \in B$ and $T^k(u) = T \circ T^{k-1}(u)$. T is Lipshitz and hence cont. We can easily show that $\{T^k u\}$ is Cauchy and hence, since this is a Banach Space, then $T^k u$ converges. Continue like this. To prove uniqueness, suppose $Tu = u$ and $Tv = v$ then $\|u-v\| \le c\|u-v\|$, which contradicts $c<1$.

So prove existence and uniqueness of a linear operator L we need to find a space B and show that L is a contraction.

Proof of exercise ①: $Tu = e^{it\Delta}\varphi - i\int_0^t e^{i(t-s)\Delta} N(u)\, ds$ with $d=3$ and $p=2$. Aff $\varphi \in H^\sigma(\mathbb{R}^3)$, $B_\varphi = \{\sup_{t\in I} \|u(t,\cdot)\|_{H^\sigma} \le 2\|\varphi\|_{H^\sigma}$

(i) I am going to assume without proof that H^σ is indeed complete. Thus, we have that $\forall \varepsilon/2 \; \exists t \ni \|u_m - u\|_B = \sup_{t\in I} \|u_m - u\|_{H^\sigma} \le \varepsilon/2 + \|u_m(t,\cdot) - u(t,\cdot)\|$ (where we will assume that $\{u_m\}$ is Cauchy). So, take $\{u_m\}$ Cauchy in $\|\cdot\|_B$ and note that $\forall t \; \|u_n(t,\cdot) - u_m(t,\cdot)\|_{H^\sigma} \le \|u_n - u_m\|_B$ and by completeness $\exists u(t,x) \ni u_n \to u$. So, from above take $m \ge M$ large enough to see that $u_m \to u$ in B.

(ii) We have by the Sobolev Embedding thm that $\|f\|_{L^\infty} \le C\|f\|_{H^\sigma}$ for $\sigma > 3/2$. Note that $N(u) = u|u|$. So,

$$\|Tu - Tv\|_{H^\sigma} \le \left\| \int_0^t e^{i(t-s)\Delta}[N(u)-N(v)]\,ds \right\|_{H^\sigma}$$

Minkowski $\rightsquigarrow$ $\le \int_0^t \|e^{i(t-s)\Delta}[N(u)-N(v)]\|_{H^\sigma}\,ds$

$\le b\int_0^t \|N(u)-N(v)\|_{H^\sigma}\,ds$ → this operator norm comes from the proof of the Strichartz estimate.

Triangle inequality, then Reverse triangle inequality

$= b\int_0^t \|u(|u|-|v|) + (u-v)|v|\|_{H^\sigma}$

$\le b\int_0^t \|u(u-v)\|_{H^\sigma} + \|(u-v)v\|_{H^\sigma}$

$\rightsquigarrow$ Take $\sigma \ge 4$ w/ $\sigma \in \mathbb{Z}$. Then,

$\|(u-v)(v)\|_{H^\sigma} = \|v(u-v)\|_{L^2} + \|Dv(u-v) + vD(u-v)\|_{L^2} + \|Dv\cdot D(u-v) + D^2v(u-v) + vD^2(u-v)\|_{L^2}$

$\uparrow \sigma = 4$

b/c for $\sigma = 4$, $\|f\|_{L^\infty} \le C\|f\|_{H^{\sigma/2}}$ and $H^\alpha \subseteq H^\beta$ $\forall \beta \le \alpha$ by Sobolev embedding

$\le C\max\{\|v\|_{H^2}, \|v\|_{H^3}, \|v\|_{H^4}\}\|u-v\|_{H^\sigma}$

$\le C\|v\|_{H^4}\|u-v\|_{H^\sigma}$

Thus, we need $u, v \in H^4$. So,

$\|Tu-Tv\|_{H^\sigma} \le D2\int_0^t \|u-v\|_{H^\sigma}\|\varphi\|_{H^\sigma}$

$\uparrow$ by def'n of $u, v \in B$

$\le D2\|\varphi\|_{H^\sigma}\, t\, \|u-v\|_B$

Thus, take t s.t. $2tD\|\varphi\|_{H^\sigma} < 1$ and we get that $\|Tu-Tv\|_B \le c'\|u-v\|_B$ w/ $I = [0,t]$ and $c' < 1$.

(iii) Thus use the Banach fixed point theorem to conclude local well posedness.

$\boxed{\text{QED}}$ $\boxed{\text{Exercise 1}}$

[redacted]

[redacted]

July. 18th/2011

There is another result that is important. It is

Prop: If $3/s = 3/r - 1$, $(r,s)\in(1,\infty)$ then $\|f\|_{L^s} \lesssim \|f\|_{L^r} + \|\nabla f\|_{L^r}$
For $r \geq 3$ then $s = \infty$.

Note: This is a Sobolev Embedding thm.

Assume $\varphi \in H^1(\mathbb{R})$ and fix an admissible pair (q,r). Define
$X^1(I) = \{ f \in C(I, H^1) \mid \|f\|_{X^1} := \|f\|_{L^\infty_t L^2_x \cap L^q_t L^r_x} + \|\nabla f\|_{L^\infty_t L^2_x \cap L^q_t L^r_x} < \infty \}$

Exercise 1: Show that $X^1(I)$ is a Banach Space.

Lemma 1: If $\varphi \in H^1$ then $e^{it\Delta}\varphi \in X^1(I)$ and
$\|e^{it\Delta}\varphi\|_{X^1(I)} \le C_0 \|\varphi\|_{H^1}$

Lemma 2: If $f, g, h \in X^1(I)$ then

$$F = \int_0^t e^{i(t-s)\Delta}[f(s)g(s)h(s)]\,ds \in X^1(I)$$

and $\|F\|_{X^1(I)} \le C|I|^\delta \|f\|_{X^1(I)} \|g\|_{X^1(I)} \|h\|_{X^1(I)}$

Proof of Lemma 2: $\|\nabla F\|_{L^\infty_t L^2_x \cap L^q_t L^r_x} \lesssim \|\nabla N\|_{L^1_t L^2_x}$

$N(s) = f(s)g(s)h(s)$ $\rightarrow$ $\|F\|_{L^\infty_t L^2_x \cap L^q_t L^r_x} \lesssim \|N\|_{L^1_t L^2_x}$

Examine $\nabla f\, gh$ and use symmetry to get it for when the gradient hits g and h.

Use Hölder twice. Note the modified Hölder $\|fg\|_{L^r} \le \|f\|_p \|g\|_q$, with $1/p + 1/q = 1/2$.

$$\|g\|_{L_t^4 L_x^\infty} \le \|g\|_{L_t^4 L_x^3} + \|\nabla g\|_{L_t^4 L_x^3} \lesssim \|g\|_{X^1}$$

and

$$\|h\|_{L_t^4 L_x^\infty} \lesssim \|h\|_{X^1}$$

Thus, $\|gh\|_{L_t^2 L_x^\infty} \lesssim \|g\|_{X^1} \|h\|_{X^1}$

Note that $\|\nabla f\|_{L_t^\infty L_x^2} \lesssim \|f\|_{X^1}$

So, $\|\nabla f g h\|_{L_t^2 L_x^2} \lesssim \|f\|_{X^1} \|g\|_{X^1} \|h\|_{X^1}$

So, $\|\nabla N\|_{L_t^1(I) L_x^2} \lesssim |I|^{1/2} \|f\|_{X^1} \|g\|_{X^1} \|h\|_{X^1}$

$\rightsquigarrow$ There is something wrong with all of our arguments. We assumed that $s=\infty$, but Sobolev Embedding doesn't hold for this. So move s away from ∞ and we see that the only term that changes is $|I|^{1/2}$ and it changes to $|I|^{1/2+\delta}$.

$\boxed{\text{QED Lemma 2}}$

Construction of non-linear solution:

$$u_1(t) = e^{it\Delta}\varphi$$

$$\vdots$$

$$u_{n+1}(t) = e^{it\Delta}\varphi - i\int_0^t e^{i(t-s)\Delta}[u_n|u_n|^2]\,ds$$

Assume $\|\varphi\|_{H^1} \le A$. The two lemmas above show that $\{u_n\}$ is a Cauchy sequence in the complete space (Y is complete b/c X^1 is complete):

$$Y(I) = \{f \in X^1(I) \mid \|f\|_{X^1(I)} \le 2C_0 A\}$$

if $|I|$ is sufficiently small (depending on A).

Need to show that (i) $u_n \in Y(I)$ $\forall n$
(ii) $\{u_n\}$ is in fact Cauchy
(iii) Conclude that $\{u_n\}$ converges.

<u>Conclusion 1</u>: Assuming $\varphi \in H^1$ there is $\varepsilon = \varepsilon(\|\varphi\|_{H^1})$ small and a unique solution $u \in X^1(-\varepsilon, \varepsilon)$ of the equation $u(t) = e^{it\Delta}\varphi - i\int_0^t e^{i(t-s)\Delta}[u(s)|u(s)|^2]\,ds$. Moreover the mapping $\varphi \to u$ is cont. from H^1 to $X^1(-\varepsilon, \varepsilon)$.

<u>Exercise 2</u>: Show that the mass an energy of the solution constructed above is conserved. That is, show

$$(i)\ \int |u|^2 dx = M$$

$$(ii)\ \tfrac{1}{2}\int |\nabla_x u|^2 + \tfrac{1}{4}\int |u|^4 dx = E$$

are conserved. Hint: constructed solution solves the Schrödinger Eqn so proceed as we did before with exercise from last lecture.

<u>Global Well Posedness</u>: Assuming $\varphi \in H^1(\mathbb{R}^3)$ and $I \subseteq \mathbb{R}$ is a bounded interval, there is a unique solution $u \in X^1(I)$ of the equation

$$u(t) = e^{it\Delta}\varphi - i\int_0^t e^{i(t-s)\Delta}\varphi\,[u(s)|u(s)|^2]\,ds.$$

Moreover, the mapping $\varphi \to u$ is cont. from $H^1 \to X^1(I)$

<u>Note</u>: changing $p=3$ to general p will break down because we required a trilinear estimate but for $p>5$ the H^1 theory breaks down and so we don't get a general p-linear estimate.

Thm: $\begin{cases}(i\partial_t+\Delta)u = u|u|^2 \\ u(x,0)=\varphi(x)\end{cases}$

Let $\varphi\in H^1(\mathbb{R}^3)$. Then is a space so that we have existance of a solution which is unique & $u(t,x)$ is defined globally.

→ Two steps to prove this
- local existance
- conserved quantities

Let $X'(I) := \{u \mid \|u\|_{L^\infty_t L^2_x \cap L^q_t L^r_x} + \|\nabla u\|_{L^\infty_t L^2_x\cap L^q_t L^r_x} \le 2\|\varphi\|_{H^1}\}$

Let $Tu = e^{it\Delta}\varphi - i\int_0^t e^{i(t-s)\Delta}N(u)\,ds$ then we can show $T: X'(I)\to X'(I)$. To do this, we need to show that $\|\nabla Tu\|_{L^\infty_t L^2_x\cap L^q_t L^r_x} \le 2\|\varphi\|_{H^1}$.

To do this, we remember that for $2/q + d/r = d/2$ then (i) $\|e^{it\Delta}\varphi\|_{L^q_tL^r_x}\le C_q\|\varphi\|_{L^2}$ and
(ii) $\|\int_{-\infty}^t e^{i(t-s)\Delta}N(s)\,ds\|_{L^\infty_t L^2_x\cap L^q_tL^r_x} \le C_q \|N(s)\|_{L^1_tL^2_x + L^{q'}_tL^{r'}_x}$.
These are our Strichartz estimates.

Thus, $\|\nabla Tu\|_{L^\infty_t L^2_x\cap L^q_t L^r_x} \le \|\varphi\|_{H^1} + \|\nabla N(u)\|_{L^1_tL^2_x}$. Next, we examine $\|\nabla f g h\|_{L^1_tL^2_x} \le \|\nabla f\|_{L^{2-}_t L^{2+}_x}[\|g\|_{L^{4+}_tL^{3-}_x}+\|\nabla g\|_{L^{4+}_tL^{3-}_x}]*$
$[\|h\|_{L^{4+}_tL^{3-}_x}+\|\nabla h\|_{L^{4+}_tL^{3-}_x}]$

→ by Sobolev Embedding

by Hölder on $\|\nabla f\|_{L^{2-}_tL^{2+}_x}$

$$\boxed{\|\varphi\|_{L^s}\le\|\varphi\|_{L^r}+\|\nabla\varphi\|_{L^r} \quad \text{for } 1/r - 1/s = 1/3}$$

Thus, $\|\nabla fgh\|_{L^1_tL^2_x}\le \|\nabla f\|_{L^{\infty-}_tL^{2+}_x}|I|^{1/2+}\|g\|_{X'(I)}\|h\|_{X'(I)}$

Continue in this manner to show that $T: X'\to X'$!

Development of Some Important Tools

Now, recall Fractional Integration. Take $f \in L^p(\mathbb{R}^d)$ for $p>1$ and let $V(f) = f * |y|^{-\gamma}$ for $0<\gamma<d$.

Prop'n: $\|V(f)\|_{L^q} \le C_q \|f\|_{L^p}$ for $1<p<q<\infty$ and

$$1/p - 1/q = 1 - \gamma/d$$

Maximal function: $Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f| \, dx$

Prop'n: (a) $\|Mf\|_{L^p} \le C_p \|f\|_{L^p}$ for $p \in (1,\infty]$
(b) $m(\{x \mid Mf > \alpha\}) \le \|f\|_{L^1}/\alpha$

Proof: for $p=\infty$, trivial. We showed in the discrete section how to prove (b).

To show (a) for $1<p<\infty$, Let $A = \{x \mid |f| > \frac{\alpha}{2}\}$

Thus, $\int |f| \chi_A \le \int_{|f| \ge 1/2\alpha} |f|^p (1/2\alpha)^{1-p} < \infty$.

So, $|f| = |f|\chi_A + |f|\chi_{A^c}$. Thus,

$$1/p \|Mf\|_{L^p}^p = \int_0^\infty |\{Mf > \alpha\}| \, \alpha^{p-1} \, d\alpha$$

Now, since $M(f+g) \le Mf + Mg$ then

$$\le \int_0^\infty d\alpha \Big(|\{ M(f\chi_A) > \alpha/2\}| + \underbrace{|\{ M(f\chi_{A^c}) > \alpha/2 \}|} \Big) \alpha^{p-1}$$

$\Rightarrow$ this set is always empty (think about it, not hard to see)

$$= \int_0^\infty d\alpha \, |\{ Mf\chi_A > \alpha/2 \}| \, \alpha^{p-1}.$$

$$\le \int_0^\infty d\alpha \, \frac{\|f\chi_A\|_{L^1}}{1/2\alpha} \alpha^{p-1}$$

$$= \int_0^{\infty} |f| \left(\int_0^{2|f|} 2\alpha^{p-2}\, d\alpha \right) dx = C \|f\|_{L^p}^p$$

$\boxed{\text{QED Prop'n}}$

$$|V(f)| = \left| \int f(x-y)\, |y|^{-\gamma}\, dy \right|$$

$$\leq \underbrace{\int_{|y|>R} |f(x-y)|\,|y|^{-\gamma}\, dy}_{\mathrm{I}} + \underbrace{\int_{|y|\leq R} |f(x-y)|\,|y|^{-\gamma}}_{\mathrm{II}}$$

I $\leq \|f\|_{L^p} \, \| |y|^{-\gamma} \chi_{|y|>R} \|_{L^q}$. Choose q so that the integral converges.

II $\displaystyle\int_{|y|\leq R} |f(x-y)|\,|y|^{-\gamma} = \sum_{k=0}^{\infty} \int_{2^{-k-1} < \frac{|y|}{R} \leq 2^{-k}} |f(x-y)|\,|y|^{-\gamma}$

$$\leq \sum_{k=0}^{\infty} (2^{-k}R)^{-\gamma}\, Mf(x)\, c\, (2^{-k}R)^d$$

$$\leq R^{d-\gamma}\, Mf(x)$$

Thus, $|U(f)| \leq R^{d-\gamma} Mf(x) + \|f\|_{L^p} \, \| |y|^{-\gamma} \chi_{|y|>R} \|_{L^q}$

Continue in this fasion to get an estimate for $|U(f)|$.

<u>exercise</u>: finish the estimate.

<u>Hint</u>: take $R \ni R^{d-\gamma} Mf = \|f\|_{L^p} R^{d/p' - \gamma}$

then $|U(f)| \leq 2c\, M(f) \left(\frac{\|f\|_{L^p}}{Mf} \right)^{p/d(d-\gamma)} \rightarrow \frac{1}{p} - \frac{1}{q} = 1 - \frac{\gamma}{d}$

Thus, $\boxed{|U(f)| \leq 2c \|f\|_{L^p}^{1-p/q} (Mf)^{p/q}}$

(6)

exercise: $\|f\|_{L^r} \le \|f\|_{L^p}^{1-\theta} \|f\|_{L^q}^{\theta}$ for $1/r = \frac{1-\theta}{p} + \theta/q$,

$1 \le p \le r < q \le \infty$

exercise: operator T, semilinear (i.e. $T(f+g) \le Tf + Tg$) w/ $\|Tf\|_{L^{q_0}} \le M_0 \|f\|_{L^{p_0}}$ and $\|Tf\|_{L^{q_1}} \le M_1 \|f\|_{L^{p_1}}$ then $\|Tf\|_{L^q} \le M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p}$

for $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $1/p = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$

Sobolev Embedding Theorem: (a) $W^{k,p} \hookrightarrow L^q$ for $1/p - 1/q = k/d$

(b) $W^{k,p}(\mathbb{R}^d) \hookrightarrow L^\infty$ for $p > d/k$

exercise: Check the scaling

Proof of Sobolev Embedding: (a) We can assume that the functions are compact and smooth as we want. Use limits to get it for the whole space.

Also, we only need to worry about $k=1$ because $\|f\|_{W^{1,p_1}} = \|f\|_{L^{p_1}} + \|\partial f\|_{L^{p_1}} \underset{\text{by induction}}{\le} (\|f\|_{L^{p_2}} + \|\partial f\|_{L^{p_2}}) + (\|\partial f\|_{L^{p_2}} + \|\partial^2 f\|_{L^{p_2}})$

$= \|f\|_{L^{p_2}} + 2\|\partial f\|_{L^{p_2}} + \|\partial^2 f\|_{L^{p_2}} \le 2\|f\|_{W^{2,p_2}}$

and we can continue this via induction (where $1/p_2 - 1/p_1 = 1/d$)

Thus, by induction, $\|f\|_{L^q} \le \|f\|_{W^{2,p}}$ for $\frac{1}{p} - \frac{1}{q} = 2/d$.

Case ①: $p > 1$

By FTC, $f(x) = -\int_0^\infty \frac{d}{dr} f(x+r\omega)\, dr$

$$= \frac{-1}{c} \int_{S^{d-1}} \int_0^\infty d/dr\, f(x+r\omega)\, dr\, d\omega$$

$$= -1/c \int_{\mathbb{R}^d} d/dr\, f(x-y)\, |y|^{-(d-1)}\, dy$$

Using the result we obtained earlier on fractional integration:

$$\|U(f)\|_{L^q} \leq C_q \|f\|_{L^p} \quad \text{for} \quad 1/p - 1/q = 1 - \gamma/d.$$

Thus, $\|f\|_{L^q} = C \| d/dr\, f * |y|^{-d-1} \|_{L^q}$

$$\lesssim \| d/dr\, f \|_{L^p}$$

$$\text{w/} \quad 1/p - 1/q = 1 - \frac{d-1}{d} = 1/d$$

Also, $d/dr = \frac{y_i \partial_i}{|y|} \Rightarrow \left|\frac{df}{dr}\right| \leq |\partial f|$

Thus, $\|f\|_{L^q} \lesssim \|\partial f\|_{L^p}$

Case ②: $p = 1$ and so $1 - 1/q = 1/d$

$$\Rightarrow q = \frac{d}{d-1}$$

- For $d=1$ $\Rightarrow q = \infty$. So, $|f(x)| = \left|\int_{-\infty}^x d/dy\, f(y)\, dy\right|$

Thus, $|f(x)| \leq \|\partial f\|_{L^1}$ $\forall x$ and so

$$\|f\|_{L^\infty} \leq \|\partial f\|_{L^1}$$

• <u>For $d=2$:</u> $|f| = \left|\int_{-\infty}^{x} \partial/\partial x f \, dx\right| \leq \int_{-\infty}^{x} |\partial/\partial x f|$

and $|f| = \left|\int_{-\infty}^{y} \partial/\partial y f \, dy\right| \leq \int_{-\infty}^{y} |\partial/\partial y f| \, dy$

Thus, $|f|^2 \leq \int_{-\infty}^{x} |\partial/\partial x f| \, dx \int_{-\infty}^{y} |\partial/\partial y f| \, dy$

$\Rightarrow \|f\|_{L^2}^2 \leq \|\partial_x f\|_{L^1} \|\partial_y f\|_{L^1}$

$\Rightarrow \|f\|_{L^2} \leq \|\partial_x f\|_{L^1}^{1/2} \|\partial_y f\|_{L^1}^{1/2} \leq c\left(\|\partial_x f\|_{L^1} + \|\partial_y f\|_{L^1}\right)$

• <u>for $d \geq 3$</u> do the same for the rest, but use Hölder's Inequality instead.

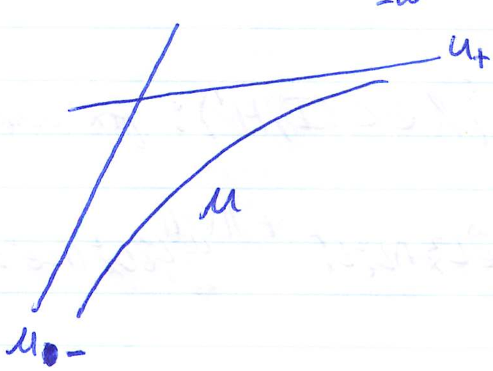
(b) Left as an exercise

$\boxed{QED}$

<u>July 19th/2011</u>

<u>Def'n:</u> We say a solution u scatters if as $t \to \pm\infty$ there is $u \in H^1$ such that

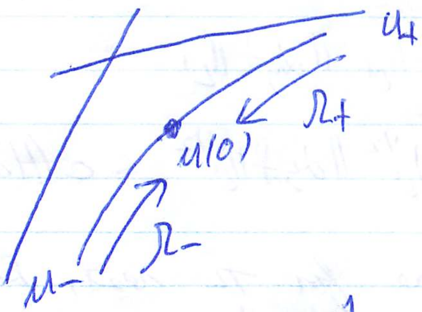
$$\lim_{t \to \pm\infty} \| e^{-it\Delta} u - u_\pm \|_{H^1} = 0$$



→ We see that for large t u looks like an asymptotically free state. That is, it looks like u_+ at $t=\infty$ and u_- at $t=-\infty$

We have $u_+ = \varphi - i \lim_{t \to \infty} \int_0^t e^{-it\Delta} [u(s)|u(s)|^2] \, ds$

Theorem: For every asymptotic state $u_+ \in H^1$ there is a unique solution in H^1 that scatters to u. Moreover, the wave operator $\Omega_+ : H^1 \to H^1$ w/ $\Omega_+(u_+) = u(0)$ is well defined and continuous.



⇝ By global well-posedness, we have that Ω_+ is injective. Surjectivity is not necessary.

Proof (Sketch): step 1 Evolution from $t=\infty$ to $t=t_0$. Solve

Guess this and check → $$u(t) = e^{it\Delta}u_+ + i\int_t^\infty e^{i(t-s)\Delta}\left[u(s)|u(s)|^2\right]ds$$

using a fixed point argument in the metric space $X^1 \cap B_\delta(Y^1)$,

$$\|f\|_{Y^1} := \|f\|_{L^5_{t,x}} + \|f\|_{L^q_t L^r_x} + \|\nabla f\|_{L^q_t L^r_x}$$

Need to show, first, that u is indeed in $X^1 \cap B_\delta(Y^1)$ for large enough t_0.

Remember: $X^1(I) = \{ f \in C(I, H^1) :$ fix admissible (q,r)

$$\|f\|_{X(I)} := \|f\|_{L^\infty_t L^2_x \cap L^q_t L^r_x} + \|\nabla f\|_{L^\infty_t L^2_x \cap L^2_t L^r_x} < \infty \}$$

Step 2: Evolution from $t=t_0$ to $t=0$ solve

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) - i\int_{t_0}^t e^{i(t-s)\Delta}\left[u(s)|u(s)|^2\right]ds$$

as in the global well posedness theory.

⊛ ⇝ Uniqueness:

⇝ Continuity: Yes, we prove that we have some sort of Lipshitz bound and hence is cont.

$\boxed{\text{QED}}$

Question: Is this operator surjective? When it is surjective then we have what is called asymptotic completeness.

So, for focusing case the answer is no; we don't have asymptotic completeness. The answer is yes for the defocusing case.

cubic NLS

Exercise 1: Let $M(t) = 2\,\mathrm{Im} \int_{\mathbb{R}^d} \partial_j a \, \bar{u} \, \partial_j u(t,x) \, dx$

Show that $\partial_t M = 4\,\mathrm{Re} \int \partial_j \partial_k a \, \partial_j \bar{u} \, \partial_k u - \int \Delta^2 a |u|^2 + \int \Delta a |u|^4 dx$

cubic NLS

Exercise 2: (Spacetime bound implies asymptotic completeness): Prove scattering assuming that $\|u\|_{L^4_{x,t}} \lesssim 1$.

cubic NLS

Exercise 3: In the defocusing case prove the spacetime bound $\|u\|_{L^4_{x,t}} \lesssim 1$

<u>Proof of Exercise</u>: Want to prove part (b) of the Sobolev embedding theorem. That is, $\|f\|_{L^\infty} \lesssim \|f\|_{W^{k,p}}$ for $p > k/d$. Thus, we will be able to conclude that $W^{k,p} \subseteq L^\infty$

WLOG, we can again assume $k=1$. This is because if $\|f\|_{L^\infty} \lesssim \|f\|_{W^{1,p}}$, then use part (a) to work your way down. Thus, we want to show that $\|f\|_{L^\infty} \lesssim \|f\|_{W^{1,p}}$.

So, we can also assume that $\text{supp}(f) \subset\subset \mathbb{R}^d$ and we can do this by just taking limits to recoup the identity $\forall f \in W^{1,p}$. So,

$$c f(x) = \int_{S^{n-1}} f \, d\sigma = -\int_0^\infty \int_{S^{n-1}} d/dr \, f(x + r\omega) \, dr \, d\omega$$

Fubini

$$= -\int_{\mathbb{R}^d} d/dr \, f(x-y) \, |y|^{-d+1} \, dy$$

Thus,

$$|f(x)| \leq 1/c \int_{\mathbb{R}^d} |d/dr \, f(x-y)| \, |y|^{-d+1} \, dy$$

$$= 1/c \int_{\mathbb{R}^d} |d/dr \, f(x-y)| \, |y|^{-d+1} \chi_{B_R}(y) \, dy$$

$\to$ b/c f has compact support. Take $B_R = B_R(x)$

$$\leq 1/c \, \|d/dr f\|_{L^p} \, \| |x-y|^{-d+1} \chi_{B_R} \|_{L^{p'}}$$

<u>Note</u>: $\| |x-y|^{-d+1} \chi_{B_R} \|_{L^{p'}}^{p'} = \int_{B_R} |x-y|^{(-d+1)p'} \, dy < \infty$

Thus, $|f(x)| \lesssim \|\partial f\|_{L^p}$

Next, take $g_R = f \eta_R$ with

$$\eta_R = \begin{cases} 0 & \text{for } |x| > R \\ 1 & \text{for } |x| \leq R/2 \end{cases}$$

and $\eta_R \in C_0^\infty$ $\forall R$.

and $|\partial \eta_R| \leq 2$ $\forall R$ sufficiently large.

So, $|g_R| \lesssim \|\partial \eta_R f\|_{L^p} + \|\eta_R \partial f\|_{L^p}$
$\lesssim \|f\|_{L^p} + \|\partial f\|_{L^p} = \|f\|_{W^{1,p}}$

$\boxed{\text{QEP Exercise}}$

July 20th/2011

Fourier Analysis on $\mathbb{T}^d = [\mathbb{R}/2\pi\mathbb{Z}]^d$

Def'n: $\hat{\mathcal{F}}f(n) = \int_{\mathbb{T}^d} f(x) e^{-in\cdot x}\,dx$, $n \in \mathbb{Z}^d$

The Fourier inversion formula holds if $f \in L^1(\mathbb{T}^d)$ and $\hat{f} \in \mathbb{T}^d$ with $f(x) = \int \hat{f} e^{inx} \sum \delta(n-m) \frac{dn}{(2\pi)^d}$

Plancherel: The function $\mathcal{F}: L^2(\mathbb{T}^d) \to L^2(\mathbb{Z}^d)$ defines an isometry.

exercise 1: Prove the Fourier inversion formula, the Plancherel theorem, and the Fourier transform identities (think products, convolutions, derivative, action of isometries)

exercise 2: (a) Assume that $1 \le p \le q \le \infty$. Prove that
$L^p(\mathbb{Z}^d) \hookrightarrow L^q(\mathbb{Z}^d)$, $L^q(\mathbb{T}^d) \hookrightarrow L^p(\mathbb{T}^d)$

(b) prove Sobolev embedding inequality: if $d/q = d/p - 1$ $(p,q) \in (1,\infty)^2$ then $\|f\|_{L^q(\mathbb{T}^d)} \lesssim \|f\|_{L^p(\mathbb{T}^d)} + \|\nabla f\|_{L^p(\mathbb{T}^d)}$

Littlewood-Paley projections: Assume $\eta: \mathbb{R}^d \to [0,1]$ is a smooth function, $\eta(\xi) = 1$ if $|\xi| \le 1$, $\eta(\xi) = 0$ if $|\xi| \ge 2$ for $k = 0,1,\dots$ define $\eta_{\le k}(\xi) = \eta(\xi/2^k)$, $\eta_k(\xi) = \eta_{\le k}(\xi) - \eta_{\le k-1}(\xi)$
$P_{\le k} f = \mathcal{F}^{-1}[\eta_{\le k}(\xi) \hat{f}(\xi)]$, $P_k = P_{\le k} - P_{\le k-1}$ where by def'n $\eta_{\le -1} = 0$ and $P_{\le -1} = 0$. So, $P_0 + P_1 + \dots + P_k = P_{\le k}$,
$P_0 + P_1 + \dots = Id$

<u>Note</u>: $\|f\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2 = \|\sum_{k=0}^{\infty} \eta_k(\xi)\hat{f}(\xi)\|_{L^2}^2$

But, $\int \left(\left|\sum_{k=0}^{\infty} \eta_k(\xi)\hat{f}(\xi)\right|\right)^2$

$$= \int \sum_{k,k'} \eta_k(\xi)\overline{\eta_{k'}(\xi)}\,\hat{f}(\xi)\overline{\hat{f}(\xi)}$$

$$= \sum_k \|P_k f\|_{L^2}^2$$

<u>Exercise 3</u>: (a) Prove the Poisson summation formula: if $f, \hat{f} \in L^1(\mathbb{R}^d)$ then $\sum f(n) = \sum \hat{f}(m)$

(b) Prove that for any $k \geq 0$ and $p \in [1,\infty]$ $\|P_{\leq k} f\|_{L^p} \lesssim \|f\|_{L^p}$

We consider the defocusing periodic NLS:

$$\begin{cases} (i\partial_t + \Delta_x)u = u|u|^{p-1} \\ u(x,0) = \varphi(x) \end{cases}$$

The quantities $M(t) = \int_{\mathbb{T}^d} |u|^2$ and $E = \frac{1}{2}\int_{\mathbb{T}^d} |\nabla_x u|^2 + \frac{1}{p+1}\int_{\mathbb{T}^d} |u|^{p+1}$ are conserved.

<u>Perturbative scheme</u>:

(i) $u_1 = e^{it\Delta}\varphi$

(ii) $u_2 = e^{it\Delta}\varphi - i\int_0^t e^{i(t-s)\Delta}[u_1(s)|u_1(s)|^{p-1}]$

⋮

(n+1) $u_{n+1} = e^{it\Delta}\varphi - i\int_0^t e^{i(t-s)\Delta}[u_n(s)|u_n(s)|^{p-1}]\,ds$

⇝ We have local well-posedness in H^σ for $\sigma \gg 1$. Would like to show local well-posedness in $H^1(\mathbb{T}^d)$. These depend on Strichartz estimates.

<u>Euclidean Strichartz Estimates</u>: $\|P_{\leq k} e^{it\Delta} f\|_{L^{(2d+4)/d}_{x,t}} \lesssim \|f\|_{L^2}$ and $\|P_{\leq k} e^{it\Delta} f\|_{L^\infty_{x,t}} \lesssim 2^{kd/2}\|f\|_{L^2}$.

By interpolation $\|P_{\le k} e^{it\Delta} f\|_{L^p_{x,t}} \lesssim 2^{k[d/2-(d+2)/p]}\|f\|_{L^2}$ for $p\in[(2d+4)/d,\infty]$.

Theorem: (Periodic Strichartz Estimates on $\mathbb{T}^d$)

$$\|P_{\le k} e^{it\Delta} f\|_{L^4_{x,t}} \lesssim A_d(k)\|f\|_{L^2}$$

where $A_2(k)=C_\varepsilon 2^{k\varepsilon}$, $A_3(k)=C_\varepsilon 2^{k(1/4+\varepsilon)}$, $A_4(k)=C_\varepsilon 2^{k(1/2+\varepsilon)}$, $A_d(k)=C\, 2^{k(d-2)/4}$ for $d\ge 5$. The L^2 estimate still holds $\|P_{\le k} e^{it\Delta} f\|_{L^\infty_{x,t}(\mathbb{T}^d\times\mathbb{T})} \lesssim 2^{kd/2}\|f\|_{L^2}$.

Proof: Want to show $\|P_{\le k} e^{it\Delta} f\|_{L^4_{x,t}} \overset{\text{want}}{\lesssim} A_d(k)\|f\|_{L^2_{x,t}}$

$$\iff \Big\|\sum_{|n|\le 2^{k+1}} e^{-it|n|^2} e^{ix\cdot n}\underbrace{\hat f(n)\eta_{\le k}(n)}_{\text{“}a(n)\text{”}}\Big\|_{L^4_{x,t}} \underset{\uparrow\text{want}}{\lesssim} A_d(k)\Big(\sum_n |a(n)|^2\Big)^{1/2}$$

$$\iff \Big\|\Big(\sum_{|n|\le 2^{k+1}} e^{-it|n|^2} e^{ixn} a(n)\Big)^2\Big\|_{L^4} \underset{\uparrow\text{want}}{\lesssim} A_d(k)^2 \cdot \sum |a_n|^2$$

$$\iff \Big\|\sum_{|n|,|m|\le 2^{k+1}} e^{ix(m+n)} e^{-it(|n|^2+|m|^2)} a_n a_m\Big\|_{L^2_{x,t}} \overset{\text{want}}{\lesssim} A_d(k)^2 \sum |a_n|^2$$

Thus $\|e^{ix\cdot p}e^{it\cdot q} c_{p,q}\|_{L^2_{x,t}} \lesssim \Big(\sum_{p,q\in\mathbb{Z}} |c_{p,q}|^2\Big)^{1/2}$

$$\lesssim \sum_{(p,q)} \Big|\sum_{(n,m)\in S_{p,q}} a_m a_n\Big|^2$$

where $c_{p,q} = \sum_{\substack{|m|,|n|\le 2^{k+1}\\ m+n=p,\ |m|^2+|n|^2=q}}$

$$S_{p,q} = \{(m,n)\in\mathbb{Z}^d\times\mathbb{Z}^d \mid m+n=p,\ |m|^2+|n|^2=q\}$$

$$\lesssim \Big[\sum_{p,q}\Big(\sum_{(m,n)\in S_{p,q}} |a_n|^2|a_m|^2\, |S_{p,q}|\Big)\Big]^{1/2} = \Big[\sum_{p,q} |S_{p,q}| \sum_{(m,n)\in S_{p,q}} |a_m|^2|a_n|^2\Big]^{1/2}$$

↑ Cauchy-Schwarz

$$\lesssim \sup_{p,q} |S_{p,q}|^{1/2}\Big[\sum_{p,q}\sum_{(m,n)\in S_{p,q}} |a_m|^2|a_n|^2\Big]^{1/2}$$

$$\lesssim \sup_{p,q}|S_{p,q}|^{1/2}\sum |a_m|^2$$

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Want $|S_{p,q}| \lesssim A_d(k)^4$

So, $m = p - n$ and $|n|^2 + |p-n|^2 = -q$

$$\sum_{j=1}^{d} [n_j^2 + (n_j - p_j)^2] = -q$$

$$\sum_{j=1}^{d} (4n_j^2 - 4n_j p_j) = -2q - 2|p|^2$$

$$\sum_{j=1}^{d} (2n_j - p_j)^2 = \underbrace{-2q - |p|^2}_{\ldots} \rightarrow \text{fixed number}$$

Thus, $|S_{p,q}| \leq 2^{dk} 2^{-2k}$

$\boxed{\text{QED}}$

<u>Key abstraction</u>: the Euclidean dispersive bound

$\|P_{\leq k} e^{it\Delta}\|_{L^1 \to L^\infty} \lesssim \min(2^{kd}, |t|^{-d/2})$ fails in the periodic case.

<u>Exercise 4</u>: Prove the $\Lambda(4)$ bound for the squares

$$\left\|\sum_{|n|\leq N} a_n e^{in^2 t}\right\|_{L^4(\mathbb{T}^d)} \lesssim_\varepsilon N^\varepsilon \left\|\sum_{|n|\leq N} a_n e^{in^2 x}\right\|_{L^2(\mathbb{T}^d)}$$

for any $\varepsilon > 0$ and $N \geq 1$.

<u>Note</u>: $\left\|\sum_{|n|\leq N} a_n e^{in^2 t}\right\|_{L^p} \lesssim \left\|\sum_n a_n e^{in^2 x}\right\|_{L^2}$

↑ $\Lambda(p)$ bound.

Proof of Exercise 2: Let $1 \le p \le q \le \infty$.

(b) Want to show $L^q(\mathbb{T}^d) \hookrightarrow L^p(\mathbb{T}^d)$

$$\|f\|_{L^p(\mathbb{T}^d)} \lesssim \|f\|_{L^q(\mathbb{T}^d)} \|1\|_{L^r(\mathbb{T}^d)} \quad \text{for } \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$$

↑ Hölder

(a) This is just the exercise of $\ell^p(\mathbb{Z}^d) \hookrightarrow \ell^q(\mathbb{Z}^d)$

$\boxed{\text{QED}}$

Note on Littlewood-Payley Theory: Remember that $\eta_k(\xi) = \eta_{\le k}(\xi) - \eta_{\le k-1}(\xi)$ with $\eta_{\le k}(\xi) = \varphi(2^{-k}\xi)$ and $\varphi = \begin{cases} 1, & |\xi| \le 1 \\ 0, & |\xi| > 0 \end{cases}$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$

So, $\widehat{P_k f} = \eta_k \hat{f}$ so $\sum_{k \in \mathbb{Z}} \widehat{P_k f} = \sum \eta_k(\xi) \hat{f}(\xi)$

$$= \sum [\varphi(2^{-k}\xi) - \varphi(2^{-k+1}\xi)] \hat{f}(\xi)$$
$$= (1-0)\hat{f}(\xi) = \hat{f}(\xi)$$

Thus, $\sum P_k = Id$.

Global Solutions (focusing case):

(a) $1 < p < 7/3$, $\varphi \in H^1(\mathbb{R}^3)$ global solution

(b) $p \in (1,5)$ $\exists \varepsilon$ s.t. if $\|\varphi\|_{H^1} < \varepsilon \Rightarrow$ global solution

(c) $p \geq 7/3$ $\exists \varphi \in H^1(\mathbb{R}^3)$ s.t. the solution blows up in finite

Global Solutions (defocusing case):

For $1 < p < 5$ we have a global solution.

Proof: We have to show that the energy $\|u\|_{H^1(\mathbb{R}^3)} \le C$

We have $M(t) = \int_{\mathbb{R}^3} |u|^2 \le C_0^2$

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} = C_1$$

Note: $\|u\|_{L_x^{p+1}} \le \|u\|_{L_x^2}^{\theta} \|u\|_{L_x^6}^{1-\theta}$ for

$$\frac{1}{p+1} = \frac{\theta}{2} + \frac{1-\theta}{6} \text{ iff } \theta = \frac{5-p}{2(p+1)}$$

So, $\|u\|_{L_x^{p+1}} \le C_0^{\theta}\left(\|u\|_{L_x^2} + \|\nabla u\|_{L_x^2}\right)^{1-\theta}$

$\Rightarrow$ let $A = \|\nabla u\|_{L^2}$ so

$$\frac{1}{2}A^2 - \frac{1}{p+1}\|u\|_{L_x^{p+1}}^{p+1} = C_1$$

$$\Rightarrow \quad A^2 \le C_1 + \left[C_0^{\theta}(C_0 + A)^{1-\theta}\right]^{p+1}$$

$$\le C_1 + C_0^{(p+1)\theta}\left[C_0^{(1-\theta)(p+1)} + A^{(1-\theta)(p+1)}\right]$$

$$\le C_0^{p+1} + C_1 + C_0^{(p+1)\theta} A^{(1-\theta)(p+1)}$$

So if $(1-\theta)(p+1) < 2 \iff p < 7/3$

Thus, $A^{2-(1-\theta)(p+1)} \le C_0^{(p+1)\theta} + (C_1 + C_0^{p+1}) A^{-(1-\theta)(p+1)}$

$\to$ for $(C_1 + C_0^{p+1}) > 0$. $\Big\{$ If $A \le 1$, then done. If $A > 1$ then $A^{-1} < 1$ and so $A^{2-(1-\theta)(p+1)} \le C$

Do opposite for $(C_1 + C_0^{p+1}) < 0$

This shows $u \in H^1$ and we've shown (a)

Alternative proof of (a) assuming (b):

We know that $(i\partial_t u + \Delta_x u) = u|u|^{p-1}$. Take $u_\lambda = \lambda^{\frac{-2}{p-1}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ and note that u_λ solves the NLS equation above.

So, $\|\lambda^{-\frac{2}{p-1}} \varphi(x/\lambda)\|_{H^1} = \lambda^{3/2 - \frac{2}{p+1}} \|\varphi\|_{L^2} + \lambda^{\frac{p-5}{2(p-1)}} \|\nabla\varphi\|_{L^2}$

For $p < 7/3$ both the powers of the λ's are negative and so we can choose $\lambda \gg 1$. Thus, $\|\varphi_\lambda\|_{H^1} \leq \lambda^{3/2 - \frac{2}{p-1}} \|\varphi\|_{H^1}$ for large λ and $\|\varphi_\lambda\|_{H^1} < \varepsilon$ for λ chosen large enough.

Thus, we can apply (b) and we're done.

$\rightarrow$ The rest of the proof is omitted. $\boxed{\text{QED}}$

July 22nd/2011

Reminder: we are considering $e^{it\Delta}\varphi \equiv \mathcal{F}^{-1}(\hat{\varphi} e^{-i|n|^2 t})$ on $\mathbb{T}^d$.

Key difficulty: the Euclidean dispersive bound $\|P_{\leq k} e^{it\Delta}\|_{L^1 \to L^\infty} \lesssim \left[\frac{2^k}{1 + 2^k |t|^{1/2}}\right]^d$ fails in the periodic case.

exercise 1: Let $K_k(x,t) = \sum_{n \in \mathbb{Z}^d} e^{-i|n|^2 t} e^{ix\cdot n} \eta_{\leq k}(n)$ denote the kernel of $P_{\leq k} e^{it\Delta}$, $x \in \mathbb{T}^d$, $t \in \mathbb{R}$. Show that

$$|K_k(x,t)| \lesssim \left[\frac{2^k}{\sqrt{q}\,(1 + 2^k [t/2\pi - a/q]^{1/2})}\right]^d \text{ if } t/2\pi = a/q + \beta,$$

$(a,q) = 1$, $q \in \{1, \dots, 2^k\}$, $|\beta| < (2^k q)^{-1}$

We consider the defocusing periodic NLS on $\mathbb{T}$:

(1) $$\begin{cases} (i\partial_t + \Delta_x)u = u|u|^2 \\ u(x,0) = \varphi(x) \end{cases}$$

The quantities $M(t) = \int_{\mathbb{T}} |u|^2 dx$ and $E(t) = \frac{1}{2}\int_{\mathbb{T}} |\nabla_x u(x,t)|^2 dx + \frac{1}{4}\int_{\mathbb{T}} |u(x,t)|^4 dx$ are conserved.

<u>Theorem</u>: The IVP (1) is globally well posed for small data in $L^2(\mathbb{T})$.

<u>exercise 2</u>: Use Sobolev embedding to prove that the IVP (1) is globally well-posed in $H^1(\mathbb{T})$

<u>Perturbation Iteration Scheme</u>:

(i) $u_1 = e^{it\Delta}\varphi$

$\vdots$

(n+1) $u_{n+1} = e^{it\Delta}\varphi - i\int_0^t e^{i(t-s)\Delta}[u_n|u_n|^2]\,ds$

<u>Step 1</u>: We have the homogeneous Strichartz estimate.

$$\|e^{it\Delta}\varphi\|_{L^4(\mathbb{T}\times\mathbb{T})} \lesssim \|\varphi\|_{L^2(\mathbb{T})}$$

$\rightsquigarrow$ take $X = C(I, H^1(\mathbb{T}))$

<u>Step 2</u>: We have the inhomogeneous Strichartz estimate

$$\left\|\int_0^t e^{i(t-s)\Delta} f(s)\,ds\right\|_{L_t^\infty L_x^2} \lesssim \|f\|_{L^{4/3}(\mathbb{T}\times I)}$$

for any $f \in L^{4/3}(\mathbb{T}\times I)$, $0 \in I$.

<u>exercise 3</u>: Prove the Strichartz estimate with loss

$$\| P_{\leq k}\, e^{it\Delta} \varphi \|_{L^6(\mathbb{T}\times\mathbb{T})} \lesssim_\varepsilon 2^{\varepsilon k} \|\varphi\|_{L^2(\mathbb{T})}$$

State and prove the analogue of the inhomogeneous Strichartz estimate in step 2.

<u>Proof of Step 1</u>: We want to show that

$$\Big\| \sum_{m\in\mathbb{Z}} e^{-itm^2} e^{imx} a_m \Big\|^2_{L^4_{x,t}} \lesssim \Big(\sum |a_n|^2\Big)^{1/2}$$

So, $\displaystyle \Big\| \sum_{m\in\mathbb{Z}} e^{-itm^2} e^{imx} a_m \Big\|^4_{L^4_{x,t}} = \Big\| \sum_{m,n} e^{i(m-n)x} e^{-it(m^2-n^2)} a_m \bar{a}_n \Big\|^2_{L^2_{x,t}}$

$$= \Big\| \sum_{p,q} e^{ipx} e^{iqx} c_{pq} \Big\|_{L^2} \sim \Big(\sum_{p,q} |c_{p,q}|^2\Big)^{1/2}$$

$$c_{p,q} = \sum_{(m,n)\in S_{p,q}} a_n \bar{a}_m \quad \text{and} \quad S_{p,q} = \{(m,n) \mid m-n=p,\ m^2-n^2=q\}$$

So, $\displaystyle \Big(\sum_{p,q} |c_{p,q}|^2\Big)^{1/2} = \Big(\sum_{p,q} \Big|\sum_{(m,n)\in S_{pq}} a_m \bar{a}_n\Big|^2\Big)^{1/2}$

$$\lesssim \sup |S_{p,q}|^{1/2} \Big(\sum |a_n|^2\Big)^{1/2}$$

$\boxed{\text{QED}}$

<u>Proof of Step 2</u>: Let $F(t) = \int_0^t e^{i(t-s)\Delta} f(s)\,ds$. Fix $t\in I$.

Need to show $\displaystyle \Big\| \int_0^t e^{-is\Delta} f(s)\,ds \Big\|_{L^2_x} \lesssim \|f\|_{L^{4/3}}$ iff

$$\int_{\mathbb{T}} \int_0^t e^{-is\Delta} f(s)\, g(x)\,ds\,dx \lesssim \|f\|_{L^{4/3}} \|g\|_{L^2}$$

So, $\displaystyle \Big| \int_0^t \int_{\mathbb{T}} e^{-is\Delta}\, g(x)\, f(s)\,dx\,ds \Big| \lesssim \|g\|_{L^2} \|f\|_{L^{4/3}}$ → justified after, in red $\boxed{\text{QED}}$

Step 3: We have the inhomogeneous estimate

$$\Big\| \int_0^t e^{i(t-s)\Delta} f(s)\,ds \Big\|_{L^4(\mathbb{T}\times I)} \lesssim \|f\|_{L^{4/3}}$$

for any $f\in L^{4/3}$, $0\in I$.

Step 4: We prove that $\{u_m\}$ is a Cauchy Sequence in the space $X(I) = C(I, L^2) \cap L^4(\mathbb{T}\times I)$. Then we use the L^2 conservation law to extend the solution to $\mathbb{T}\times\mathbb{R}$. We conclude that for any $\varphi \in L^2$ with $\|\varphi\|_{L^2} \le \varepsilon_0$ there is a unique solution $u \in C(\mathbb{R}, L^2) \cap L^4_{loc}(\mathbb{T}\times\mathbb{R})$ of the equation

$$u(t) = e^{it\Delta}\varphi - i\int_0^t e^{i(t-s)\Delta}[u|u|^2]\,ds$$

The mapping $\varphi \to u$ is cont. from L^2 to $X(I)$ ~~for any~~ bounded interval I.

Proof of Step 3: Want to show that $\|\int_0^t e^{i(t-s)\Delta} f(s)\,ds\|_{L^4} \lesssim \|f\|_{L^{4/3}}$

We have the IVP $\begin{cases} (i\partial_t + \Delta_x)u = f \\ u(x,0) = \varphi(x) \end{cases}$ on $\mathbb{T}\times I$

Def'n: For any n, $P_n = \{0 = t_{n,0} < t_{n,1} < \dots < t_{n,k_n} = T\}$

$1 = \|f\|_{L^{4/3}} \longrightarrow \|f|_{[t_{n,i}, t_{n,i+1}]}\|_{L^{4/3}} \in [2^{-n}, 2^{-n+1}]$ with

$|k_n| \lesssim 2^{4/3 n}$. We chose $t_{n,\phi}$ to be the time

where $\left(\int_{[t_{n,0}, t_{n,1}]} |f|^{4/3}\right)^{3/4} \in [2^{-n}, 2^{-n+1}]$ and do this

this over and over.

Thus, $P_n \subseteq P_{n+1}$. Let $U\varphi = \mathbb{1}_{[0,T)}(t)\, e^{it\Delta}\varphi$

$$V_n = \sum_{i=0}^{k_n - 1} \mathbb{1}_{[t_{n,i}, t_{n,i+1}]}(t) \times e^{i(t - t_{n,i})\Delta} \times u_{n-1}(t_{n,i})$$

$$u_m = u_{m-1} - v_m$$
$$u_0 = u - v_0$$

We are going to show that $\|u_{n-1}\|_{L^2} \lesssim C\, 2^{-n}$

$$\|v_m\|_{L^4}^2 \leq \sum_{i=0}^{k_n-1} \|\mathbb{1}_{[t_{n,i},\, t_{n,i+1}]} v_m\|_{L^4}^2$$

$$\leq \sum_{i=0}^{k_n-1} \|u_{n-1}(t_{n,i})\|_{L^2}^2 \lesssim 2^{-2/3 n}$$

$$u = u_0 + v_0 = u_1 + v_1 + v_0 = \dots = u_n + v_n + \dots + v_0$$

$\boxed{\text{QED}}$

Note: the above argument used to prove step 3 is called the Christ-Kiselev (some people may call it something else, but the general idea is there).

Proof of exercise 1 of today: Want to show that

$$|k(x,t)| \lesssim \frac{2^k}{\sqrt{q}\,(1+2^k[a/q - t/2\pi]^{1/2})}$$

Consider $$|k(x,t)|^2 = \sum_{n,m} \varphi(2^{-k}n)\,\varphi(2^{-k}n)\, e^{-itn^2 + inx + im^2t - ixm}$$

$$= \sum_{n,m} \varphi(2^{-k}n)\,\varphi(2^{-k}m)\, e^{it(m^2-n^2) + itx(n-m)}$$

$$= \sum_{p,q} \varphi\left(2^{-k}\frac{p+q}{2}\right)\varphi\left(2^{-k}\left(\frac{p-q}{2}\right)\right) e^{-itpq + ixq}$$

$$\leq \sum_p \left|\sum_q \varphi\left(2^{-k}\frac{p+q}{2}\right)\varphi\left(2^{-k}\frac{p-q}{2}\right) e^{iq(x-pt)}\right|$$

Next, Summation by parts is $\sum_{k=0}^{N} a_k b_k = \sum_{k=0}^{N-1}(a_k - a_{k+1}) S_k + a_N S_{N-1}$

where $S_m = \sum_{k=0}^{m} b_k$. (shown last week)

<u>exercise</u>: finish the estimate. <u>Hint</u>: use summation by parts; being careful of course!

Justification for $\int_{\mathbb{T}}\int_0^t e^{-is\Delta} f(s)\, g(x)\, ds\, dx$

$$= \int_{\mathbb{T}}\int_0^t (e^{-is\Delta} g)\, f \, ds\, dx$$

$$\int_{\mathbb{T}} (e^{it\Delta} f)\, g = \int_{\mathbb{T}} (k * f)\cdot g \, dx$$

$$= \int_{\mathbb{T}}\int_{\mathbb{T}} k(x-y)\, f(y)\, g(x)\, dy\, dx$$

$$= \int_{\mathbb{T}}\int_{\mathbb{T}} K(y-x)\, g(x)\, dx\,)\, dy$$

$$= \int_{\mathbb{T}} (k * g)\, f \, dy$$