Tadahiro Oh Notes on Bourgain's NF + I Method paper.

(Notes by A. Selvitella)

Monday 23 May 2010 HIRO 17.00

$$q_t = \frac{\partial H}{\partial \bar{q}} \qquad H(q) = \frac{1}{2}\int |\check{q}|^2 + \frac{1}{6}\int |\check{q}|^6$$

$$\overset{\shortparallel}{\sum |m|^2 |q_m|^2} + N(q)$$

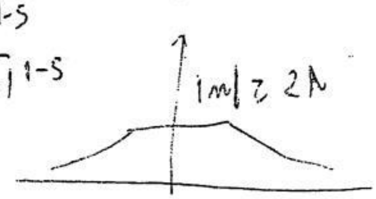
$$N(q) = \sum_{m_1 - n_2 + n_3 - n_4 + m_5 - n_6 = 0} q_{m_1}\bar{q}_{n_2} q_{n_3} \bar{q}_{m_4} q_{m_5} \bar{q}_{m_6}$$

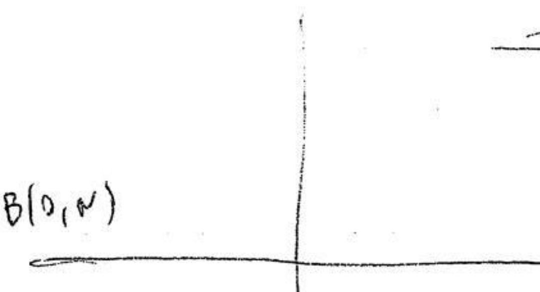
Goal: find Γ symplectic s.t. do what we want 1

$$\|\Gamma q\|_{H^1} \sim \|q\|_{H^1}$$

$$s \in (0,1) \qquad m_N(m) = \left(\frac{N^2}{N^2 \vee |m|^2}\right)^{\frac{(1-s)}{2}} = \begin{cases} 1 & |m| \le N \\ \frac{N^{1-s}}{|m|^{1-s}} & |m| \ge 2N \end{cases}$$

$1 - s > 0$





$Iq = q$ if $\operatorname{supp} q \subset B(0,N)$

$$\|Iq\|_{H^1} \le N^{1-s} \|q\|_{H^s} \qquad (1.6)$$

$$(LHS)^2 = \sum m_N^2 m^2 q_n^2 = \underbrace{\sum_{|m| \le N} m^2 q_m}_{\substack{N^{1-s} \sum_{|m| \le N} |m|^{2s} q_m^2 \\ \le (RHS)^2}} + N^{1-s} \sum_{|m| \ge N} |m|^{2s} |q_m|^2$$

$H(q) \sim \|q\|^2_{\dot{H}^1} + \|q\|^6_{L^6}$

If $\|q\|^6_{L^6} \gtrsim \underset{\underset{q_0^2}{\vee\!\!/}}{q_0^6} + \|q - P_0 q\|^6_{L^6}$

If $q_0 << 1$, then it may happen that $q_0^2 >> \|q\|^6_{L^6}$

(An issue here!)

$\|q\|^6_{L^6} \lesssim \|q\|^6_{H^{1/3}}$

$s > \frac{1}{3} \Rightarrow \|q\|^6_{L^6} \lesssim \|q\|^6_{H^{1/3}}$

$\Downarrow$

$\|Iq\|^6_{L^6} \lesssim \|Iq\|^6_{H^{1/3}}$

Assume $\|q\|_{H^s} < C$ (*)

$H(Iq) = \frac{1}{2}\int |\nabla(Iq)|^2 + \frac{1}{6}\int |Iq|^6$

Want to prove $H(Iq) \lesssim N^{2(1-s)}$.

If we assume $s \geq \frac{1}{3}$

$\|Iq\|^6_{L^6} \lesssim \|Iq\|^6_{H^{1/3}} \leq \|Iq\|^4_{H^{1/3}} \|Iq\|^2_{H^1} \lesssim N^{2(1-s)} C^2 \|q\|^4_{H^{1/3}}$

$\leq C$

by assumption (*)

Poincaré - Wirtinger ??

②

: $\swarrow$ I indep. on time.

$$\frac{d}{dt}\left[H(Iq, I\bar{q})\right] = \frac{\partial H}{\partial q} I\dot{q} + \frac{\partial H}{\partial \bar{q}} I\dot{\bar{q}} = (*)$$

• Smoothing but not Hamiltonian

$$\dot{q} = -i H_{\bar{q}} \qquad (Iq)_t = I q_t = -i I H_{\bar{q}}$$

? Adiabatic Invariants

$\Updownarrow$

Keep ∃ hamiltonian

$$(*) = \frac{1}{i}\Big\{ \underset{z_1}{H_q} \cdot I \underset{z_2 \text{ (second argument of the Hamiltonian)}}{H_{\bar{q}}} - H_{\bar{q}} I H_q \Big\}^{(\Delta)} \quad \dot{\bar{q}} = i H_q \quad \Leftarrow \begin{pmatrix} q \\ \bar{q} \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_q \\ H_{\bar{q}} \end{pmatrix}$$

$$H(Iq) = \sum |n|^2 \underbrace{m^2(n)\,|q_n|^2}_{|Iq_n|^2} + N(Iq, \overline{Iq})$$

Let's plug in,

$$\frac{\partial H}{\partial q_n}(Iq) = \sum |n|^2 m^2(n)\, \bar{q}_n + \frac{\partial N}{\partial q_n}(Iq)\, m(n)$$

$\Updownarrow$ ↓ more precisely

$$\frac{\partial H}{\partial z_1} = \sum |n|^2 m\, \bar{q}(n) + \frac{\partial N}{\partial (Iq)}$$

$$(\Delta) \sim \sum_n \left[m_n |n|^2 \bar{q}_n + \frac{\partial N}{\partial z_1} \right] m_n \left[|n|^2 q_n + \frac{\partial N}{\partial \bar{q}_n} \right]$$

Analogously the 2nd part $\cancel{\sum_n [m(n)] |n|^2 \left(\bar{q}_n \frac{\partial N}{\partial \bar{q}_n} - q_n \frac{\partial N}{\partial q_n} \right)}$

$$- \sum_n \left[m(n) |n|^2 q_n + \frac{\partial N}{\partial q}(I\bar{q}) \right] m(n) \left[|n|^2 \bar{q}_n + \frac{\partial N}{\partial \bar{q}_n} \right] =$$

$$= \sum m_n^2 |n|^2 \left(\bar{q}_n \frac{\partial N}{\partial z_2} - q_n \frac{\partial N}{\partial z_1} \right)$$

[STAFFILANI - PAVLOVIC - DA SILVA - TZIRAKIS] ⇐ correction by Vedran Sohinger?

Basic I-method < pointwise estimates on multiplier
Multilinear estimates

Collecting all the terms

$$\frac{d}{dt} H(Iq, I\bar{q}) = (1.9) \sum m_n^2 |n|^2 \left(\bar{q}_n \frac{\partial N}{\partial z_2} - q_n \frac{\partial N}{\partial z_1} \right) +$$

$$(1.10) \quad \sum m_n^2 |n|^2 \left(q_n \frac{\partial N}{\partial q_n}(Iq) - \bar{q}_n \frac{\partial N}{\partial \bar{q}_n}(I\bar{q}) \right)$$

$$+ \sum m_n^2 |n|^4 |q_n|^2 - m_n^2 |n|^4 |q_n|^2$$

$$+ \sum m_n \left[\frac{\partial N}{\partial q_n}(Iq) \frac{\partial N}{\partial \bar{q}_n} - \frac{\partial N}{\partial q} \frac{\partial N}{\partial \bar{q}_n}(I\bar{q}) \right]$$

Suppose $\operatorname{supp}(q) \subset B(0, N)$ $(\Leftrightarrow q_n = 0, \ |n| > N)$

$$\Rightarrow Iq = q$$

$$\left[\frac{\partial N}{\partial \bar{q}_n}(q, \bar{q}) = C \sum q_{n_1} \bar{q}_{n_2} q_{n_3} \bar{q}_{n_4} q_{n_5} \right.$$

do not care

$$\frac{\partial N}{\partial \bar{q}_n} = \sum Iq_{n_1} \overline{Iq_{n_2}} \cdots Iq_{n_5}$$

$$* = \{ n_1 - n_2 + n_3 - n_4 + n_5 - n_6 \}$$

$$N(q, \bar{q}) = \sum_{(*)} q_{n_1} \bar{q}_{n_2} q_{n_3} \bar{q}_{n_4} q_{n_5} \bar{q}_{n_6}$$

(3)

$$\boxed{\frac{\partial N}{\partial \bar{q}_n} = 3 \sum_{n = m_1 - m_2 + m_3 - m_4 + m_5} q_{m_1} \bar{q}_{m_2} q_{m_3} \bar{q}_{m_4} q_{m_5}} \quad (*) \qquad \frac{\partial \bar{q}_m}{\partial \bar{q}_n} = \delta_{mn}$$

When $m_2 = m_4$, take for example

$$\frac{\partial}{\partial \bar{q}_{m_2}} \bar{q}^2_{m_2} \cdot \bar{q}_{m_6} = 2 \cdot \bar{q}_{m_2} \cdot \bar{q}_{m_6} = \underbrace{\bar{q}_{m_2} \bar{q}_{m_6}}_{\frac{\partial}{\partial q_{m_4}}} + \underbrace{\bar{q}_{m_4} \bar{q}_{m_6}}_{\frac{\partial}{\partial q_{m_2}}}$$

When $u_2 = u_4 = u_6 \Rightarrow \frac{\partial}{\partial \bar{q}_n} \bar{q}^3_n = 3\bar{q}_n = \bar{q}_{m_2} \bar{q}_{m_4} + \bar{q}_{m_4} \bar{q}_{m_6} + \bar{q}_{m_6} \bar{q}_{m_2}$

So $(*)$ is correct!

$(1.11) = 0$ too

$\frac{d}{dt} H(\mathbb{I}q, \mathbb{I}\bar{q}) = 0 \Rightarrow H^1$ norm of $\mathbb{I}q$ is bdd (done!)

$\Rightarrow$ forget low frequency part interaction

$\Rightarrow$ at least one frequency is to be high

$\Rightarrow \max(n_j) > N$

nonlinearity on partial derivatives

1st term coming from previous computations

$$(1.9) = 3 \sum_{m_1 - m_2 + m_3 - m_4 + m_5 - m_6 = 0} m^2(m_6) |m_6|^2 q_{m_1} \bar{q}_{m_2} q_{m_3} \bar{q}_{m_4} q_{m_5} \bar{q}_{m_6} \qquad c(\bar{m})$$

symmetries

$$= \sum_{*} \left(m^2(m_2) |m_2|^2 + m^2(m_4) |m_4|^2 + m^2(m_6) |m_6|^2 \right) q_{m_1} \cdots \bar{q}_{m_6}$$

1st term $\bar{q}_n \frac{\partial N}{\partial \bar{z}_1} (q)$

2nd term:

$$- q_n \frac{\partial N}{\partial z_1} = -3 \sum_{*} m^2(n_6) |n_6|^2 \bar{q}_{n_1} q_{n_2} \dots \bar{q}_{n_5} q_{n_6}$$

$$= -\sum_{*} \left(m^2(n_1)|n_1|^2 + \dots + m^2(n_5)|n_5|^2 \right) q_{n_1} \dots \bar{q}_{n_6}$$

(Missing - sign in (1.14)!)

Check 1.1?, same computations as before but with (Iq) instead of q.

Is the sign correct now?

<u>Remember:</u> we have to estimate (1.11) (1.14) (1.15)

forget the other things

(2.2) general, (2.3) is better, but for quints

$(n_3^*)^{2+\varepsilon} \lesssim K$ proved in the sequel

$(n_3^*)^\varepsilon \, |n_1^2 - n_2^2 + n_3^2 - n_4^2 + n_5^2 - n_6^2|^* \leq K$ proved in the Appendix

$\Rightarrow$ Appendix 7

Γ_F Lie Transform $\quad \frac{\partial F}{\partial \bar q} \simeq \mathcal{X}$ of my lecture

$$H_1(q,\bar q) = \sum_{\substack{j_1 - j_2 + j_3 - j_4 + j_5 - j_6 = 0 \\ * n}} c(j_1, \dots, j_{2r})\, q_{j_1} \dots \bar q_{j_{2r}}$$

NR: $\alpha_r = j_1^2 - \dots - j_{2r}^2 \quad |\alpha_r| > K$

$H = H_0 + H_1 + \text{h.o.t.}$

$$H \circ \Gamma_F \overset{\text{Lemma (CHECK!)}}{=} H_0 + H_1 + \{ H_0 + H_1, F \} + \text{h.o.t.} =$$

$$q_t = -i \underbrace{X_F}_{\partial F / \partial \bar q} \qquad H_0 = \sum j^2 |q_j|^2$$

$$= H_0 + \{F, H_0\} + H_1$$

$c(\vec n)$ symmetric in the arguments

$$\{F, H_0\} = \sum \frac{\partial F}{\partial q_m}\frac{\partial H_0}{\partial \bar q_m} - \frac{\partial F}{\partial \bar q_m}\frac{\partial H_0}{\partial q_m}$$

$$F = \sum f_{j_1 \dots j_{2r}}\, q_{j_1} \dots \bar q_{j_{2r}}$$

$$\frac{\partial H_0}{\partial \bar q_m} = |m|^2 q_m \qquad \frac{\partial F}{\partial q_m} = \dots$$

$$\dots = \sum f_{j_1 \dots j_{2r}} (j_1^2 + j_3^2 + \dots + j_{2r-1}^2)$$

$$- \sum f_{j_1 \dots j_{2r}} (j_2^2 + \dots + j_{2r}^2)\, q_{j_1} \dots \bar q_{j_{2r}}$$

$$\Rightarrow \alpha_r(\vec j) \cdot f_{\vec j} = c(\vec j) \qquad [\,\supset \text{NLS},$$

(2 Resonant decomposition

Proof of $\|e^{it\Delta}\phi\|_{L^6(\mathbb{T}\times\mathbb{T})} \leq \left(\exp\left(C\frac{\log N}{\log\log N}\right)\right)\|\phi\|_2$

$\operatorname{supp}\hat{\phi} \subset [-N, N]$ $\quad e^{it\Delta}\phi = \sum \hat{\phi}_j\, e^{i(jx+j^2t)}$

$\|e^{it\Delta}\phi\|^6 =$

$\int S(t)\phi \quad L^6(\mathbb{T}_x\times\mathbb{T}_t)$

$\eta(t) \leftarrow$ time cut off (smooth, time periodic)

$= \|\eta\, S(t)\phi\, \overline{S(t)\phi}\, S(t)\phi\|^2_{L^2} = \Big\| \underbrace{\sum_{m=m_1-m_2+m_3} \hat{\phi}_{m_1}\overline{\hat{\phi}_{m_2}}\hat{\phi}_{m_3} e^{i(mx)} e^{i(m_1^2-m_2^2+m_3^2)t}}_{} \Big\|^2_{L^2_{t,x}}$

$\Rightarrow F(m,\tau) = \sum_{\substack{m=m_1-m_2+m_3\\ |m_i|\leq N}} \hat{\phi}_{m_1}\overline{\hat{\phi}_{m_2}}\hat{\phi}_{m_3}\,\delta\left(\tau-(m_1^2-m_2^2+m_3^2)\right)$ $\quad F(m,\tau)$

$= \|F(m,\tau)\|_{\ell^2_{m,\tau}}$ $\qquad m\in\mathbb{Z},\ \tau\in\mathbb{Z}$

$|F(m,\tau)| \leq \left(\sum |\hat{\phi}_{m_1}|^2|\hat{\phi}_{m_2}|^2|\hat{\phi}_{m_3}|^2\right)^{1/2}$

$\times\left(\sum \delta^2\left(\tau-(m_1^2-m_2^2+m_3^2)\right)\right)^{1/2} = \mathrm{I}\times\mathrm{II}$

$\Rightarrow \|F(m,\tau)\|_{\ell^2_{m,\tau}} \lesssim \left(\sum_m |\mathrm{I}|^2\right)^{1/2} \times \sup_{m,\tau}(\mathrm{II})^{1/2}$

$\|\,\|\phi\|^3_{L^2}\|_{\ell^2_\tau} = \ell^2_\tau$ finite $\qquad \sup_{m,\tau} \#\left\{(m_1,m_2,m_3)\,\middle|\, m=m_1-m_2+m_3,\ \tau=m_1^2-m_2^2+m_3^2\right\}$

Now just to compute:

$$n_1^2 - n_2^2 + n_3^2 - n^2 = \underbrace{\tau - n^2}_{\text{fixed! } (n \text{ fixed})}$$

$$\overset{?}{=} 2\,(n-n_1)(n-n_3)$$

$$n - n_1 \mid \tau - n^2 \Rightarrow O\big((\tau-n^2)^{\varepsilon}\big) \text{ many choices for } n-n_1$$

$\to |\tau - n^2| \le N^2$

$\Rightarrow$ same # for n_3

FACT: given N $d(N) \lesssim N^{\varepsilon}$ $\forall \varepsilon > 0$ $\lesssim e^{c \ln N / \ln\ln N}$

2D L^4, but you have to count $\tau = |m_1|^2 - |m_2|^2 = m_1 m_2$

$m_j \in \mathbb{Z}^2$ $(\bar{m}_1 + i\bar{m}_2)$

can reduce on counting on the circle $\to$ $m = n_1 - n_2$ $(\bar{m}_1 - i\bar{m}_2)$

Gaussian integers

$\Downarrow$

back to pag. 15h

$$u(x,t) = \sum_{j,m \in \mathbb{Z}} \hat{u}(j,m)\, e^{i(jx + mt)}$$

$$\|u\|_{0,p} = \Big[\sum_m \Big(\sum_j |\hat{u}(j, j^2 + m)|^2\Big)^{p/2}\Big]^{\frac{1}{p}}$$

$$\hat{u}(j, j^2+m) = \widehat{e^{it\Delta} u}(j,m)$$

$$\big(e^{-it\Delta} u\big)^{\wedge}(m,t) = \hat{u}(j,t)\, e^{-ij^2 t} \Rightarrow \text{linear flow in the negative time direction}$$

$$\big(e^{-it\Delta} u\big)^{\wedge}(m,t) = \int \underbrace{e^{-i\tau t}\, e^{-ij^2 t}}_{e^{-i(\tau + j^2)t}}\, \hat{u}(j,t)\, dt$$

$S(-t)$

Digression

$$\|u\|_{X^{s,b}} = \|S(t)u\|_{H^s_x H^b_t}$$

$\|u\|_{0,p}$ $\quad 1 \le p \le +\infty \quad$ $\|u\|_{L^\infty_t L^2_x} \le \|u\|_{0,1}$ (1)

$\shortparallel$

$\|S(-t)u\|_{\mathcal{F}L^p_t L^2_x}$ $\quad \left(\|f\|_{\mathcal{F}L^p} = \|\hat{f}(n)\|_{\ell^p_n} \right.$

(1):

$\|u\|_{L^\infty_t L^2_x} \le \|u\|_{\mathcal{F}L^1_t, L^2_x}$, in the sense that $\|f\|_{L^\infty_t} \le \|\hat{f}\|_{L^1_\tau}$

$\shortparallel$

$\|u_{0,1}\|$

$$\|f\|_{\mathcal{F}L^{s,p}} := \|\langle n\rangle^s \hat{f}(n)\|_{\ell^p_n}$$

$$H^{-s}_x \supset L^2_x \supset H^s \quad s>0$$

$\cap \quad \supset$

$\mathcal{F}L^{0,p} \quad \mathcal{F}L^{0,p}$

$p>2 \quad\quad p<2$

$p > q$ $\|u\|_{a,p} \leq \|u\|_{a,q}$ ~ convexity

$a^2 + b^2 \leq (a+b)^2$ (convexity)

by induction $\sum a_n^2 \leq \left(\sum |a_n|\right)^2$

We ~~can~~ bounded $L^6_{x,t}$ by L^2_x. Now we want a bound by something in $\|u\|_{b,1}$

supp $\hat{u}(n,t) \subset [-N,N]$

$u(x,t) = \int e^{it\tau} f_\tau(x,t)\,d\tau$, ?? $u \in \exists L^1_t L^2_x$

Assume: $b > \frac{1}{2}$, $u \in X^{s,b}$. Then $\exists\, f_{\tau,0} \in L^2(\mathbb{R}: H^s)$ s.t.

$$u(x,t) = \int \frac{e^{it\tau} f_\tau(x,t)}{\langle \tau \rangle^b}\,d\tau = \int \frac{e^{it\tau} e^{it\Delta} f_{\tau,0}}{\langle \tau \rangle^b}\,d\tau$$

where $\begin{cases} i\partial_t f_\tau - \partial_x^2 f_\tau = 0 \\ f_\tau|_{t=0} = f_{\tau,0} \end{cases}$

Try to prove Bourgain

$u(x,t) = \sum \hat{u}(n,\tau)\, e^{i(t\tau + xn)}$. Let $\tau = \tau' + n^2$

and $f_{\tau'}(x) = \sum_n \hat{u}(n, \tau' + n^2)\, e^{inx}$

$\Rightarrow S(t) f_{\tau'}(x) = \sum_n \hat{u}(n, \tau' + n^2)\, e^{in^2 t}\, e^{inx}$

$$\Rightarrow u(t) = \sum_{\tau'} e^{it\tau'} f_{\tau'}$$

$$\|u\|_{L^6_{x,t}} \lesssim \sum_{\tau'} \|S(t) f_{\tau'}\|_{L^6_{x,t}} \lesssim C_N \|f_{\tau'}\|_{L^2_x}$$

$$\sim C_N \sum_{\tau'} \sum_{n} |\hat{u}(n, \tau' + n^2)|^2 \Big) = C_N \|u\|_{0,1}$$

Tuesday, May 25, 2010

$u_t = i\{H, u\}$

Our initial $H_1(q) = \sum_{j_1 - j_2 + \dots - j_6 = 1} q_{j_1} \bar{q}_{j_2} \dots \bar{q}_{j_6}$

$$\Rightarrow \int_{\mathbb{T}} H_1(q(t))\, dt << \underset{<< N^{\varepsilon}}{C_N} \|q\|^6_{0,1}$$

$$\int_{\mathbb{T}} H_1(q_1, \dots, q_6)\, dt << C_N \prod_{j=1}^{6} \|q_j\|_{0,1} \quad \text{multilinear version (7.11)}$$

$q = \sum q_j(t) e^{ijx}$ ← assume time dep.

(*) (~)

Impose: $\mathcal{H}_1 = \int_{\mathbb{T}} H_1(q(t))\, dt << 1$ ← Hamiltonian appearing in the later step

↑ induction

$$H_r(q) = \sum_{j_1 + \dots - j_{2r}} c(\bar{j})\, q_{j_1} \cdots \bar{q}_{j_{2r}}$$ (*) assuming $\|q\|_{0,1} \ll C$

$$\overset{(\sim)}{\tilde{C}_N} \|q\|_{0,1}^{2r}$$

$$(\Leftarrow) \int H_r(q_1, \dots, q_{2r})\, dt \ll C_N \prod_{j=1}^{2r} \|q_j\|_{0,1} \quad (7.12)'$$

assumption (NOT PROVEN)

$\mathcal{H}_1$

Q: $\mathcal{H}_1 = H_1(q(t))$

$$\frac{\partial H_1}{\partial q_j} = \sum_{j = j_2 - j_3 + \dots + j_{2r}} c(j, j_2, j_3, \dots, j_{2r})\, \bar{q}_{j_2}^{(t)} q_{j_3}^{(t)} \cdots \bar{q}_{j_{2r}}(t)$$

$$\mathcal{H}_1(q(t)) = \sum_{\substack{j_1 - j_2 + \dots - j_{2r} = 0 \\ \tau_1 - \tau_2 - \dots - \tau_{2r} = 0}} c(\bar{j})\, q_{j_1}(\tau_1)\, \bar{q}_{j_2}(\tau_2) \cdots \bar{q}_{j_{2r}}(\tau_{2r})$$

$$\frac{\partial \mathcal{H}_1}{\partial q_{j,\tau}} = \sum_{\substack{j = \dots \\ \tau = \dots}} c(j, \dots, j_{2r})\, \bar{q}_{j_2} \cdots \bar{q}_{j_{2r}}(\tau_{2r})$$

$$(7.13) \quad \mathcal{H}_1(\tilde{q}_1, q_2, \dots, q_{2r}) \underset{(7.12)'}{\ll} C_N \|\tilde{q}_1\|_{0,1} \prod_{j=2}^{2r} \|q_j\|_{0,1}$$

$$\int_{\mathbb{T}} H_1(\tilde{q}_1, q_2, \dots, q_{2r})\, dt = \int_{\mathbb{T}} \sum_j \frac{\partial H_1}{\partial q_j} \tilde{q}_j\, dt =$$

$$= \left\langle \frac{\partial \mathcal{H}_1}{\partial q}, \tilde{q}_1 \right\rangle_{x,t,\, \omega_{j,\tau}} \leftarrow \text{duality} \quad =: (RHS)$$

Take $\sup\limits_{\|\tilde{q}_1\|_{q_1} = 1}$ $\Rightarrow$ $\sup (RHS) = \left\| \frac{\partial \mathcal{H}_1}{\partial q} \right\|_{q, +\infty}$

$\mathcal{J}$ same

$$\mathcal{J} \simeq \frac{\mathcal{H}}{\alpha} \quad \left\| \frac{\partial \mathcal{J}}{\partial q} \right\|_{q,1}$$

$$= \left(\sum_m \left(\sum_j \left| \sum \frac{c(j, j_2, \dots, j_{22})}{j^2 - j_2^2 + \dots - j_{22}^2} \bar{q}_{j_2} \cdots \bar{q}_{j_{22}} \right|^2 (j, j^2 + m) \right)^2 \right)^{1/2}$$

. $\|q^{(\ell)}\|_{q,1} < C$

WLOG $q^{(\ell)}(t) = \left(\underbrace{e^{it\Delta} \phi^{(\ell)}}_{S(t)} \right) e^{im_\ell t}$

$$u(x,t) = \int \hat{u}(\xi, \tau)\, e^{i(x\tau + \xi t)}\, d\xi\, d\tau$$

$$f_{\tau'}(x) = \int \hat{u}(\tau' + \xi^2, \xi)\, e^{ix\cdot\xi}\, d\xi$$

$$\Rightarrow u(t) = \int e^{it\tau'} S(t) f_{\tau'}\, d\tau'$$

$$\Rightarrow q^{(\ell)}(t) = \sum_{m_\ell} \left(S(t) \phi^{(\ell)}_{m_\ell} \right) e^{itm_\ell}$$

What is $\|q^{(\ell)}\|_{0,1}$?

$$\|q^{(\ell)}\|_{0,1} = \sum_m \Big(\sum_j \underbrace{|\hat{q}(j, j^2+m)|^2}_{\widehat{S(-t)q}(j,m)} \Big)^{1/2} \quad \approx \text{like } X^{s,b}\text{-norm}$$

$$= \sum_m \Big(\sum_j |\phi_m^{(\ell)}(j)|^2 \Big)^{1/2} = \sum_m \|\phi_m^{(\ell)}\|_{\ell^2_j} = \sum \|\phi_m^{(\ell)}\|_{L^2_+}$$

Minkowski $\Rightarrow$ Summation outside

14.00 CLAUDIO MUNOZ

"On the soliton dynamics under slowly varying medium: cubic KdV equation"

Afternoon Hilo 14.30 25 Maggio 2010

$$\begin{cases} q^{(\ell)}(t) = \sum_m \left(S(t)\phi_m^{\ell}\right) e^{itm} \\ \|q^{(\ell)}\|_{0,1} = \sum_m \|\phi_m^{(\ell)}\|_{L^2_x} \end{cases}$$

$\Rightarrow$ Assume $q^{(\ell)}(t) = S(t)\phi^{(\ell)} e^{itm_\ell}$ $(*)$

$\phi^{(\ell)}$ in $L^2(\mathbb{T})$

$Q(t)$ (Minkowski integral inequality $\Leftarrow$ just because we have ℓ^1)

$\shortparallel$

$\overline{q^{(2)}_{j_2}}\, q^{(3)}_{j_3} \dots \overline{q^{(22)}_{j_{22}}}$ of the form $(*)$

$\longrightarrow$ split frequencies

$\downarrow$ (for new Bourgain's notation)

$e^{it(j_2^2 + m_2)}\phi^{(2)}\, e^{-it(j_3^2+m_3)}\phi^{(3)}$

$\downarrow$ Fourier transf in time

$+\overline{m} = m_2 - m_3 + \dots + m_{27}$

$$\prod_{j=2}^{22} \phi^{(j)}\, \delta\left(j_2^2 - j_3^2 + \dots + j_{22}^2 + \overline{m}\right)$$

$\Leftarrow$ Bourgain conjugates

$\Leftarrow$ (remember appropriate conjugates)

evaluate $\left(Q(t) \right)^{\wedge}(m') \neq 0$

$\Rightarrow m' = \overline{m} + \left(j_2^2 - j_3^2 + \dots + \overline{m}\right)$

~~Need~~ Not importance the conjugates, because I have a modulus and want $c = c(J)$ $\to$ modulus of order

$\bar{m} - m = j_2^2 - j_1^2 + \dots - j_{22}^2$ Note we fixed (9)

$\Downarrow$

(7.14)

$$\left\| \frac{\partial \mathcal{F}}{\partial q} \right\|_{0,1} = \sum_{|m-\bar{m}|>K} \frac{1}{|m-\bar{m}|} \times \Big[\qquad \Big]^{1/2} \quad \bar{m} \text{ fixed}$$

By assumption $|\bar{m} - m| > K$

$$= \sum_{|m-\bar{m}|>K} \frac{1}{|m-\bar{m}|} \Big[\sum_{j} \qquad \Big]^{1/2}$$

$\Leftarrow$ absolute value inside ok because $\| \|_{0,1}$ has absolute value $\Rightarrow$ always think Fourier side positive

$$= \left[\sum_{j} \left| \left(\frac{\partial \mathcal{H}_1}{\partial q} \right)^{\wedge} (j_1, j^2 + m) \right|^2 \right]^{1/2}$$

put sup in m $\sup_m [\ ] \leq \sum_{|m-\bar{m}|>K} \frac{1}{|m-\bar{m}|} \left\| \frac{\partial \mathcal{H}_1}{\partial q} \right\|_{0,\infty} < 1$ (NK)

by def. of the norm

$|m - \bar{m}| = | j^2 - (\qquad) | \leq 2N^2$

(*)

$\Rightarrow \ < \ln N \quad C_N \sim 1$

Assuming (7.12) we have all these things

$$\{ H_1, F \} = i \sum_{j} \left[\frac{\partial H_1}{\partial q_j} \frac{\partial F}{\partial \bar{q}_j} - \frac{\partial H_1}{\partial \bar{q}_j} \frac{\partial F}{\partial q_j} \right]$$

$$\frac{\partial H_1}{\partial q_j} (q(t)) =: \frac{\partial \mathcal{H}_1}{\partial q_j} (q(t))$$

$$\left|\int_{\mathbb{T}} \left[\sum_j \frac{\partial \mathcal{H}_1}{\partial q_j}\frac{\partial \mathcal{F}}{\partial \bar q_j}\right] dt\right| = \left|\left\langle \frac{\partial \mathcal{F}}{\partial q}, \frac{\partial \mathcal{H}_1}{\partial q}\right\rangle\right| \le \left\|\frac{\partial \mathcal{H}_1}{\partial q}\right\|_{0,\infty} \left\|\frac{\partial \mathcal{F}}{\partial q}\right\|_{0,1}$$

$\ll N^{\varepsilon}$

assuming (7.12)

$$q_j(t) = q_j\, e^{ij^2 t}$$

$$\Rightarrow \quad q_j\,\delta(n - j^2)$$

F.T. in time

F.T. of $(j, j^2+m) \Rightarrow q_j\,\delta(m)$

$$\Rightarrow \|q\|_{0,1} = \sum_m \|\hat q(j, j^2+m)\|_{\ell^2_j} = \sum_n \|q_j\,\delta(n)\|_{\ell^2_j} = \|q_j\|_{\ell^2_j}$$

$$q^{(\ell)}(t) = \left(e^{it\Delta}\phi^{(\ell)}\right) e^{itm_\ell}$$

$\Downarrow$ $\leftarrow$ extra shift

$$q^{(\ell)}_j(t) = e^{itj^2}\phi^{(\ell)}_j\, e^{itm_\ell}$$

$$\|q^{(\ell)}\|_{0,1} = \sum_m \|\phi^{(\ell)}_j\,\delta(m - m_\ell)\|_{\ell^2_j} = \|\phi^{(\ell)}_j\|_{\ell^2_j} = \|\phi^{(\ell)}\|_{L^2_x}$$

m_ℓ fixed

choice of variables

$$\frac{\partial F}{\partial \bar q} = \left.\frac{\partial \mathcal{F}}{\partial \bar q}\right|_{t=0}$$

$$\frac{\partial F}{\partial \bar q}(q) = \sum_{j = j_2 - j_3 + \dots + j_{2n}} \frac{c(j, j_2, \dots \bar q_{j_{2n}})}{j^2 - j_2^2 + \dots - j_{2n}^2} \bar q_{j_2} \cdots \bar q_{j_{2n}}$$

exchange conjugate

$$\Rightarrow \frac{\partial F}{\partial \bar q} = \frac{\partial \mathcal{F}}{\partial \bar q}\Big|_{t=0} \Rightarrow \left\| \frac{\partial F}{\partial \bar q} \right\|_2 \leq \left\| \frac{\partial \mathcal{F}}{\partial \bar q} \right\|_{L^\infty_t l^2_x} \leq$$

$$\leq \left\| \frac{\partial \mathcal{F}}{\partial \bar q} \right\|_{0,1} << 1$$

Explanation start with $q \in l^2(\mathbb{Z})$ indep. of t

construct $\tilde q(t,x)$ by $\tilde q_j(t) = q_j e^{ij^2 t}$

and $\tilde q_j(0,x) = q(x)$ and $\|\tilde q\|_{0,1} = \|q\|_{l^2_x}$

In particular $\frac{\partial F}{\partial \bar q}(q) = \frac{\partial \mathcal{F}}{\partial \bar{\tilde q}}(\tilde q(t))\Big|_{t=0}$

$$<< C_n \|\tilde q\|_{0,1}^{2n-1} = \|q\|_{l^2_x}$$

Page 1353 Lemma $\sum (-1)^{\ell} j_\ell = 0$

$$|\sum (-1)^{\ell} j_\ell^2| < k$$

① Strichartz

② $\int |m - \bar{m}| = | j_1^2 - j_2^2 + j_3^2 - \dots + j_{22} |^2 \lesssim N^2$

$$\iint u_1 u_2 \dots u_6 = \int \sum_{j_1 - j_2 + \dots - j_6} e^{it j_1^2} \phi \dots e^{-it t_6^2} \bar{\phi}_{j_6} dt$$

$\Rightarrow j = j_1 - j_2 + j_3 = j_4 - j_5 - j_6$

$\Big\downarrow$ t dep

Assume

$$\int e^{it(j_1^2 - j_2^2 + \dots - j_6^2)} dt$$

$\Downarrow$ $j \leq j_1^*$

$j_1^* = j_1$

$j_2^* = j_2$ or j_3

$\uparrow$ different signs

$$t = j_1^2 - j_2^2 + j_3^2 = j_4^2 - j_5^2 + j_6^2$$

$$\Rightarrow t \lesssim (j_3^*)^2$$

$$\Rightarrow t - j^2 \leq (j_3^*)^2$$

Wednesday 26, May 2010 HIRO 16.30

Want to prove that the same thing that we have at step 0 is at step r, but with $6 \Rightarrow 2r$!

Think of H_r as $\{H, F\}$ or $\{\{..\{H, F\}..\}$

$\Rightarrow \{H_r, F\} = \{\{H, F\}, F\}$. We argued by induction

Assume $\mathcal{H}_r << C_N\, r\, (j_3^*)^2$

$\Rightarrow_{\text{duality}} \left\| \frac{\partial \mathcal{H}_r}{\partial q} \right\|_{0,\infty} << (j_3^*)^2$

as before

Now I want the same estimate but with the third largest index of step $r+1$ (j_3^* can be different index)

. One of the terms we should look at is

$$\int_{\mathbb{T}} \sum_j \frac{\partial \mathcal{H}}{\partial q_j} \frac{\partial \mathcal{F}}{\partial q_j}\, dt \qquad J_1, \dots, J_{2n} \sim \{\mathcal{H}_r, \mathcal{F}\}$$

$j_1, \dots, j_{2r} \qquad \mathcal{F}\ \tilde{j}_1 \dots \tilde{j}_{2r}$

If $J_1^* = j_1^*$ and $J_2^* = j_2^*$ $\Rightarrow$ $\tilde{j}_1^* \lesssim J_3^* \Rightarrow$

$$\Rightarrow \left\| \frac{\partial \mathcal{F}}{\partial q} \right\|_{0,1} << (\tilde{j}_1^*)^{\varepsilon} \lesssim (J_3^*)^{\varepsilon} \qquad (7.14)$$

similar

(verify $<< \tilde{\alpha}_2^{\tilde{\ }*}$ $\qquad \tilde{\alpha} = \tilde{j}_1^2 - \dots - \tilde{j}_{2\tilde{r}}$

Assume: q time independent (we can reduce to it by yesterday argument)

$$\Rightarrow \hat{q}(x,t) = \sum_j q_j \, e^{ij^2 t} e^{ijx}$$

$$j_1 - \dots - j_{2r} + \tilde{j}_1 - \tilde{j}_2 - \dots - \tilde{j}_{2\tilde{r}} = 0$$

$$j_1^2 - j_2^2 + \dots - j_{2r}^2 + \tilde{j}_1^2 - \dots - \tilde{j}_{2\tilde{r}}^2 = 0$$

Assume $J_1^* = \tilde{j}_1^*$ $\quad j_2^* = \tilde{j}_2^*$

Note: $\tilde{\alpha}_* - \alpha = O\left[(J_3^*)^2\right]$

[$\to$ cases in which the two biggest indeces come from not both $\mathcal{H}$ not both $\tilde{\mathcal{F}}$

Try to fix the remaining cases of the previous estimate

$$\begin{array}{ccc} H & F & \mathcal{J} \sim \{H, F\} \\ \uparrow & \nwarrow & \nwarrow \\ j & \tilde{j} & \tau \end{array}$$

indices

$$\int \sum_j \frac{\partial \mathcal{H}}{\partial q_j} \cdot \frac{\partial \mathcal{J}}{\partial q_j}\, dt$$

take a derivative $\Rightarrow$ a frequency is missing (possibly even or odd)

$$j_1 - j_2 + \dots - \check{j} - \dots - j_{2r} - \tilde{j}_1 + \tilde{j}_2 \quad \check{j} + \tilde{j}_{2\tilde{r}} = 0$$

$2(r+\tilde{r})-2$ frequencies

Since product structure I can insert j and $-j$ (mis. freq. are the same)

because they have opposite signs

Similarly

$$j_1^2 - j_2^2 + \dots (\check{j})^2 - j_{2r}^2 - \tilde{j}_1^2 + \dots (\check{j})^2 - (\tilde{j}_{2\tilde{r}})^2 = 0$$

$\Rightarrow$ add j^2 and $-j^2$ $\Rightarrow r = \tilde{r}$ $\Rightarrow$ apply the previous argument (the one with all the frequencies showed before, today) if ① $J_1^* = j_1^*$, $J_2^* = j_2^*$

or ② $J_1^* = \tilde{j}_1^*$, $J_2^* = \tilde{j}_2^*$

<u>Assume</u> : $\{ J_1^*, J_2^* \} = \{ j_1^*, \hat{j}_1^* \}$

Say $j_1^* = j_1$ ⓐ $\tilde{j}_1^* = \hat{j}_1$

odd or even $(+, -)$ sign

From the equation with "squares" I get

$$(j_1 - \tilde{j}_1)(j_1 + \tilde{j}_1) = \text{small freq} = O\left[(J_3^*)^2\right]$$

ⓐ If $j_1 \neq \pm j_1$ then $(\text{HS}) \geq J_1^*$

$$\underbrace{}_{(J_3^*)^2}$$

$\Rightarrow (J_3^*) \geq \sqrt{J_1^*}$ but $\left\| \frac{\partial \mathcal{J}}{\partial q} \right\|_{0,1} \lesssim (\tilde{J}_1^*)^{?} \lesssim (J_1^*)^{?} \lesssim (J_3^*)^{?}$

b) $\tilde{j}_1^* = \tilde{j}_2$ easier

$(J_1^*)^2 \leq j_1^2 + j_2^2 = O\left((J_3^*)^2\right) \Rightarrow (J_3^*) \geq J_1^*$ good

Miss complementary of (ⓐ)

Back to ⓐ : $\underbrace{j_1 - \tilde{j}_1}_{= 2j_1 \text{ if } j_1 = -\tilde{j}_1} - j_2 + \dots - j_{2r} + \tilde{j}_2 - \dots + \tilde{j}_{2r} = 0$

$= O(J_1^*) \Rightarrow J_3^* \sim J_1^*$

Now last case: $j_1 = \tilde{j}_1$

$j = j_1 - j_2 \quad \check{j} \quad - j_{2r}$

$\shortparallel$

$\tilde{j}_1 - \tilde{j}_2 \quad \check{j} \quad - \tilde{j}_{2r} \Rightarrow \tilde{j}_1 - \tilde{j}_2 \pm j \quad - \tilde{j}_{2r} = 0$

$= J_1^* = J_2^*$

$J_1^* = \tilde{j}_1^* \Rightarrow$

$\Rightarrow \tilde{J}_n^* = O(J_1^*)$

$\wedge$

(J_3^*)

Back to section 2. Want to prove 2.3

$$\sum_{m_1 - m_2 + \dots - m_{2r} = 0} |c(\bar{m})| \prod_{k=1}^{2r} |q_{m_k}| \lesssim \max_{\bar{m}} \left(n_3^{*\alpha} |D(\bar{m})| \right) \|q\|_{\ell^2}^{2r}$$

prove it

(2.3) holds, • for initial $\int |\phi|^6$

• for $\{H_1, F\}$ ← Appendix; $H_1 \ll \left\|\frac{\partial H}{\partial q}\right\|_{0,\infty} \left\|\frac{\partial F}{\partial q}\right\|_{0,1}$ etc

← let me call it

But also there appears something like

$\{H_0, F\} = -N_1$ etc.

$\{\{H_0, F\}, F\} = -\{N, F\}$ etc.

just a second: $\| \frac{\partial \mathcal{H}}{\partial q} \|_{0,1}$ ~~etc.~~ $\gtrsim \| \frac{\partial \mathcal{H}}{\partial q} \|_{\ell^2}$ $\leftarrow$ appendix

N_0 contribution $|D(\bar{n})| < k$

N_1 contribution $|D(\bar{n})| > k$

$$\{H, F\} \approx \sum_j \frac{-\partial H}{\partial q_j} \frac{\partial F}{\partial q_j} \underset{C.S.}{\lesssim} \left\| \frac{\partial H}{\partial q} \right\|_{\ell^2} \left\| \frac{\partial F}{\partial q} \right\|_{\ell^2} \lesssim$$

$$\lesssim \left\| \frac{\partial \mathcal{H}}{\partial q} \right\|_{0,1}$$. But if H is N_1 (i.e. nonresonant contribution)

$\|\phi\|_{\ell^\infty} \lesssim \|\phi\|_{H^{1/2+}} \lesssim n_j^{1/2+}$

Prove 2.5 first

Initial nonlinearity: $\int \phi_1 \dots \overline{\phi_6} \underset{\text{CS Hölder}}{\lesssim}$

$$\lesssim \|\phi_1\| \|\phi\|_2 \prod_3^6 \|\phi_j\|_\infty \underset{\text{Sobolev}}{\lesssim} (n_3^*)^{2+\varepsilon} \|\phi\|^6_{\ell^2}$$

In general nonlinearity: $\mathcal{N}' \sim \{N, F\}$ where we assume N satisfies (2.5) (induction)

$$\{N, F\} \sim \sum \frac{\partial N_m}{\partial q_m} \frac{\partial F}{\partial q_m} \underset{C.S.}{\lesssim} \left\| \frac{\partial N_m}{\partial q_m} \right\|_{\ell^2_m} \left\| \frac{\partial F}{\partial q_m} \right\|_{\ell^2_m}$$

$\hookrightarrow$ by duality $\sim \sum C_{\mathcal{N}}\, \bar{q}_{m_2} \dots \bar{q}_{m_{2r}} \cdot p_{m_1}$

$\| p_{m_1} \|_{\ell^2} \leq 1$

by 2.5 (Induction assumption)

$\lesssim \max (n_3^*)^{2+q} \, \| q \|_2^{2r-1}$

$\uparrow$ for $\mathcal{N}$

3rd largest for $\mathcal{N} \leq n_3^* =$ 3rd largest freq. for $\{N, F\}$

$\hookrightarrow \lesssim (n_3^*)(F))^q$ after using product structure

(7.20) $$\sum_{D(J)=a} C(J) \prod_{k=1}^{2r} (q_{j_k}) = \int H_1 (|q_{j_1}| e^{i j_1^2 t} e^{-i a t}, |q_{j_2}| e^{i t j_2^2 t}, \dots, |q_{j_{2r}}| e^{i j_{2r}^2 t}) \, dt$$

If we integrate in time we get

$\delta(D(J)-a)$ i.e. $D(J)=a$

LHS $\lesssim (j_3^*)^\varepsilon$ by 7.12

previous computation

LHS of (2.3) $= \sum_a$ LHS of (7.20)

$\leq \max a \cdot (j_3^*)^\varepsilon \lesssim \max D(J)\, (j_3^*)^\varepsilon$

$(2.3) \Leftarrow (7.20)$

check 2nd from Section 2 Bour

(15)

Thursday, May 27, 2010 h. 13.30

Section 3

Fix T, let $N = N(T)$ (to be choosen later) and

q s.t. (3.1) $\|q\|_2 < C$

(3.2) $\|Iq\|_{H^1} < N^{1-s}$

For the Hamiltonian function we define

$$\|\mathcal{N}\| \underset{(\Delta)}{\overset{\text{def}}{=}} \sum |c(\bar{m})| \prod_1^{2r} |q^{(j)}_{m_j}|$$

$(*) = \{ \|q^{(j)}\|_2 < C \ \forall j \quad \|Iq^{(j)}\|_{H^1} < N^{1-s} \ \forall j$ except at most 2 indices $\}$

Want to show (WTS): $\|\mathcal{N}\| < N^{2(1-s)}$

$\hookrightarrow$ Hamiltonian appearing in the reduction

- $\int |\phi|^6 \lesssim \underbrace{\|\phi\|_2^2}_{C} \ \underbrace{\|\phi\|_\infty^4} \lesssim \left(\sum |q_n| \right)^4$

$m^2(n) n^2 = \begin{cases} n^2 & |n| \le N \\ N^{2(1-s)} |n|^{2s} & |n| \ge N \end{cases}$

- $\sum |q_n| \lesssim \sum_{|n| \le N^{1-s}} + \sum_{|n| \ge N^{1-s}} \frac{1}{m(n) \cdot n} m(n) n |q_n|$

$\underset{C.S.}{\lesssim} N^{\frac{1-s}{2}} \|q\|_2 + \sum$

$$\left(\sum_{N^{1-s}<|m|\leq N} \frac{1}{m^2}\right)^{1/2} \sim \frac{1}{N^{\frac{1-s}{2}}}$$

$$\frac{1-s}{9} < \frac{1}{4}$$

$$\wedge$$

$$\left(\sum_{|m|\geq N} \frac{1}{N^{2(1-s)}\,|m|^{2s}}\right) \sim \frac{1}{N^{1-s}} \cdot \underset{s\geq 1}{\ } \frac{1}{N^{s-1/2}} \sim \frac{1}{N^{1/2}}$$

worst

$$\Rightarrow \int |\phi|^6 \lesssim \left(\sum |q_n|\right)^4 \leq N^{2(1-s)} + \frac{1}{N^{2(1-s)}} \underbrace{\|\mathcal{I}q\|^4_{H^1}}_{\lesssim N^{4(1-s)}}$$

To prove this is also the case for the iterated nonlinearities we need 3.6

$$\|\{H_1, H_2\}\| \lesssim \|H_1\|\ \|H_2\|$$

$$\sim \sum_n \frac{\partial H_1}{\partial q_n} \frac{\partial H_2}{\partial \bar{q}_n} \leftarrow \text{not making use of cancelation}$$

possible improvement

$$\frac{\partial H_1}{\partial q} \frac{\partial H_2}{\partial \bar{q}} - \frac{\partial H_1}{\partial \bar{q}} \frac{\partial H_2}{\partial q}$$

Need $\left\| \frac{\partial H_2}{\partial \bar{q}} \right\|_2 \lesssim \| H_2 \|$

↳ one of the factors may be assumed to be only bdd in L^2 (i.e. not satisfying 3.2)

(3.8)

$$\left\| I \frac{\partial H_2}{\partial \bar{q}} \right\|_{H^1} \lesssim N^{1-s} \| H_2 \|$$

2 → 1 because the duality factor is only in L^2

all factors satisfy (3.1) – (3.2)

<u>Proof</u> (3.7)

$$\left\| \frac{\partial N}{\partial \bar{q}} \right\|_2 \lesssim \sup_{\| p^{(2r)} \|_2 = 1} \sum |c(\bar{n})| \prod_{j=1}^{2r-1} | q^{(j)}_{n_j} | \, |p|$$

duality variable

$\lesssim \| N \|$ by definition if all

$(q^{(j)}, j = 1, \dots, 2r-1$ except at most 1 satisfy (3.2)

$\Rightarrow$ (3.7)

(LHS) of (3.8)

duality variable

$$\sim \sum |n| \, |m(n)| \, |c(\bar{n})| \, |q^{(1)}_{n_1}| \dots | q^{(2r-1)}_{n_{2r-1}} | \times |q_n|$$

Assume wlog $|n| \lesssim |n_1|$ $\quad n = n_1 - n_2 + \dots \pm n_{2\ell-1}$

$$r_{n_1} := |n| \, |m(n)| \, |c(\bar{n})| \, |q^{(1)}_{n_1}|$$

$$\|r\|_{L^2} = \| I q \|_{H^1} \lesssim N^{1-s}$$
$$\Rightarrow \tilde{r} := \frac{1}{N^{1-s}} r \Rightarrow \|\tilde{r}\|_2 < C$$

↳ a set $\lesssim N^{1-s} \Rightarrow$ Loss of N^{1-s} in the factor of the summation

$$\hookrightarrow \sum c(\bar{n}) \, |\tilde{r}_{n_1}| \, |q^{(2)}_{n_2}| \dots | q^{(2\ell-1)}_{n_{2\ell-1}} | \, |q_n| \lesssim \|N\|$$

only in L^2

To prove these estimates need the exception of 2 factors

Let's go back to (3.6)

(WTS) $\| \{H_1, H_2\} \| \lesssim \|H_1\| \, \|H_2\|$

(i) Both exceptional factors appear in $\frac{\partial H_1}{\partial q}$

say $q^{(1)}_{n_1}$, $\bar{q}^{(2)}_{-n_2}$ in $\frac{\partial H_1}{\partial q}$

$$\left\| \sum_n \frac{\partial H_1}{\partial q_n} \frac{\partial H_2}{\partial \bar{q}_n} \right\| \overset{C.S.}{\lesssim} \| \quad \|_{\ell^2} \| \quad \|_{\ell^2}$$

can't use 3.7, because it admits only one exceptional factor

But write

$$\sum_n \frac{1}{n\, m(n)} N^{1-s} \frac{\partial H_1}{\partial q} \times \frac{m(n) \cdot n}{N^{1-s}} \frac{\partial H_2}{\partial \bar{q}}$$

Consider one of them

$$\left(\sum_n \left| \frac{1}{m(n) n} N^{1-s} \frac{\partial H_1}{\partial q} \right|^2 \right)^{1/2} \sim \underset{\substack{\uparrow \\ \text{norm by duality}}}{\sum} c(n) \prod_{j=1}^{2r-1} |q^{(j)}_{n_j}| \underbrace{\frac{N^{1-s}}{m(n)|n|} \underset{\text{only in } \ell^2}{|q_n|}}_{=: p}$$

$$\Rightarrow \| I_1 \|_{H^1} = \| p \|_2 \ N^{1-s} \lesssim N^{1-s} \Rightarrow (3.2)$$

need 3.1

$$\frac{1}{m(n) \cdot n} = \begin{cases} \dfrac{1}{n} & |n| \le N \\[2ex] \dfrac{1}{N^{1-s} |n|^s} & |n| \ge N \end{cases}$$

$$\cdot\ |n| \ge N^{1-s} \Rightarrow \frac{N^{1-s}}{m(n) \cdot n} \lesssim 1$$

$$\Rightarrow \| p \|_{\ell^2} \le \| q \|_{\ell^2} < C$$

$\leftarrow$ I multiplied in the first $\Rightarrow$ can divide in the 2nd factor

$$(LHS) \text{ of } (3.6) \lesssim \frac{1}{N^{1-s}} \|H_1\| \underbrace{\left\| I \frac{\partial H_2}{\partial q} \right\|_{H^1}}_{\leq \|H_2\|}$$

(3.8) because H_2 have no exceptional modes

- $|m| \leq N^{1-s}$ $(\leq N)$

duality now q_m satisfies only $\|q\|_2 \lesssim 1$

but I is the identity

$$\|Iq\|_{H^1} = \|q\|_{H^1} \lesssim N^{1-s}\|q\|_2 \leq N^{1-s} \Rightarrow \text{sat. } (3.7)$$

$$\Rightarrow (3.6) \lesssim \left\| \frac{\partial H_1}{\partial q} \right\|_2 \left\| \frac{\partial H_2}{\partial q} \right\|_2 \quad \underbrace{\text{apply}}_{\text{proof of}} \ (3.7) \text{ for both}$$

Case (ii) easy, case (iii) easy

$\leftarrow$ can apply (3.7) once in the duality argument have just 2 except. factor (1 duality, the other is of H)

$\rightarrow$ symmetric to (i), so just change H_1, H_2 in the proof of (i).

Back to the NLS

Assume: $H(q) = \underbrace{\sum n^2 |q_n|^2}_{H_0} + \mathcal{N}_0 + \mathcal{N}_1(q) + \mathcal{N}_2(q)$

$(3.9): |D(\bar{n})| < N^{2(1-s)+\varepsilon}$

$(3.10): |D(\bar{n})| > N^{2(1-s)+\varepsilon}$

error $\|\mathcal{N}_2\| < N^{-C}$ for large C

define F s.t. $\{H_0, F\} = -\mathcal{N}_1$ etc

new Hamiltonian: $H' = H_0 \Gamma^{-1} = H_0 + \mathcal{N}_0 + \cancel{\mathcal{N}_1} + \mathcal{N}_2 \circ \Gamma^{-1} +$

$$+ \cancel{\{H_0, F\}} + \{\mathcal{N}_0, F\} + \{\mathcal{N}_1, F\} + \text{h.o.t. in } F$$

$$\|F\| \lesssim \frac{\|\mathcal{N}_1\|}{N^{2(1-s)+\varepsilon}} \lesssim N^{-\varepsilon}$$

$\|\mathcal{N}_1\| \leq N^{2(1-s)}$ by inductive assumption

error term: $\|\mathcal{N}_2 \circ \Gamma^{-1}\| \leq \underbrace{\|\mathcal{N}_2\|}_{N^{-C}} + \underbrace{\|\{\mathcal{N}_2, F\}\|}_{\leq N^{-\varepsilon} N^{-C}} + \dots$

$\lesssim N^{-C} \leftarrow$ by (3.6) not precise

$$\left\| I \frac{\partial F}{\partial \bar{q}} \right\|_{H^1} \sim \left\langle I \partial_x \frac{\partial F}{\partial \bar{q}_n}, q_n \right\rangle \lesssim$$

$$\lesssim \| I_q \|_{H^1} \left\| \frac{\partial F}{\partial \bar{q}_n} \right\|_2$$ and apply l^2 Cauchy Schwarz

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \| \quad (3.7)$

$$\lesssim \| Iq \|_{H^1} \quad \| F \| \lesssim N^{1-s-\varepsilon}$$

by def. of F on Γ

$$q(1) = q(0) + \int_0^1 \frac{\partial F}{\partial \bar{q}} \, dt$$

$\hookrightarrow$ take $\| I \cdot \|_{H^1}$

$$\Rightarrow \| I q(1) \|_{H^1} \leq \| I q(0) \|_{H^1} + N^{1-s-\varepsilon}$$

$\quad \|$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ (see 1.4)

$I \Gamma q$

$\Rightarrow \Gamma_q$ satisfies (3.2)

and $\| \Gamma_q \|_{l^2} = \| q \|_{l^2} + O(N^{-\varepsilon})$ $\quad \left\| \frac{\partial F}{\partial \bar{q}} \right\| \leq \| F \| \lesssim N^{-\varepsilon}$

<u>Note:</u> if $\| q \|_{l^2} = C$

$$\frac{1}{2} C \leq \| \Gamma q \|_2 \leq 2C$$ $\leftarrow$ finite and not zero

$\Rightarrow$ (1.7) possible to prove for N large the argument

19

$$H(q) = \|q\|_{\dot{H}^1}^2 + \|q\|_{L^6}^6$$

$$\gtrsim \|q\|_{L^2}^6 \geq \|q\|_{L^2}^2$$ because of

$$\frac{1}{2}C \leq \|\Pi q\|_2 \leq 2C$$

$$H(q) \gtrsim \|q\|_{H^1}^2$$

By Sobolev and interpolation:

$$\|q\|_{L^6} \leq \|q\|_{H^{1/3}}$$

$\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

$\frac{1}{3} = \theta \cdot 1 + (1-\theta) \cdot 0$

interpolation

$\Rightarrow \theta = 1/3$

$$\leq \|q\|_{H^1}^{1/3} \|q\|_{L^2}^{2/3}$$

$$\Rightarrow \|q\|_{L^6}^6 \leq \|q\|_{H^1}^2 \cdot \|q\|_{L^2}^4 \lesssim C \|q\|_{H^1}^2$$

$$\Rightarrow H(q) \sim \|q\|_{H^1}^2$$

$\hookrightarrow$ depends on $\|q\|_2$

~~Friday~~ Thursday 27 May 2010 17.00 Mio

$$H' = H \circ \Gamma^{-1} = H_0 + \mathcal{N}_0 + \{\mathcal{N}_0, F\} + \{\mathcal{N}_1, F\}$$
$$+ \text{h.o.t.} + \mathcal{N}_7 \circ \Gamma^{-1}$$

$\leftarrow \|\cdot\| \lesssim N^{-C}$

h.o.t. $\{ \dots \{N_0, F\}, F\} \dots \}$

? $k-1$ times

h.o.t. $\| \downarrow \| \overset{\circledast}{\underset{3.6}{\Rightarrow}} \|N_0\| \, \|F\|^k \leq$

$\lesssim N^{2(1-S)} N^{-k\varepsilon} \lesssim N^{-C}$ for large $k(\varepsilon)$

inductive assumption

Same thing for the $\mathcal{N}_1$

$\mathcal{G}$ = remaining terms $\{N_0, F\}, \{\{N_0, F\}, F\}$, etc...

" dep $\leq k$ "

(WTS) $\|\mathcal{G}\| < N^{-\varepsilon} \|\mathcal{N}_1\|$

$\|\{N_1, F\}\| \lesssim$

20

- $\| \{ N_1, F \} \| \leq \|F\| \, \|N_1\| < N^{-\varepsilon} \|N_1\|$
- $\| \{ N_0, F \} \| \leq \underbrace{\|N_0\|}_{\leq N^{2(1-s)}} \|F\| < N^{2(1-s)} \dfrac{\|N_1\|}{N^{2(1-s)+q}} \qquad \left(F \sim \dfrac{N_1}{D(\bar{n})} \right)$

$\| \{ \{ N_0, F \}, F \} \| \leq \|N_0\| \, \|F\|^2 \lesssim \|N_0\| \, \|F\| \lesssim N^{q} \|N_1\|$

etc.

$\| \{ \{ H_0, F \}, F \} \|$

Rmk

coming from the Lie transform

$H_0 \circ \Phi^{-1} = H_0 + \{ H_0, F \} + \frac{1}{2} \{ \{ H_0, F \}, F \}$

$N_1 \circ \Phi^{-1} = N_1 + \{ N_1, F \} + \{ \{ N_1, F \}, F \}$ etc.

$- \{ \{ H_0, F \}, F \}$ can't cancel each other

Write $\mathcal{G} = \overline{N_0} + N_1'$

$\overline{N_0}$: D small; N_1': D large ((3.10))

Let $N_0' := N_0 + \overline{N_0}$

N_0: old; $\overline{N_0}$: coming from $\mathcal{G}$

Need to check it satisfies inductive hypothesis

$$\|N_0'\| \leq \|N_0\| + \|\overline{N_0}\|$$
$$\lesssim N^{2(1-s)} \qquad \lesssim N^{-\varepsilon}\|N_1\| \lesssim N^{2(1-s)-\varepsilon}$$

3.15

Also, $\|N_1'\| < N^{-\varepsilon}\|N_1\|$

$\Rightarrow$ Repeat Normal form reductions many times

$$\Rightarrow \|N_1^{(\ell)}\| < N^{-\ell\varepsilon} \underbrace{\|N_1\|}_{\lesssim N^{2(1-s)}} < N^{-C}$$

can put this into the error term N_2

$\Rightarrow$ we can assume the Hamiltonian has the form

$\Rightarrow$ Assume $\quad H = \sum n^2 |q_n|^2 + N_0 + N_2$

where

$$\|N_0\| \lesssim N^{2(1-s)} \text{ and } \|N_2\| < N^{-C}$$

$N = N_0 + N^2$

Apply $\frac{d}{dt}\, H(Jq)$

21

Come back to the proof of (1.11)

$$(1.11) = \sum m(n) \left[\frac{\partial N}{\partial q_n} (Iq) \frac{\partial N}{\partial \bar{q}_n} - \frac{\partial N}{\partial q_n} \frac{\partial N}{\partial \bar{q}_n}(Iq) \right]$$

Assume (1.13) $n_1^* > N \Rightarrow n_2^* \geq N$ (sum = 0)

$\Rightarrow$ at least one factor of $\frac{\partial N}{\partial q} \gtrsim N$

• Say $|j| > N \Rightarrow m(j) = N^{1-s}/|j|^{1-s}$

otherwise everything cancels

(3.2) $\| Iq \|_{H^1} \lesssim N^{1-s}$

$$\left\| \frac{N^{1-s}}{|j|^{1-s}} |j| q_j \right\|_{\ell^2_j} \Rightarrow \| |j|^s q_j \|_{\ell^2_j} \lesssim 1$$

$$\Rightarrow \| q \|_{L^2} < N^{-s}$$

$$\left\| \frac{\partial N}{\partial q} \right\|_2 \overset{(3.7)}{\lesssim} \frac{1}{N^s} \| N \|$$

$$|(1.11)| \lesssim \frac{1}{N^{2s}} \| N \|^2 \leq N^{4-6s}$$

$$(1.14) = -\sum_{n_1 - n_2 + \dots - n_{22} = 0} c(\vec{n}) \left(m(n_1)^2 |n_1|^2 - m(n_2)^2 |n_2|^2 \right) q_{n_1} \bar{q}_{n_2} \cdots \bar{q}_{n_{22}}$$

Let $\varphi(n^2) = m(n)^2 |n|^2$

$$\varphi(u) = \begin{cases} u & |u| \leq N^2 \\ N^{2(1-s)} u^s & |u| \geq N^2 \end{cases}$$

$\Rightarrow \varphi'(u) \lesssim \varphi(u)/u$

N_0 – part of (1.14) – resonant part

$|D(\bar{n})| \overset{(3.9)}{<} N^{2(1-s)+\varepsilon}$ $\left(K < N^2 < |n_1^*|^2\right)$

$(n_1^* > N)$ 1.13

reed condition

$\Rightarrow (n_1^*)^2 \sim (n_2^*)^2 \Rightarrow n_2^* \sim n_1^*$ $n_1^* > N$ (1.13)

$(n_1^*)^2 = (n_2^*)^2 + O\left((n_3^*)^2 + N^{2(1-s)+\varepsilon}\right)$ $\because$ because

$(n_1^*)^2 - (n_2^*)^2 = D(\bar{n}) - n_3^2 + n_4^2$ $\therefore$ therefore

drop star can assume $n_1^* = n_1$, $n_2^* = n_2$

MVT

$|\varphi(n_1^2) - \varphi(n_2^2)| \lesssim \varphi'(n_1^2)\,|n_1^2 - n_2^2| \lesssim$

$\lesssim \underbrace{\varphi(n_1^2)/n_1^2}_{\text{u by definition}}\, O(n_3^{*2} + N^{2(1-s)+\varepsilon})$

u by definition $m(n_1)^2 = \dfrac{N^{2(1-s)}}{n_1^{2(1-s)}}$ $|n_1| > N$

Let's prove (3.19)

<u>Note</u>: $\varphi(a)$ is increasing in a.

This means $\varphi(n_j)^2 \leq \varphi((n_3^*)^2)$ $\quad n_j \lesssim n_3^*$

$R(\bar{n}) := n(n_1)^2 \ldots$

$$R(\bar{n}) \lesssim |\varphi(n_1)^2 - \varphi(n_2^2)| + C\varphi(n_3^*)^2$$

$$\lesssim \underbrace{\frac{\varphi(n_1^2)}{n_1^2}(n_3^*)^2}_{\leq \varphi(n_3^*)^2} + \underbrace{\frac{\varphi(n_1^2)}{n_1^2}}_{\leq 1} N^{2(1-s)+\varepsilon}$$

$$\Rightarrow R\bar{n} \lesssim \varphi(n_3^*)^2 + N^{2(1-s)+\varepsilon} \lesssim (n_3^*)^2$$

$|n_3^*| > N$ $\quad \dfrac{N^{2(1-s)}}{n_1^{2(1-s)}} (n_3^*)^2 \lesssim \dfrac{N^{2(1-s)}}{n_3^{*\,2(1-s)}} (n_3^*)^2 = N^{2(1-s)} n_3^{*\,2s} = \varphi(n_3^*)^2$

$|n_3^*| < N$ $\quad \begin{array}{c} \wedge \\ (n_3^*)^2 = \varphi(n_3^{*2}) \end{array}$

Two cases

(3.20) $\quad n_3^* < N^{1-s+2\varepsilon}$

$(1.14) = \sum \mathrm{dR}(\bar m) \prod q_{m_j} \leq \underset{\substack{\uparrow \\ \text{control on } (n_3^*)^2}}{N^{2(1-s)+4\varepsilon}} / N_0 4 \cdot \frac{1}{N^{2s}}$

$N_0 = \underset{1/N^{2s}}{\sum} c(\bar m) \overset{2r}{\underset{j=1}{\prod}} q_{m_j} \swarrow N^{2s}$

Recall $n_1, n_2 > N \Rightarrow \|q^{(1)}\|_2 \ \|q^{(2)}\|_2 < N^{-s}$

Let $p^{(1)} = N^s q^{(1)} \Rightarrow \|p^{(1)}\| < C$ etc.

$\hookrightarrow \ \lesssim N^{4-6s+4\varepsilon}$

(3.21)
When $n_3^* \gtrsim N^{1-s+2\varepsilon} \underset{(3.9)}{>>} \sqrt{D(\bar m)}$

of $A - B = o(n_1) \Rightarrow \lesssim \varepsilon G \Rightarrow$ cannot kill $A - B$

$(n_1 - n_2)(\cancel{n_4} + n_2) = -n_3^2 + o(n_3^2)$ Assume now:

$n_1 - n_2 = -n_3 + n_4 = -n_3 + o(n_3)$

$(-n_3 + o(n_3))(n_1 + n_2) \Rightarrow n_1 + n_2 = n_3 + o(n_3)$

$n_1 + n_2 - n_3 = o(n_3)$ but

$+ \ n_1 - n_2 + n_3 = o(n_3) \Rightarrow_{\text{summing}} 2n = o(n_3)$ ↯

(23)

hence $n_3^* \sim n_4^*$

$\Rightarrow$ ~~...~~ $R(\bar{u})$ LHS 3.19

$|n(\bar{u})| \lesssim (n_3^*)^2 \sim n_3^* n_4^*$

Use (2.3) $\max (n_3^*)^{2+\varepsilon} \underbrace{\cancel{D(\bar{n})}}_{\lesssim N^{2(1-s)+\varepsilon}} (\ln m_1^*)^2$ guess of $H_{1/2}$

~~...~~ 1

$B \sim n_3^*$

$(1.14) \le B^{2+}$ ~~...~~ $N^{2(1-s)+\varepsilon} \underbrace{C(\|q\|_2)}_{= \prod_{j=1}^{4} \|q\|_2} (\ln m_1^*)^2$ maybe

$\underline{B > N}$: $\|q^{(j)}\|_2 < \frac{1}{|m_j|^s} \lesssim \frac{1}{B^s}$ $j = 1, 2, 3, 4$ b/c $m_3^* \sim n_4^* \sim B > N$

$\left(\| |n|^s q_n \|_{\ell^2_n} \lesssim 1 \right)$

(1.14) ~~...~~

$(1.14) \le \frac{B^{2+}}{B^{4s}} N^{2(1-s)+\varepsilon} = B^{2-4s+} < N^{2-4s+} N^{2(1-s)+\varepsilon}$ $s > \frac{1}{2}$, $2-4s<0$

$= N^{4-6s+\varepsilon+}$

$(\ln m_1^+)^?$ can be absorbed in $\frac{1}{|m_1^+|^\beta}$

Allowed to add $(\ln m_1^+)$!

<u>$B < N$</u> $\quad \| q^{(j)} \|_2 < N^{-s} \quad j = 1,2$

$j = 3,4$: $\| I q \|_{H^1} \leq N^{1-s}$

$B \| q \|_2 \quad \because \; m(m_3) = 1$

$$\Rightarrow \| q^{(j)} \|_2 < \frac{N^{1-s}}{B} \qquad j = 3,4$$

$$\Rightarrow (1.14) \lesssim B^{2+} N^{2(1-s)+\varepsilon} \frac{1}{N^{2s}} \frac{N^{2(1-s)}}{B^2} =$$

$$\sim B^{0+} N^{4-6s+\varepsilon} \lesssim N^{4-6s+\varepsilon+}$$

Now N_2 contribution of (1.14) – non-resonant part

- N_2

$R(\bar{n}) \lesssim \varphi(m_1^2) \sim m(m_1)|m_1| \; m_2(m_2)|m_2|$ $\qquad \prod_{j=?} |q_{m_j}|$

$$\Rightarrow (1.14) \lesssim \sum |c(\bar{n})| \; m(m_1)|m_1| q_{m_1} \cdot m(m_2)|m_2| |q_{m_2}| \; i)$$

$$p(1) = \frac{m(m_1)|m_1| q_{m_1}}{N^{1-s}}$$
$$p(2) = \frac{m(m_2)|m_2| q_{m_2}}{N^{1-s}}$$

$\Rightarrow$ can multiply all the sum by $N^{2(1-s)}$

$$\| p^{(i)} \|_{\ell^2} = \frac{1}{N^{1-s}} \| Iq \|_{H^1} \le C$$

Now can use the definition of norm

$$\lesssim N^{2(1-s)} \| N_2 \| < N^{2(1-s)-C} < N^{-C/2}$$

$$\text{Now } 1.15 = \sum c(\bar{m}) D(\bar{m}) \prod_{j=1}^{2r} (Iq)_{m_j}$$

<u>Resonant part</u>

$$m(m) \le 1 \qquad \| Iq \|_2 \le \| q \|_2 \le C$$

$$\| I(Iq) \|_{H^1} \le \| Iq \|_{H^1} < N^{1-s}$$

$$(1.13) \lesssim N^{2(1-s)+\varepsilon} \underbrace{\sum c(\bar{m}) \prod (Iq)_{m_j}}_{N_0(Iq)} \lesssim$$

$$\lesssim N^{2(1-s)+\varepsilon} \underset{\substack{\wedge \\ N^{2-2s}}}{\| N_0 \|} \frac{1}{N^{2s}} \lesssim N^{4-6s+\varepsilon}$$

Non resonant part.

$$D(\bar{n}) \gtrsim (m_1^*)^2 \sim n_1^2 n_2^2$$

$$\Rightarrow (1.15) \lesssim \sum C(\bar{n}) \frac{m(n_1) n_1 q_{n_1}}{N^{1-s}} \frac{m(n_2) n_2 q_{n_2}}{N^{1-s}} \prod_{j=3}^{2r} I q_{n_j}$$

$N^{2(1-s)}$

$\Rightarrow$ same computation of before (even better cause we have Iq)

$$< N^{2(1-s)-c} < N^{-c/2}$$

Last part

$$\left| \frac{d}{dt} H'(Iq(t)) \right| < N^{4-6s+\varepsilon}$$

assuming $\|q\|_2 < C$; $\|Iq\|_{H^1} < N^{1-s}$

$$\Rightarrow H'(I_q(t)) \lesssim$$

$$\lesssim H(Iq(0)) + T N^{4-6s+\varepsilon} \quad |t| < T$$

$$\| \quad H(\Gamma Iq(0))$$

$$= H(\Gamma Iq(t))$$

equivalence with the original Hamiltonian

$$\|\underbrace{Iq^{(t)}}_{P}\|^2_{H^1} \lesssim \|\Gamma \underbrace{Iq^{(t)}}_{P}\|^2_{H^1} + \underbrace{N^{2(1-s)}}_{\text{loose}}$$

$$\sim H(\Gamma Iq(t)) + N^{2(1-s)} \leq$$

$$\leq H(\underbrace{\Gamma I q(0)}) + N^{4-6s+\varepsilon} + N^{2(1-s)}$$

$$\|\Gamma I q(0)\|^2_{H^1} \sim \|Iq(0)\|^2_{H^1}$$

$$\Rightarrow \|Iq(t)\|^2_{H^1} \leq C\left(\|Iq(0)\|^2_{H^1} + TN^{4-6s+\varepsilon} + N^{2(1-s)}\right)$$

Consider Cauchy problem for

$$\begin{cases} i\partial_t Iq_t + \Delta Iq - I(|q|^4 q) = 0 \\ q|_{t=0} = q_0 \end{cases} \qquad \|Iq\|^2_{H^1}$$

Now choose local existence time $\delta = \delta(3CN^{2(1-s)})$

Then $\overset{0}{|}\ \overset{\delta}{|}\ \overset{2\delta}{|}\ \overset{3\delta}{|} \rightarrow$ can iterate

LWP argument until $\|Iq(t)\|^2_{H^1} \lesssim 2CN^{2(1-s)}$

$\Rightarrow \left(\|Iq(0)\|^2_{H^1} \lesssim N^{2(1-s)}\right) \Rightarrow$ can iterate local steps

$\underbrace{M}_{\#\text{ of LWP step}} N^{4-6s+\varepsilon}$

$M \; N^{4-6s+\varepsilon} \sim N^{2(1-s)} \quad \Rightarrow M \sim N^{-2+4s-\varepsilon}$? ← doesn't matter

$\Rightarrow N \sim M^{\frac{1}{4s-2-4\varepsilon}}$

$T = \delta M \quad \Rightarrow \quad T \sim N^{-2+4s-\varepsilon}$

[Scaling ⇒ subcritical problem ⇒ scale to time 1 existence]

$\Rightarrow$ GWP $\quad s > \frac{1}{2}$

II statement:

$\|Iq(t)\|_{H^1}^2 \lesssim N^{2(1-s)} \sim T^{\frac{2(1-s)}{4s-2-4\varepsilon}}$

$\|Iq\|_{H^1}^2 = \sum_{|n| \leq N} |n|^2 |q_n|^2 + \sum_{|n| \geq N} N^{2(1-s)} |n|^{2s} |q_n|^2$

$\geq \|q\|_{H^s}$

$\Rightarrow \|q\|_{H^s} \lesssim T^{\frac{2(1-s)}{4s-2-4\varepsilon}}$

$|t| \leq T \quad \Rightarrow$ also true for $\frac{1}{2} T \leq |t| \leq T$

$\Rightarrow \|q\|_{H^s} \lesssim |t|^{\frac{1-s}{4s-2-4\varepsilon}}$

Mon, May 31, 2010

Lemma 4.1 $\quad N_1 > N_3^{1+\delta} \quad N_1 \geq N_2 \geq N_3$

$D > N_1^{\delta} \qquad S_j := \operatorname{supp} \phi_j \subset [N_j, 2N_j]$

Then
$$(4.5) \int_0^{2\pi/D} \int_{\mathbb{T}} \prod |S(t)\phi_j|^2 \, dx\,dt \lesssim N_1^{-\delta} \prod_1^3 \|\phi_j\|_2^2$$

Rmk (replace D by 1)
On all the space cannot be true, because $\|S(t)\phi\|_{L^6_{x,t}} \leq \underbrace{e^{\frac{c \ln N}{\ln \ln N}}}_{C_N} \|\phi\|_2$

Reduction:

$$\text{LHS of } (4.5) \leq \underbrace{\|S(t)\phi_1\|_6 \, \|S(t)\phi_2\|_6^2}_{\leq} \, \|S(t)\phi_1 \, (S(t)\phi_3)\|_{L^2_{x,t}}$$

$$\leq C_{N_1}^3 \|\phi\|_2^3 \, \|S(t)\phi_1 \, S(t)\phi_3\|_{L^2} \qquad (\mathbb{T} \times [0, \tfrac{2\pi}{D}])$$

have $N_2 = N_3$

$\Rightarrow$ enough to prove (4.5) under (4.7) $N_1 > (N_2 + N_3)^{1+\delta}$

(*) $\Rightarrow \quad N_1 \geq [N_2 + N_3]^{1+\delta}$ do we have this for (*)

Yes because $N_1 > (N_3 + N_3)^{1+\delta}$

Assume (4.7) thanks to the reduction

By contradiction assume (4.5) fails

$$\int_0^{2\pi/D} \int \dots \, dx\,dt > N_1^{0-}\ (-\varepsilon)$$

Assume also $\|\phi_j\|_2 = 1$

$$\wedge\!\!|$$

$$\underbrace{\frac{1}{D}\|\phi_1\|_2^2 \|\phi_2\|_{+\infty}^2 \|\phi_3\|_\infty^2}_{\leq \frac{1}{D}\Delta_2\Delta_3}$$

Cauchy Schwarz

$$\|\phi_2\|_\infty \leq \|\hat{\phi}_2\|_{\ell^2} \leq \Big(\sum_{n\in S_2} 1\Big)^{1/2} \|\phi\|_{L^2} \leq \Delta_2^{1/2}$$

$$\Rightarrow D < N_1^{0+}\Delta_2\Delta_3$$

$$F(\bar{n},\bar{m}) = n_1^2 - m_1^2 + n_2^2 - m_2^2 + n_3^2 - m_3^2$$

$$\text{LHS of } (4.3) = \int_0^{1/D} \sum_{\substack{n_j, m_j \\ \sum n_i = \sum m_j}} e^{iF(n,m)t}\, \hat{\phi}_j(n_j)\, \overline{\hat{\phi}_j}(m_j)$$

$$\leq \frac{1}{D}\sum_{*} \prod_{j=1}^{3} |\hat{\phi}_j(n_j)|\,|\hat{\phi}_j(m_j)|$$

$\Rightarrow$ Define $f(x)$ s.t. $\hat{f}(n) = |\hat{\phi}(n)|$ $\leftarrow$ on case L^2

$$=) = \frac{1}{D}\int_\pi |f_1 f_2 f_3|^2 \, dt$$

Note: $\|f\|_2 = \|\phi\|_2$ $(=1)$

$$\leq \frac{1}{D}\|f_1\|_2^2 \|f_2\|_\infty^2 \|f_3\|_\infty^2 \text{ etc. as before}$$

$$(4.10) \quad m_1 + m_2 + m_3 = n_1 + n_2 + n_3$$

$$(4.11) \quad m_1^2 + m_2^2 + m_3^2 = n_1^2 + n_2^2 + n_3^2 + O(D) \Leftarrow \text{CHECK}$$

For now let's assume

Note: $m_1 - n_1 = -m_2 + n_2 + n_2 - m_3 = O(N_2) \ll O(N_1)$ (4.7)

$\Rightarrow$ cancellations $m_1 - n_1$ be of the same size

$\Rightarrow$ By (4.11) $m_1 - n_1 = \underbrace{\frac{1}{m_1+n_1}}_{\sim \frac{1}{N}} \left(-m_2^2 + \dots + n_3^2 + O(D) \right)$

$$(m_2^2 - n_2^2) = (m_2 - n_2)(m_2 + n_2)$$

Δ_2 $\quad$ N_2

both if they have some of difference size

$$\Rightarrow \lesssim \frac{1}{N_1} \left(\Delta_2 N_2 + \Delta_3 N_3 + N_1^{0+} \Delta_2 \Delta_3 \right) \lesssim N_1^{-\delta +} \underbrace{(\Delta_2 + \Delta_3)}_{=: \Delta}$$

$$\frac{N_2}{N_1} \lesssim N_1^{-\delta'} \qquad \Delta_2 \leq N_2 \Rightarrow \frac{N_1^{0+}}{N_1} < N_1^{-\delta'}$$

Write $S_1 = \bigcup_\alpha S_{1,\alpha}$ = partition of $\mathbb{Z}$ into size $\sim \Delta$

Let define $\phi_{1,\alpha}$ by $\hat{\phi}_{1,\alpha} = \hat{\phi}_1\big|_{S_{1,\alpha}}$ $\tilde{\phi}_{1,\alpha} = \phi_{1,\alpha} / \|\phi_{1,\alpha}\|_2$

$$\Rightarrow \phi_1 = \sum_{\alpha} \|\phi_{1,\alpha}\|_2 \tilde{\phi}_{1,\alpha} = \sum \phi_{1,\alpha}$$

by definition

and $\sum_{\alpha} \|\phi_{1,\alpha}\|_2^2 = 1$

$$1 = \|\phi_1\|_2^2 = \left(\sum_{\alpha} \phi_{1,\alpha}, \sum_{\alpha'} \phi_{1,\alpha'}\right)_{L^2} = \sum_{\alpha} \|\phi_{1,\alpha}\|_2^2$$

$$S_{1,\alpha'} \cap S_{1,\alpha'} = \emptyset \text{ if } \alpha \neq \alpha'$$

$$\int_0^{1/D} \int_{\mathbb{T}} \prod_j \left| e^{it\Delta} \phi_j \right|^2 dx\,dt \lesssim$$

$$e^{it\Delta}\phi_1 \cdot \overline{e^{it\Delta}\phi_1} =$$

$$= e^{it\Delta} \sum \tilde{\phi}_{1,\alpha} \|\phi_{1,\alpha}\|_2 \times \overline{e^{it\Delta} \sum_{\beta} \tilde{\phi}_{1,\beta} \|\phi_{1,\beta}\|_2}$$

$$\lesssim \sum_{\alpha} \sum_{\beta} \|\phi_{1,\alpha}\| \, \|\phi_{1,\beta}\| \left| \int_0^{1/D} \int_{\mathbb{T}} \underbrace{e^{it\Delta}\tilde{\phi}_{1,\alpha}}_{n_1 \in S_{1,\alpha}} \, \underbrace{\overline{e^{it\Delta}\tilde{\phi}_{1,\beta}}}_{n_1 \in S_{1,\beta}} \prod_{j=2}^{3} |S(t)\phi_j|^2 \, dx\,dt \right|$$

But by (4.12) $|m_1 - n_1| \leq \Delta$

$$\Rightarrow \beta = \alpha \text{ or } \alpha \pm 1$$

Since basically there is no summation in β, so

$$\leq \sum_\alpha \sum_{k=-1}^{1} \|\phi_{1,\alpha}\| \, \|\phi_{1,\alpha+k}\| \int \dots \lesssim$$

$$\lesssim \underbrace{\sum_\alpha \|\phi_{1,\alpha}\|^2}_{=1} \times \int \dots$$

$$\leq C \sum_\alpha \sum_{k=-1}^{1} \left(\|\phi_{1,\alpha}\|_2^2 + \|\phi_{1,\alpha+k}\|_2^2 \right)$$

then switch summation in k and α

put $\max_\alpha$

$\Rightarrow \exists\, \alpha$ s.t. $(4.13) > N_1^{0-}$ (by abs. of our not contradiction assumption

$$\Rightarrow \int_0^{1/D} \int |e^{it\Delta} \tilde{\phi}_{1,\alpha}|^2 \prod_{j=2}^{3} |e^{it\Delta} \phi_j|^2 \, dx\, dt \leq$$

{ & similarity with blow-up proof of Bourgain
↗ Alessandro Ruck

$$\underset{(*)}{\leq} \|(S(t)\tilde{\phi}_{1,\alpha})(S(t)\phi_3)\|^2_{L^2(\mathbb{T} \times [0, 1/D])} \underbrace{\|S(t)\phi_2\|_6^2 \, \|S(t)\phi_3\|_6}_{C_{N_1}^3}$$

Can absorb C_N^3 $\Rightarrow$ do not loose power, but log type N_1 !

$$\Rightarrow \int_0^{1/D}\int_{\mathbb{T}} |S(t)\tilde{\phi}_{1,\alpha}|^4 |S(t)\phi_3|^2 \, dx\, dt > N_1^{0-}$$

(In the induction process go on loosing $e^{c\frac{\ln N}{\ln\ln N}}$ but still good case N_1^{0-} can absorb it

<u>Note:</u> $m_1 - n_1 + m_1' - n_1' + m_3 - n_3 = 0$ $\quad m_1, n_1, m_1', n_1' \in S_{1,\alpha}$
$m_3, n_3 \in S_{m_3}$

By (4.12) $|(m_1 - n_1)| \leq \Delta$, $|(m_1' - n_1')| \leq \Delta$ $\Rightarrow |m_3 - n_3| \leq 2\Delta$

Write $S_3 = \bigcup_\alpha S_{3,\alpha}$ with $\dim S_{3,\alpha} \leq \Delta$

and $\phi_3 = \sum_\alpha \|\phi_{3,\alpha}\|^2 \tilde{\phi}_{3,\alpha}$

$$\Rightarrow \text{As before, } \int_0^{1/D}\int_{\mathbb{T}} |S(t)\tilde{\phi}_{1,\alpha_1}|^4 |S(t)\tilde{\phi}_{3,\alpha_3}| \, dx\, dt > N_1^{0-}$$

By Hölder as in (*) $\quad (\tilde{\phi}_{1,\alpha}) \quad \tilde{\phi}_{1,\alpha} (\tilde{\phi}_{3,\alpha})^2$

$\wedge$
$||$

$C_N^3 \times L^6$-Strichartz

$$\Rightarrow \int_0^{1/D} \int_{\mathbb{T}} |S(t)\tilde{\phi}_{1,\alpha}|^2 \, |S(t)\tilde{\phi}_{3,\gamma}|^4 > N_1^{0-}$$

(4.16) satisfies $\tilde{N}_1 > (\tilde{N}_2 + \tilde{N}_3)^{1+\delta}$

because $\tilde{N}_1 = N_1$

$\tilde{N}_2 = \tilde{N}_3 = N_3$

We reduced $\phi_1, \phi_2, \phi_3, N_1, N_2, N_3, \Delta_1, \Delta_2, \Delta_3$

into same functions $\tilde{\phi}_{1,\gamma}, \tilde{\phi}_{2,\gamma}, \tilde{\phi}_{3,\gamma}, N_1, N_3, N_3 \quad \Delta, \Delta, \Delta$

Keep in mind $\Delta < N_1^{-\delta/2}(\Delta_2 + \Delta_3)$

Restart argument with (4.16) ~~instead~~ as contr. hypothesis instead of (4.8)

$\Rightarrow D < N_1^{0+} \Delta^2$ (compare this with 4.9)

By (4.12) Choose $\Delta^{(2)}$ to be

$\Delta^{(2)} = N_1^{-\delta+} 2\Delta$ go to (4.16)

Restart argument $\Rightarrow D < N_1^{0+} \left(\Delta^{(2)}\right)^2 \left(< N_1^{-\delta/2} \cdot \Delta \right)$

Repeat K times $\Rightarrow D < N_1^{0+} \left(\Delta^{(k)}\right)^2 < \underbrace{N_1^{0+} N_1^{-k\delta}}_{N_1^{-C}} (\Delta_2 + \Delta_3)$!!

$\Rightarrow$ contradiction because $D > N_1^{\delta}$

Need just lemma $\Rightarrow$ Corollary

Wednesday 2 June 2010 HIRO's computations

SECTION 5. post GWP $s < \frac{1}{2}$

<u>Difficulty</u>. $\|N\| \lesssim N^{2(1-s)}$ assume $\|\varphi\|_2 < C$

for $s<\frac{1}{2}$ $\|I\varphi\|_{H^1} \lesssim N^{1-s}$

Now, assume $\operatorname{supp} \hat{\varphi} \subset [-N_1, N_1]$ $N_1 > N$

$$\|\varphi\|_\infty \leq \sum_{|m|\leq N_1} |\hat{\varphi}_m| = \sum_{|m|\leq N^{1-s}} + \sum_{N^{1-s}\leq |m| \leq N} + \sum_{N\leq |m| \leq N_1}$$

$$\leq_{C.S.} N^{(1-s)/2} \|\varphi\|_{L^2} + \sum \frac{1}{m(n)|m|} m(n)\hat{\varphi}_m$$

pg 131

$m_N(n) = \begin{cases} 1 & |n| \leq N \\ N^{1-s}|n|^{s} & |n| \geq N \end{cases}$

$\leq_{C.S.} \|I\varphi\|_{H^1} \sum_{|m|\geq N^{1-s}} \left(\frac{1}{n^2}\right)$

$N^{1-s/2}$

3rd term $\left(\sum_{N_1 \geq |m| \geq N} \frac{1}{N^{2(1-s)} N^{2s}} \right)^{1/2} \|I\varphi\|_{H^1}$ $\|I\varphi\|_H N^{-(1-s)} < C$

$\sim N^{-(1-s)} N_1^{\frac{1}{2}-s}$ $\lambda_{|m|}^{1-2s>0}$

$\Rightarrow \|\varphi\|_\infty \lesssim N^{(1-s)/2} + N_1^{1/2-s}$ does $N^{\frac{1-s}{2}} = N_1^{1/2-s}$

$\Rightarrow$ optimize $\Leftarrow N_1 = N^{\frac{1-s}{1-2s}}$

$\Rightarrow \int_{\mathbb{T}} \prod_{j=1}^{6} \varphi_j \leq \underbrace{\|\varphi_1\| \, \|\varphi_2\|}_{\lesssim C} \prod_{3}^{6} \|\varphi_j\|_\infty$

$\lesssim N^{2(1-s)}$

or $\|N_1\|$

Fix T let $N = N(T)$ (to be chosen later)

First, consider (5.3) $iu_t + u_{xx} - \mathbb{P}_{N_1}(u|u|^4) = 0$

$\hookrightarrow$ Hamiltonian

Note $N_1 \gg N^k$ if $s \nearrow \frac{1}{2}$

<u>Hamiltonian</u> of (5.3)

$$H(q) = \sum_{|n| \leq N_1} n^2 |q_n|^2 + \sum_{n_1 - n_2 + \dots - n_6 = 0} q_{n_1} \overline{q_{n_2}} \cdots \overline{q_{n_6}}$$

$\hookrightarrow \| \quad \| \leq N^{2(1-s)}$

From p133–134 (not used $s > \frac{1}{2}$, but yes in 135–136)

$\Rightarrow H' = \sum n^2 |q_n|^2 + N_0 + N_2$

(31)

(5.5) : $|D(\bar{n})| < N^{2(1-s)+\varepsilon}$

(5.6) $\|\mathcal{N}_0\| < N^{2(1-s)}$ for $\mathcal{N}_0$

(5.7) $\|\mathcal{N}_2\| < N^{-C}$ $\quad C \gg 1$

Recall we used $s > \frac{1}{7}$ on pp 135-136 $\Leftarrow$ need to improve this

Bourgain: cannot do it Frequency cut off

Let $\mathcal{N}_0^* = \mathcal{N}_0\Big|_{n_1^* \gtrsim N}$ $\quad \mathcal{N}_0^{**}\Big|_{\substack{n_1^* \gtrsim N \\ n_6^* \leq N^{9/10}}}$

<u>Claim</u>: (5.8) $|\mathcal{N}^{**}(q)| \lesssim N^{2(1-s)-\delta} C(\|q\|_2)$

(5.9) $\|\mathcal{N}_0^*\| < N^{2(1-s)-c\delta}$ $\quad \delta > 0$

We take $\varepsilon \ll \delta$

(5.8): initial step

$$\mathcal{N}_0^{**} = \sum_{\substack{\sum(-1)^j n_j = 0 \\ |D(\bar{n})| < N^{2(1-s)-\varepsilon} \\ n_1^* \gtrsim N \\ n_6^* \lesssim N^{9/10}}} q_{n_1} \bar{q}_{n_2} \dots \bar{q}_{n_6}$$

$$| \quad | \underset{\uparrow}{\lesssim} N^{2(1-s)+\varepsilon} \, N^{-\delta'} \underset{\uparrow}{\lesssim} N^{2(1-s)-\delta'/2}$$

Cor 4.17 $\qquad$ every thing $\qquad$ 4.9

By Corollary 4.17 ~~(a)~~ we can take ~~sup~~ $D(\vec{n}) \in I$ with

$|I| < N^{2(1-s)+\varepsilon}$ $\qquad$ assume $k_1^* \gtrsim N$, $\quad k_6^* \leq N^{9/10}$

$N^{**} = \{N, F\} \sim \sum_n \frac{\partial F}{\partial q_n} \frac{\partial N}{\partial q_n}$

$\underset{C.S.}{\lesssim} \left\| \frac{\partial N}{\partial q_n} \right\|_{\ell^2_n} \left\| \frac{\partial F}{\partial q_n} \right\| \overset{\text{duality}}{\lesssim} \left| \underset{m_j}{N(q)} \right| \left| \underset{k_j}{F(q)} \right|$

Suppose $m_1^* = k_1^* \Rightarrow$ 6th largest freq of $\frac{\partial N}{\partial q} < k_6^* \leq N^{9/10}$

$\qquad\qquad\qquad\qquad\qquad\qquad$ m_6^* or m_7^*

If $m_6^* < N^{9/10}$ good $\qquad$ and $m_6^* > N^{9/10}$

otherwise we have $m_7^* < N^{9/10}$ and $\exists j \in \{1, \dots, 6\}$

s.t. we differentiated in $q_{m_j^*} \Rightarrow \exists |n| \geq m_6^*$ in

the q_n factor lost in $\frac{\partial N}{\partial q_n}$ (and in $\frac{\partial F}{\partial q_n}$)

Recall $\qquad \underbrace{m_1 - m_2 + \dots}_{\text{freq of } \partial N/\partial q_n} = \underbrace{k_1 - k_2 + \dots}_{\text{freq of } \partial F/\partial q_n} = \pm n >> N^{9/10}$

(2)

What we have proved here is
Assume (*) for $\{N, F\}$. Then (*) is true holds for N
(or F)

$\Rightarrow \exists$ freq in $\frac{\partial F}{\partial q_m}$ s.t. $k_1^* \gtrsim N^{9/10}$

$\Rightarrow h_6^f \gg N^{9/10}$ $\not$ Contradiction

Restart (5.8) Recall $|D_F| > N^{2(1-s)+\varepsilon}$

Write $D_F \in [D, 2D] = \cup I_\alpha$ $|I_\alpha| = N^{2(1-s)+\varepsilon}$

• Note: $D_{\{N,F\}} \in I$ and $D_{\{N,F\}} = D_N + D_F$
$\not=$ size $N^{2(1-s)+\varepsilon}$

Fix I
$\Rightarrow$ $\forall$ size α s.t. $D_N \in I_\alpha \Rightarrow \exists$ at most $3\ \alpha$'s s.t.
$D_F \in I_{\alpha'}$

$$\{N, F\} \sim \sum_m \frac{\partial N}{\partial q_m}\frac{\partial F}{\partial q_m} \sim \sum_\alpha \sum_{\substack{m \\ D_F \in I_\alpha}}$$

$\# \alpha' \lesssim \frac{P}{N^{2(1-s)+q}}$

$\underbrace{\sum_{n}}$

C-S. as before $\sum_{\alpha} |N|_{D_N \in I_\alpha} \left(|F|_{D_N \in I_\alpha} \right)$

$< \cancel{\frac{D}{N^{2(1-s)+q}}} \cdot N^{2(1+s)-\delta} \cdot \cancel{\frac{1}{D}} \quad |I_{\alpha'}| \, (n_1^*)^{0+}$

by induction $\left(I_\alpha \text{ size} < N^{2(1-s)+\varepsilon} \right)$

from F (D = size of D_F)

$\left(F \sim \frac{N'}{D_F} \right.$

$N' = \sum_{D_F = a} 1 \cdot \max_a N'|_{D_F = a}$

use 5.13

being $D_F \in I_{\alpha'}$

$e^{\frac{\ln n_1^*}{\ln n_1^* \ln n_1^*}} \leq e^{\ln N_1 / \ln\ln N_1} < N^{0+}$

$\Rightarrow \left| \{ N_1 F \}^{**} (q) \right| < N^{2(1-s)-\delta+} \, C(\|q\|_2)$

$\Rightarrow$ (5.8)

<u>Note</u>: $n_1^* \geq N$ (not already in N, but $n_1^* \leq N_1 = N^k$)

Thursday, 3 June 2010, HPO 14.30

LHS of corollary: $(I = [a, a+D-1])$

$$\text{get} = \sum_{k=0}^{D-1} \underbrace{\sum_{n_j} \prod_{1}^{6} \hat{f}_j(n_j) \int_0^{2\pi} e^{i(F(n)-a-k)t}\,dt}$$

$$\overset{\shortparallel}{D}$$

$$= \sum_{n_j} \int_0^{2\pi} \overbrace{S(t)\, e^{-i(a+kt)}}\, \phi_1(n_1) \prod^{6} \overbrace{S(t)\phi_j}\,(n_j)$$

$$e^{-i(a+k)t}$$

$$\overset{C.S.}{\leq} \| S(t)\phi_1\, S(t)\phi_4\, S(t)\phi_6 \|_{L^2(\mathbb{T}\times\mathbb{T})} \times \| S(t)\phi_3 \ldots \|_{L^2(\mathbb{T}\times\mathbb{T})}$$

$D \sup_k$

Write $\mathbb{T}_t = \sum_{\ell=0}^{D-1} \left(\frac{2\pi}{\ell}, \frac{2\pi}{D}(\ell+1)\right) = \sum I_\ell$

$$\int_0^{2\pi} e^{i\tilde{F}t}\,dt = \sum_{\ell} e^{i\tilde{F}t}\,dt = \sum_{\ell} e^{i\frac{2\pi\ell}{D}\tilde{F}} \int_0^{2\pi/D} e^{i\tilde{F}t}\,dt$$

$$|1^{st}\text{ term}|^2 \leq \sum_{\ell=0}^{D-1} \Bigg| \sum_{\substack{n_i, m_i \\ i=1,4,6}} \hat{\phi}(n_i)\overline{\hat{\phi}_i(m_i)}\, e^{i\frac{2\pi\ell}{D}\tilde{F}} \int_0^{\frac{2\pi}{D}} e^{i\tilde{F}(m,n)}\,dt$$

apply Lemma 4.

$$= \sum \prod \hat{f}_i \bar{f}_j \int_0^{2\pi/D} e^{iF(n,n)t}\,dt$$

where $\hat{f}_j(n) = e^{i2\pi\ell/D\, n^k}\hat{\phi}_j(n)$

LHS of Cor 4.17

$$\lesssim \underbrace{D^{3/2} N_1^{-\delta} C N_1^{3}}_{\wedge (2.35)} \ll \underbrace{D^{3/2} N_1^{-\delta+}}_{M_1}$$

$\wedge (2.35)$ interpolate

$\underbrace{D \cdot C_{N_1}^{6}}_{M_2}$ $\leq \Rightarrow < M_1^{\theta} M_2^{1-\theta}$, ($\theta$ small)

$< D^{1+\varepsilon} N_1^{-\delta'}$ $\varepsilon \ll 1$

Pf of (5.9)

$\|N_0^{+}\| < N^{2(1-s)-\delta}$ $m_1^{+} \gtrsim N \Leftarrow (1.13)$

$|D| < N^{2(1-s)+\varepsilon}$ $n_6^{+} > N^{9/10}$

$$\|N_0^{+}\| \leq \sum_{a} \underbrace{\sum_{D=a} c(\bar{m})\, q_{m_1} \cdots q_{m_{22}}}_{\text{Use } 5.13}$$

$$\leq N^{2(1-s)+\varepsilon} (m_1^{*})^{0+} \prod \|q_{m_j}\|_{l^2}$$

$j = 1, \ldots, 6$ $\| \pm q_j \| < N^{1-s}$

$\Rightarrow$ let $P_j = \frac{1}{N^{1-s}} \cdot \left(\frac{1}{m(m)m}\right)^{-1} q_m \Rightarrow \|f_m\|_2 < C$

Note

~~Replace~~ ~~...~~ at least 4 factors of q_m, $j = 1, \ldots, 6$ satisfy (3.2)

34

Replace those by $\{n_j\}$

$$\Rightarrow \|N_0^*\| \lesssim N^{2(1-s)+\varepsilon} \frac{N^{4(1-s)}}{\prod_3^6 m(n_j^*)\, n_j^\alpha}$$

$$m(n)\cdot n = \begin{cases} n & |n| \le N \\ N^{1-s} n^s & |n| \ge N \end{cases} \qquad \ge N^{9/40} \text{ if } |n| \ge N^{9/10}$$

n_6 satisfies

that means

$$\lesssim N^{6(1-s)+\varepsilon} / N^{36/10} < N^{2(1-s)-\frac{1}{2}} \quad s > \frac{1}{3}$$

$$6(1-s) + \varepsilon - \frac{18}{5} = \underbrace{2-2s+\varepsilon}\, + \underbrace{4-4s-\frac{18}{5}}_{< \frac{-3}{15} < -\frac{1}{2} \text{ for } s > \frac{1}{3}}$$

$=$

Estimation of 1.14

(i) N_1^* contribution

$$(3.19) \quad \left| m(n_1)^2 n_1^2 - m(n_2)^2 n_2^2 + \dots \right| \lesssim \varphi\left((n_3^*)^2\right) + N^{2(1-s)+\varepsilon}$$

WLOG assume $|n_1| = n_1^*$

Since $|D(\bar{n})| < \tau N^2 \lesssim (n_1^*)^2$
$\quad |n_1^2 - \dots|$ $\Rightarrow$ recall condition $n_2^* = |n_2| \sim n_1^*$

Case 1: $n_3^* < N^{1-\cancel{8}s+\varepsilon}$ $(\in (3.20))$

$\Rightarrow n_6^* \lesssim n_3^* < N^{9/10}$ $s > \frac{1}{10}$ example

$\Rightarrow$ use (5.8) (N^{0+} resonants)

$LHS(3.19) =: R$

$R \lesssim (n_3^*)^2 + N^{2(1-s)+\varepsilon} \lesssim N^{2(1-s)+4\varepsilon}$

$(1.14) < N^{2(1-s)-\delta} N^{2(1-s)+\varepsilon} N^{-2s}$

Recall $\|Iq\|_{H^1} < N^{1-s}$ $\quad$ $\cancel{\top}$ $n_1^* \sim n_2^* > N$

$|n| \gtrsim N$ $\quad \| N^{1-s} |n|^s q_n \|_{\ell^2_n}$ $\Rightarrow \|q\|_2 < N^{-s}$ if $|n| > N$

$\geq N^s N^{1-s} \|q\|_2$

Hence $\lesssim N^{6(1-s)-\delta+4\varepsilon-2} = N^{4-6s-\delta+4\varepsilon}$

Case 2: $n_3^* \gtrsim N^{1-s+2\varepsilon}$

$\Rightarrow n_4^* \sim n_3^*$ (see

Case 2: $n_3^* \geq N^{1-s+2\varepsilon} \Rightarrow n_3^* \sim n_4^*$

$\Rightarrow R \lesssim n_3^* m(n_3^*) n_4^* m(n_4^*)$

Recall $\varphi(n^2) = n^2 m(n)^2$ $\quad m(n)n = \begin{cases} |n| & n < N \\ N^{1-s} n^s & n \geq N \end{cases}$

$= (n_3^*)^2 \qquad n_3^* \leq N \quad \geq N^{2(1-s)+\delta}$

$N^{2(1-s)} (n_3^*)^{s} \quad n_3^* \leq N$

if $n_6^* < N^{9/10}$, use 5.8

$N^{2(1-s)-\delta}\, N^{-2s}\, N^{2(1-s)}$

l^2-norm

(3.1) $\|q\|_2 < C$

(3.2) $\|\boxed{I} q\| \lesssim N^{1-s} \Rightarrow \|q\|_2 \lesssim N^{-s}$ $\quad j = 4,1$

If $n_3^* \sim n_4^* \geq N \Rightarrow \cancel{N^{2(1-s)}} (n_3^*)^{2s}$

$\Rightarrow R \|q_3\|_2 \|q_4\|_2 \leq N^{2(1-s)} (n_3^*)^{2s} (n_3^*)^{-2s} \lesssim 1/\delta$

When $\cancel{n_3^*} \sim N \quad n_3^* \leq N \qquad \|Iq_3\|_{H^1} \lesssim N^{1-s}$

$\|q\|_2 \leq N^{1-s} |n|^{-1} \quad \Leftarrow \quad \|n \cdot m\|_{\ell^2_n}$

$\Rightarrow R \cdot \|q_3\| \|q_4\| \lesssim (n_3^*)^2 N^{2(1-s)} \cancel{(n_3^*)}^{-2}$

if $n_6^* > N^{9/10}$: use (5.13) $\left(|D| < N^{2(1-s)+\varepsilon}\right)$

$$\sum_a \sum_{D=a} C(n)\, q^{\circ}_{n_1} \cdots \overline{q}_{n_{2r}} \qquad \|q_5\|\,\|q_6\|$$

$$\lesssim N^{2(1-s)+\varepsilon} \underbrace{N^{2(1-s)}}\, N^{-2s}\, N^{2(1-s)} / N^{9/5}$$

$R \|q_3\|_2 \|q_4\|_2$

showed before

$\|q_1\|_2 \|q_2\|_2$

$$< N^{8(1-s)+\varepsilon - \frac{9}{5}}$$

(ii) N_2 - contribution

see (3.24)

Rmk: 5-6 section, 4- : 1) high-low freq dec $\Rightarrow$ 2) Normal form $\Rightarrow$ 3) I-method

$$(1.15) = \left| \sum C(\bar{n})\, D(\bar{n}) \prod_1^{2r} I q_{n_j} \right|$$

$$\lesssim N^{2(1-s)+\varepsilon}$$

36

$\|Iq\|_2 < N^{-s}$ (case $\|Iq\|_2 \leq \|q\|_2 < N^{-s}$ $j = 1,2$ because $n_1^* \sim n_2^* > N$)

call $P_{n_j} = N^{s} I q_{n_j}$ $j = 1,2$ exceptional ones

$$(1.15) \leq N^{2(1-s)+\varepsilon} N^{-2s} \|N_0\| < N^{2(1-s)-\delta} \quad 5.9$$

$$< N^{4-6s+\varepsilon-\delta}$$

[Rmk $\|N_0\| \lesssim N^{2(1-s)}$ in the previous round

but now we have $\|N_0^*\| < N^{2(1-s)-\delta}$ from

improvement!

(ii) N_2 contribution $\leftarrow$ same as before

$$(1.11) = \sum_n m(n) \left(\frac{\partial N}{\partial q_n}(Iq) \frac{\partial N}{\partial \bar{q}_n} - \dots \right)$$

$$\lesssim N^{-2s} \left\| \left\{ \frac{\partial N}{\partial q_n}(Iq), N \right\} \right\| \leq$$

from n_1^* either k_1^* or $m_1^* = n_1^* \geq N$

3.6

$$\lesssim N^{-2s} \|N(Iq)\| \, \|N\|$$

m_j k_j

say $m_1^* = n_1^*$

$$\leq N^{-2s}\left(\|N_0^*\| + \|N_2\|\right)\overbrace{\|N\|}^{< N^{2(1-s)}}$$

$\leftarrow m_1^* \geq N$

$\leftarrow$ def. of $*$

$\leftarrow < N^{-c}$

$\leftarrow$ (5.9)

$< N^{4-6s-\delta}$

Collecting (5.16) – (5.21) leads to
$N^{4-6s+4\varepsilon-\delta}$ as biggest term (coming from (5.16))

This is the energy increment

$$\Rightarrow \left|\frac{d}{dt} H'(Iq)\right| < N^{4-6s+4\varepsilon-\delta}$$

Impose $N^{4-6s+4\varepsilon-\delta}\, T < N^{2(1-s)}$

$$N^{2(1-s)} \sim \|Iq(0)\|_{H^1}^2$$

Satisfied for
$s > \underbrace{\frac{1}{2} - \frac{\delta}{4} + \varepsilon}_{s^*}$

$$\Rightarrow \|u(t)\|_{H^s} < N^{1-s} \text{ for } |t| < T$$

$\leftarrow iu_t + u_{xx} - P_{N_1}(|u|^4 u) = 0$

$$N_1 = N^{(1-s)/(1-2s)}$$

$$N \gg T^{\frac{1}{4s-2+\sigma-4\varepsilon}} = T^{\frac{1}{4(s-s^*)}}$$

Sec. 6 (6.1)

$$\begin{cases} iu_t + u_{xx} - u|u|^4 = 0 \\ u(0) \in H^s(\mathbb{T}) \end{cases}$$

WTS: (6.1) is GWP on $[0,T]$

Truncated equation: $iw_t + w_{xx} - P_{N_1}(|w|^4 w) = 0$

$$H_1(\phi) = \sum_{|n|\leq N} n^2 |\hat{\phi}(n)|^2 + \sum_{|n_j|\leq N_1} \hat{\phi}(n_1) \dots \overline{\hat{\phi}(n_6)}$$

$\Rightarrow$ we obtained $\mathcal{H} = H_1 \circ \Gamma$

$$\mathcal{H}(I_q(0)) \sim H_1(\Gamma I_q(0)) \sim \|\Gamma I_q(0)\|^2_{H^1}$$

$$\cancel{H(I_q(0)) + N^{2(1-s)}} \underset{(\#)}{\lesssim} \|I_q(0)\|^2_{H^1} + N^{2(1-s)-2\varepsilon}$$

$$\lesssim N^{2(1-s)}$$

$$\Gamma I_q = I_q\Big|_{\tilde{t}=1} = I_q\Big|_{\tilde{t}=0} - i\int_0^1 \frac{\partial F}{\partial \bar{\phi}}(I_q)\, dt$$

time of the auxiliary Hamiltonian

$$\left\| \frac{\partial F}{\partial q_n} \right\|_{H^1} = \sum |n|^2 \, C(\bar{m})$$

$$\Downarrow$$

$$\sum_{n = n_1 - n_2 + \dots + n_{2r-1}} \frac{C(\bar{m})}{D(\bar{m})} I q_{n_1} \dots I q_{m_{2r-1}} \, p_n$$

$\cancel{}$ only in ℓ^2

stability

Can assume WLOG $|m_1| \gtrsim |m|$

$$\sim \sum \frac{C(\bar{m})}{D(\bar{m})} \cdot \underbrace{n_1 I q_{m_1}}_{N^{1-s}} \cdot \underline{I q_m} \dots I q_{m_{2r-1}} \cdot \underline{p_m}$$

only in L^2

$$\sim N^{1-s} \|F\| \sim N^{1-s-\varepsilon}$$

by def. 2 excep. terms $\qquad$ 3.14 proof of (*)

without switching Γ and I

$$|t| < T \qquad \left| \frac{d}{dt} H(I q(t)) \right| < N^{4-6s+4\varepsilon-\delta} < N^{2(1-s)} T^{-1}$$

$T \approx N^{0-}$ because $N = T^{\frac{1}{4(s-s_0)}}$ close to 0

$$< N^{2(1-s)-k}$$

(6.6) $\|\Gamma q\|_2 = \|q\|_2$

Let $G(q) = \|q\|^2_{\ell^2} = \sum |q_m|^2$

$$\frac{\partial G}{\partial t} = c\{F, G\} = 0$$

$\hookrightarrow F = F$

(6.7) $\|q - \Gamma q\|_{H^1} \lesssim N^{-k} \|q\|_{H^1}$

$$q - \Gamma q = c\int_0^1 \frac{\partial F}{\partial \bar{q}}\, dt$$

& take H^1 norm

because $|m| \lesssim |m_1|$

$$\|q - \Gamma q\| \leq \left\|\frac{\partial F}{\partial \bar{q}_m}\right\|_{H^1} \lesssim \sum \frac{C(\bar{m})}{D(\bar{m})} \frac{|m_1|\, q_{m_1} \bar{q}_{m_2} \cdots q_{m_{2r-1}}}{\|q\|_{H^1}} \, \underset{\text{only in } \ell^2}{\|q\|_{H^1}} \, \rho_m$$

$\|q\|_{H^1}$

$$\lesssim \|F\| \|q\|_{H^1} < N^{-\varepsilon} \|q\|_{H^1}$$

(6.8) $\|Iq - I\Gamma q\|_{H^1} \lesssim N^{-k} \|Iq\|_{H^1}$

same as before (see p. 18 in the notes)

use $\left\| I\frac{\partial F}{\partial \bar{q}_m}\right\| \lesssim \|Iq\|_{H^1} \|F\|$ below (3.14)

$$(6.9) \quad \| \Gamma I q - \Gamma I q' \|_{H^1} \;\; \| I \Gamma q - I \Gamma q' \|_{H^1}$$

$$\lesssim \| I q - I q' \|_{H^1} + N^{-\delta} \| q - q' \|_2$$

$$\Leftarrow \| \Gamma I (q - q') \|$$

Correction to (6.7)

$$\sup_{\tilde{t} \in [0,1]} \| q(\tilde{t}) \|_{H^1} \leq \tilde{t} N^{-\varepsilon} \sup_{t' \in [0, \tilde{t}]} \| q(t') \| + \| q(0) \|_{H^1}$$

$$\sup_{t \in [0,1]} \| q(t) \|_{H^1} < t N^{-\varepsilon} \sup_{t' \in [0,t]} \| q(t') \|_{H^1} + \| q(0) \|_{H^1}$$

$$\leq \sup_{t' \in [0,1]}$$

Choose N large s.t. $N^{-\varepsilon} < \frac{1}{2}$

$$\Rightarrow \sup_{t \in [0,1]} \| q(t) \|_{H^1} < 2 \| q(0) \|_{H^1}$$

(6.8)

$$\Leftarrow \| I \Gamma (q - q') \|_{H^1} \lesssim \| I (q - q') \|_{H^1} + N^{1-s} \| I (q - q') \|_{H^1}$$

$$\lesssim 2 \| I (q - q') \|_{H^1}$$

$$\Gamma I(q-q') - I(q-q') = c\int_0^1 \frac{\partial F}{\partial \bar q_n}\left(I(q-q')(\tilde t)\, d\tilde t \right)$$

$$\Rightarrow \left\| \frac{\partial F}{\partial q}\left(I(q-q') \right)\right\|_{H^1} \lesssim N^{1-s}\sum \frac{c(n)}{D(\bar n)} \frac{|n_1|}{N^{1-s}}\, I(q-q')_{n_1}\, I(q-q')_{n_2} \cdots I(q-q')_{n_{2r-1}}\ f_n$$

only in Γ

and $\dfrac{\| I(q-q')\|_2}{\| I(q-q')\|_2}$

$$\Rightarrow \left\| \frac{\partial F}{\partial q} I(q-q')\right\|_{L^2} \leq \|f\|\, N^{1-s}\, \| I(q-q')\|_2$$

not correct

$$\leq N^{1-s-\varepsilon}\, \| q-q'\|_2$$

$$\Gamma I q = I q + c\int_0^1 \frac{\partial F}{\partial \bar q}\left(I q(t)\right) dt$$

$$\Gamma I q' = I q' + c\int_0^1 \frac{\partial F}{\partial \bar q'}\left(I q'\right.$$

$$\left\| \frac{\partial F}{\partial \bar q}(I q) - \frac{\partial F}{\partial \bar q}(I q')\right\|_{H^1}$$

Friday 14.00 4-June-2010

(6.4) $\| \Gamma I q - \Gamma I q' \|_{H^1} + \| I \Gamma q - I \Gamma q' \|_{H^1} \leq$

$\leq N^{1-s} \| q - q' \|_{H^s}$

<u>Pf.</u> $\Gamma I q = I q(1) = I q(0) \mp C \int_0^1 \frac{\partial F}{\partial \bar{q}} (I q) dt$

$\Gamma I q' = I q'(1) = I q'(0) + \int_0^1 \frac{\partial F}{\partial \bar{q}'} (I q) dt$

(v)

$\cdot \| \Gamma I q - \Gamma I q' \|_{H^1} \lesssim \| I q - I q' \|_{H^1} + \sup_{t \in [0,1]} \Big\| \frac{\partial F}{\partial q} (I q(t)) - \frac{\partial F}{\partial \bar{q}} (I q'(t)) \Big\|_{H^1}$

$\| \frac{\partial F}{\partial \bar{q}} (I q) + \frac{\partial F}{\partial \bar{q}} (I q') \|_{H^1} \lesssim$

$\lesssim \sum \frac{C(\bar{m})}{D(\bar{m})} |m_1| \; I(q_{m_1} - q'_{m_1}) I q_{m_2} \cdots I q_{m_{2j-1}} \, \rho_m \cdot \| I q - I q' \|_{H^1}$

$+ \sum \frac{C(\bar{m})}{D(\bar{m})} |m_1| \; \overset{\shortparallel}{I}(q_{m_2} - \overset{\shortparallel}{q'_{m_2}}) \overset{H^1}{} \; I \bar{q}_{m_1} \cdots I q_{m_{2j-1}} \cdot \rho_m \quad \cancel{\| I q - I q' \|_{H^1}}$

$\times N^{1-s} \| q - q' \|_{H^s} \quad L^2 \quad N^{1-s} \; \cancel{\| I q' \|_{H^1}} \; \| q - q' \|_{H^s} \quad L^2$

$\lesssim \| F \| \; \| I q - I q' \|_{H^1} + N^{1-s} \| q - q' \|_{H^s} \cdot \| F \|$

Check $\dfrac{I(\bar{q}_{m_2} - \bar{q}'_{m_2})}{\| q - q' \|_{H^s}}$

(40)

Check the exceptions are just 2 in the section

$$\frac{\| I q_{m_1} - I q'_{m_2} \|_2}{\| q - q' \|_{H^s}} \leq \frac{\| q - q' \|_2}{\| q - q' \|_s} \lesssim 1 \quad \Rightarrow (3.4)$$

sloppy notation

$$(3.27) \quad \frac{\| I (I q_{m_1} - I q'_{m_2}) \|_{H^1}}{\| q - q' \|_{H^s}} \lesssim \frac{N^{1-s} \cancel{\| q - q' \|_{H^s}}}{\cancel{\| \quad \|}} = N^{1-s}$$

Consider $\| I \Gamma_q - I \Gamma_{q'} \|_{H^1}$

$$I \Gamma_q = I q(t) = I q(0) + C \int_0^t \frac{\partial F}{\partial \bar{q}}(q) \, dt$$

$$\left\| I \frac{\partial F}{\partial \bar{q}}(q) - I \frac{\partial F}{\partial \bar{q}}(q') \right\|_{H^1} \lesssim \sum \frac{C(\bar{m})}{D(\bar{m})}$$

$$m(n) n = \begin{cases} n & |n| \leq N \\ N^{1-s} |n|^s & |n| \geq N \end{cases} \quad \text{if } |n| \leq |n_1| \quad |m(n) n| \leq |m(n_1) n_1|$$

By assumption $N_1 \geq N$

$$m(n_1) n_1 (q_{n_1} - q'_{n_1}) \bar{q}_{n_2} \cdots p_n$$

as before multiply and divide by $\| I_\varepsilon - I_\varepsilon \|_{H^s}$ — same part

$$+ \sum \frac{C(\bar{n})}{D(\bar{n})} \frac{m(n_1) n_1 q_{n_1}}{N^{1-s}} \frac{(\bar{q}_{n_2} - \bar{q}'_{n_2})}{\| q - q' \|_{H^s}} q_{n_3} \cdots p_n \; ; \; N^{1-s} \| q - q' \|_{H^s}$$

$$\lesssim \text{RHS of } (6.9) \times N^{-\varepsilon}$$

low freq

let $k(\psi) = H_1 \circ \Gamma \circ I \circ \Gamma^{-1}\psi$

$$\Rightarrow \left| k(\omega(t)) - k(\omega(t'))\right| \leq N^{2(1-s)-k} |t-t'|$$

$$(6.5.1) \quad \left| \frac{d}{dt} \underbrace{\mathcal{H}}_{H_1 \circ \Gamma \circ I \circ \Gamma^{-1} \circ \Gamma q} \Big(I \underset{\psi}{q}(t) \Big) \right| < N^{2(1-s)-k}$$

and integrate in time

Here depends on $\|q\|_2$, $\|Iq\|_{H^1}$
$\wedge \quad \wedge$
$C \quad CN^{1-s}$

$$k(\psi) \sim \|I\psi\|^2_{H^1}$$

$$K(\psi) = H_1 \circ \Gamma \circ I \circ \Gamma^{-1}\psi$$

$$\sim \|\Gamma I \Gamma^{-1}\psi\|^2_{H^1} \overset{\Gamma \text{ preserves}}{\sim} \|I\Gamma^{-1}\psi\|^2_{H^1} \overset{(6.8)}{\sim} \|I\psi\|^2_{H^1}$$

$$\frac{1}{2}\|q\|_{H^1} \leq \|\Gamma q\|_{H^1} \overset{(6.7)}{\leq} 2\|q\|_{H^1} \leftarrow N \text{ large}$$

similar to (6.8)

$$(6.11) \quad |k(\psi) - k(\psi')| \lesssim N^{3(1-s)} \|\psi - \psi'\|_{H^s}$$

$\underline{H^1\text{-part:}}$ $\|\Gamma I \Gamma^{-1}\psi\|^2_{H^1} - \|\Gamma I \Gamma^{-1}\psi'\|^2_{H^1} \lesssim$

$$\leq \|\Gamma I \Gamma^{-1}\psi - \Gamma I \Gamma^{-1}\psi'\|_{H^1} \left(\| \quad \|_{H^1} + \| \quad \|_{H^1}\right)$$

$$\overset{6.9}{\lesssim} N^{1-s} \|\Gamma^{-1}\psi - \Gamma^{-1}\psi'\|_{H^s} \times N^{1-s}$$

$$\lesssim N^{2(1-s)} N^{s(1-s)} \| \psi - \psi' \|_{H^s}$$

(*)

At some computation as before:

$$\begin{cases} \Gamma_q = q(1) = q(0) + C \int_0^1 \frac{\partial F}{\partial q}(t)\, dt \\ \Gamma_{q'} = q(1) = \dots \end{cases}$$

$\Downarrow$

$$\left\| \frac{\partial F}{\partial q}(q) - \frac{\partial F}{\partial q}(q') \right\|_{H^s} \lesssim \sum \frac{C(\bar{m})}{D(\bar{m})} |m_1|^s (q_{m_1} - q'_{m_1}) \bar{q}_{m_2} \dots p_n \quad \text{1st}$$

$$+ \sum \frac{C(\bar{m})}{D(\bar{m})} |m_1|^s q_{m_1} (q_{m_2} - q'_{m_2}) q_{m_3} \dots p_n \quad \text{2nd}$$

1st: $\dfrac{\| q - q' \|_{H^s}}{\| q - q' \|_{H^s}}$

2nd $\quad \dfrac{|m_1|^s \, 1 \quad q_{m_1}}{\underset{\text{1st element}}{N^{s(1-s)}} \; \underset{\text{one of the other elements}}{\| q - q' \|_{H^s}}} \cdot N^{s(1-s)} \| q - q' \|_{H^s}$

$$(\quad) \lesssim \left(1 + N^{s(1-s)} \right) \| q - q' \|_{H^s} \cdot N^{-\varepsilon}$$

We have this 2 conditions $\qquad$ For $|n| \geq N_1 \quad w(n) \cdot n = N^{1-s} |n|^s$

$\| * \|_2 \lesssim 1 \qquad \qquad \| P_{\leq N} q \|_{H^s} \lesssim 1$

$\| I q \|^2_{H^1} \lesssim N^{1-s}$

For $|n| \lesssim N$, $\|P_N q\|_{H^1} \lesssim N^{1-s}$ (3.2')

$\Rightarrow$ Interpolate $\|q\|_{H^s} \lesssim (N^{1-s})^{\theta}\, 1^{1-\theta}$ $s = 1-\theta = 0-(1-s)$

$$= N^{s(1-s)}$$

- L^6-part

$$\|\Gamma I \Gamma^{-1}\psi\|_{L^6}^6 - \|\Gamma I \Gamma^{-1}\psi'\|_{L^6}^6 =$$

factorize

$$\lesssim \|\Gamma I \Gamma^{-1}\psi - \Gamma I \Gamma^{-1}\psi'\|_{L^6} \left(\| \quad \|_{L^6}^5\right) \quad \frac{1}{2}-\frac{1}{6}=\frac{1}{3}$$

$$\lesssim \|I\Gamma^{-1}\psi - I\Gamma^{-1}\psi\|_{H^{1/3}} \lesssim$$

$\|q\|_{L^6} \lesssim \|q\|_{H^{1/3}}$

Sobolev

$$\left[\text{Use } \|\Gamma^{-1}\psi - \Gamma^{-1}\psi'\|_{H^\theta} \quad N^{\theta(1-s)}\|\psi - \psi'\|_{H^\theta}\right]$$

$$\frac{1}{p} - \frac{1}{q} = \frac{s}{n}$$

$$\|\psi\|_{L^q} \lesssim \|\psi\|_{\dot W^{s,p}}$$

$$\frac{1}{p} - \frac{1}{q} = \frac{s}{n}$$

$$\lesssim \|\Gamma^{-1}\psi - \Gamma^{-1}\psi'\|_{H^{1/3}} \cdot N^{2(1-s)}$$

$$\lesssim \|\psi - \psi'\|_{H^{1/3}}\, N^{\frac{7}{3}(1-s)}$$

$$s > \frac{1}{3}$$

Let $\phi = \phi_0 + \phi_1$ $\quad \|\phi\|_2 < C$; $\|I\phi_0\|_{H^1} < C N^{1-s}$

 $\mathbb{P}_{N_1}\phi$ $\quad \|\phi_1\|_{H^s} < ?$ $\quad ? > 1$

w sol. of the low. freq. eq. $\begin{cases} i w_t + w_{xx} - \mathbb{P}_{N_1}(|w|^4 w) = 0 \\ w|_{t=0} \qquad t \in [0, \tau] \end{cases}$

$\Rightarrow \|w(t)\|_2 = \|\phi_0\|_2$

$$|K(w(t)) - K(\phi_0)| \overset{(6.10)}{<} N^{2(1-s)-n}\, t, \quad |t| < \tau$$

$v = u - w$ $\quad$ diff. eq. $\quad$ proj to high freq

$$i v_t + v_{xx} \underbrace{- 3|w|^4 v - 2|w|^2 w^2 \bar{v} + O(|v|^2) - \mathbb{P}^C_{N_1}(|w|^4 w)}_{\substack{-|u|^4 + \mathbb{P}_{N_1}(|w|^4 w) \\ = -(w+v)(\bar{w}+\bar{v})(\bar{w}+\bar{v})(w+v) + \\ |w|^4 w - \mathbb{P}^C_{N_1}(|w|^4 w)}} = 0$$

$P^C_{N_1} = Id - P_{N_1}$ $\quad \phi_1 = P^C_{N_1}\phi$

$\xi(t) = \int_{\mathbb{T}} |w|^4(t)\, dx$ $\quad \Omega(t) = e^{-3i\int_0^t \xi(t')\,dt'}$ Removing effect Ampl ons ($\Rightarrow$ 0 norm)

$A = |w|^4 - \xi$ $\quad \Leftarrow$ norm 0 . $\quad \hat{A}(0)(t)$ space Fourier trans

$A(n,t)$

$B = |w|^2 w^2 \bar{\Omega}^2(t)$ $\quad V := \Omega v$ $\quad |V| = |v|$

Let $V = \bar{\Omega} v$

$$\begin{cases} iV_t + V_{xx} - 3AV - B\bar{V} + O(|v|^2) - \bar{\Omega}\, \mathbb{P}_{N_1}^{c}(|w|^4 w) = 0 \\ V(0) = \phi_1 \end{cases}$$

$$iV_t = -3\xi(t)\bar{\Omega} v + i\bar{\Omega} v_t =$$

$$= -3\xi \bar{\Omega} v - \bar{\Omega} v_{xx} + 3\bar{\Omega}|w|^4 v +$$

$$+ 2\bar{\Omega}|w|^2 w^2 \bar{v} + O(|v|^2) + \bar{\Omega}\, \mathbb{P}_{N_1}^{c}(|w|^4 w)$$

$$V_{xx} = \bar{\Omega} v_{xx}$$

$$-3AV = -3\left(|w|^4 - \xi\right)\bar{\Omega} v$$

$$-2B\bar{V} = -2|w|^2 w^2 \underbrace{\Omega^2 \Omega}_{\bar{\Omega}} V$$

$$\|f\|_{\sigma, b} = \left(\sum_m \int \langle m\rangle^{2\sigma} \langle \lambda - m^2\rangle^{2b} |\hat{f}(m,\lambda)|^2 d\lambda\right)^{\frac{1}{2}}$$

f on $\mathbb{T}_x \times I_t$

$$\|f\|_{X^{\sigma,b}(I)} = \|f \cdot \chi_I\|_{X^{\sigma,b}(\mathbb{R})}$$

Fix $0 < \sigma_0 < \frac{1}{2}$ $\quad \frac{1-\sigma_0}{2} < b < \frac{1}{2}$

for $b < \frac{1}{2}$ norm doesn't see jumps

$b \geq \frac{1}{2}$ $\forall$ fund is cont

(a3)

$$\|f\|_{X^{\sigma,b}(I)} \quad \cancel{\|\ldots\|}$$

$$= \inf_g \left\{ \|g\|_{X^{\sigma,b}(\mathbb{R})} : g|_I = f \right\}$$

L^6-Strichartz

$$\| P_{\leq N} S(t)\phi \|_{L^6} \lesssim N^{\varepsilon} \|\phi\|_{L^2} \Rightarrow \|S(t)\phi\|_{L^6} \lesssim \|\phi\|_{H^{\varepsilon}}$$

[Ref. sharp cut-off with the I-method ⇒ derivative NLS, one about it]
partition of unity decreasing in time axis

$$\Rightarrow \|u\|_{L^6_{x,t}} \leq \|u\|_{X^{\varepsilon,\frac{1}{2}+}}$$ Transference principle

$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$ Hölder

$$\|u\|_{L^6_{x,t}} \lesssim \|\hat{u}\|_{L^{6/5}_{x,T}} \lesssim \left\| \underbrace{\|\langle \tau - n^2\rangle^{-\frac{1}{3}-}\|_{L^3_\tau}}_{< C} \|\langle \tau - n^2\rangle^{\frac{1}{3}+}\hat{u}\|_{L^2_\tau} \right\|_{\ell^{6/5}_n}$$

← Hausdorff Young

$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$ Hölder $\cancel{\ldots}\, N^{1/3}$

$$\lesssim N^{1/3} \wedge \|u\|_{X^{1/3,\frac{1}{3}+}}$$

and interpolate this two.

$\sigma_0 = \theta \cdot \frac{1}{3} + (1-\theta)\cdot 0 \quad b = \theta \cdot \frac{1}{3} + (1-\theta)\cdot \frac{1}{2}$

$$\Rightarrow \|u\|_{L^6_{x,t}} \leq \|u\|_{\sigma_0,b} \qquad \Rightarrow b > \sigma_0 + \frac{1-3\sigma_0}{2} = \frac{1-\sigma_0}{2}$$

Can take this $b < \frac{1}{2}$

$$(6.25) \quad \left\| \int_0^t S(t-\tau)(F_1 \bar{F}_2 F_3 \bar{F}_4 F_5)(\tau)\, d\tau \right\|_{\sigma, 1-b}$$

$$\lesssim \max_{j_1, j_2} \left(\|F_{j_1}\|_{\sigma, b} \|F_{j_2}\|_{4\sigma_0, b} \prod_{j \neq j_1, j_2} \|F_j\|_{\sigma_0, b} \right)$$

$\sigma_0 \to 4\sigma_0$ increases

$$\lesssim \max_j \|F_j\|_{\sigma, b} \prod_{j' \neq j} \|F_{j'}\|_{4\sigma_0, b}$$

early point

$$\text{LHS of } (6.25) \lesssim \| F_1 \dots F_5 \|_{\sigma, -b}$$

$$\sim \int F_1 \dots F_5 |\partial_x|^{\sigma+\sigma_0} v \, dx\, dt$$

v in $X^{0,b}$

$(X^{s,b})^* = X^{-s,-b}$ assume $|n_1| \geq |n|$

$n = n_1 - n_2 + \dots + n_5$

$n \to v$, $n_1 \to F_1$

L^6-Hölder L^6-Strichartz

$$\sim \int (|\partial_x|^{\sigma+\sigma_0} F_1) \bar{F}_2 \dots F_5 v \, dt \lesssim \|F_1\|_{X^{\sigma+2\sigma_0, b}} \prod_2^5 \|F_j\|_{X^{\sigma_0, b}} \|v\|_{X^{\sigma_0, b}}$$

N_1 too large: no place to distribute the derivatives

BAD

proof for cubic NLS

No derivative loss in the cubic case so this computation gives LWP directly, but here we have to modify the argument

Assume ~~[illegible]~~ $n_1 = n_1^*$

$D_{n_1^*} = \bigcup_\alpha Q_\alpha \quad |Q_\alpha| = |D_{n_2^*}| \longrightarrow$ distribute remaining derivatives

$$2^{k_1 \sigma} \int \sum_{n=n_1-n_2 \cdots + n_5} \hat{F}_1(n_1, \tau_1) \cdots \hat{F}_5(n_5, \tau_5) \frac{c_{n,\tau}}{\langle \tau - n^2 \rangle^b} \, d\tau =$$

$(n_1^*)^\sigma$

$$= 2^{k_1 \sigma} \sum_\alpha \int \sum_n \hat{F}_{1,\alpha}(n_1, \tau_1) \hat{F}_2(n_2, \tau_2) \cdots \hat{F}_5(n_5, \tau_5) \frac{c_{n,\tau,\alpha}}{\langle \tau - n^2 \rangle^b} \, d\tau$$

where $\hat{F}_{1,\alpha} = \hat{F}_1 \big|_{Q_\alpha} \quad c_{n,\tau,\alpha} = c_{n,\tau} \big|_{Q_\alpha}$

$\|u\|_{L^6} \lesssim \|u\|_{X^{s_0,b}} \longleftarrow \|u\|_{L^6} \lesssim \|u\|_{X^{0+,\frac{1}{2}+}}$

$\|u\|_{L^6} \lesssim \underline{N^{1/3}} \|u\|_{X^{0,\frac{1}{3}+}}$

\# of freq in interval

$\|F_{1,\alpha}\|_{L^6} \lesssim (n_2^*)^{1/3} \|F_{1,\alpha}\|_{X^{0,\frac{1}{3}+}}$

<u>Must show</u> $\quad \|F_{1,\alpha}\|_{L^6} \lesssim (n_1^*)^{0+} \|u\|_{X^{0,\frac{1}{2}+}}$

We proved $\|u\|_{L^6_{x,t}} \lesssim (N_3^*)^\varepsilon \|u\|_{X^{0,\frac{1}{2}+}}$

$$\Updownarrow$$

$$\|S(t)\phi\|_{L^6_{x,t}} \lesssim (N_3^*)^\varepsilon \|\phi\|_{L^2}$$

$$= (N_1^*)^\sigma \sum_\alpha \|F_{1,\alpha}\|_{L^6_{x,t}} \prod_2^5 \|F_j\|_{L^6} \left\| \left(\frac{C_{m,\tau,\alpha}}{\langle \tau - m^2 \rangle^b}\right)^\vee \right\|_{L^6_{x,t}}$$

$$\lesssim (N_1^*) \prod_2^5 \|F_j\|_{X^{\sigma_0,b}} \sum_\alpha (N_1^*)^{3\sigma_0} \|F_{1,\alpha}\|_{X^{0,b}} \cdot \|g_\alpha\|_{X^{0,b}} \lesssim (N_1^*)^\sigma \prod_{3}^{5} \|F_j\|_{X^{\sigma_0,b}} \|F_2\|_{X^{3\sigma_0,b}} \cdot$$

Assume $|N_2| = |N_1^*|$

C.S.

$$\cdot \underbrace{\left(\sum_\alpha \|F_{1,\alpha}\|^2_{0,b}\right)^{1/2}}_{X^{0,b}} \left(\sum_\alpha \|g_\alpha\|^2_{X^{0,b}}\right)^{1/2}$$

$$\|F_1\|_{\sigma,b}$$

$$\lesssim \|F_1\|_{\sigma,b} \|F_2\|_{X^{3\sigma_0,b}} \prod_3^5 \|F_j\|_{\sigma_0,b}$$

$$\|g\|_{0,b} = \left(\int \sum_m \langle \tau - m^2 \rangle^{2b} |C_{m,\tau}|^2 / \langle \tau - m^2 \rangle^{2b} \, d\tau\right)^{1/2} =$$

$$= \|C_{m,\tau}\|_{L^2_{m,\tau}} = 1$$

Choose I small s.t.

$$\|S(t)\phi_0\|_{4\sigma_0,b} = o(1) \qquad \sigma = s_1 < s < \tfrac{1}{2}$$

$$\sigma_0 = 10^{-3}$$

$$\|S(t)\phi_0 \chi_I\|_{4\sigma_0,b} \lesssim \|\phi_0\|_{H^{4\sigma_0}} \|\chi_I\|_{H^b_t} \lesssim N^{4\sigma_0/(1-s)} |I|^{\overbrace{1/2-b}^{>0}}$$

$$\widehat{S(t)\phi}(n,\tau) = \hat{\phi}(n)\delta(\tau - n^2)$$

$$\|S(t)\phi\,\chi_I\|^2_{s,b} = \sum_n \int \left| \left(\widehat{S(t)\phi}(n,\cdot) * \hat{\chi}_I \right)(\tau) \right|^2 \langle \tau - n^2 \rangle^{2b} =$$

$$= \int_{\tau = \tau_1 + \tau_2} \widehat{S(t)\phi}(n,\tau_1)\, \hat{\chi}_I(\tau_2)\, d\tau_1 \qquad \langle \tau - n^2\rangle^{2b} \lesssim \langle \tau_1 - n^2 \rangle^{2b} + \langle \tau_2 \rangle^{2b}$$

$$\sim \left\| \left\| \int \hat{\phi}(n) \langle \tau_1 - n^2 \rangle^b \delta(\tau_1 - n^2) * \hat{\chi}_I(\tau_2)\, d\tau_2 \right\|_{L^2_\tau} \right\|_{\ell^2_n}$$

$$\le \left\| \hat{\phi}(n) \langle \tau_1 - n^2 \rangle^b \delta(\tau_1 - n^2) \right\|_{\ell^2_n L^1_t} + \|\hat{\chi}_I\|_{L^2_t}$$

Young $\frac{1}{2} + 1 = \frac{1}{2} + 1$

$$\sim \|\phi\|_{L^2} + \|\chi_I\|_{L^2}$$

2nd term $\sim \left\| \int \hat{\phi}(n)\delta(\tau_1 - n_1^2) \langle \tau_2 \rangle^b \hat{\chi}_I(\tau_2)\, d\tau_2 \right\|_{\ell^2_n L^2_t}$

$$\le \left\| \|\hat{\phi}(n)\delta(\tau_1 - n^2)\|_{L^1_{\tau_1}} \|\langle \tau_2 \rangle^b \hat{\chi}_I(\tau_2)\|_{L} \right\|_{\ell^2_n}$$

$$\sim \|\phi\|_{L^2} \|\chi_I\|_{H^b}$$

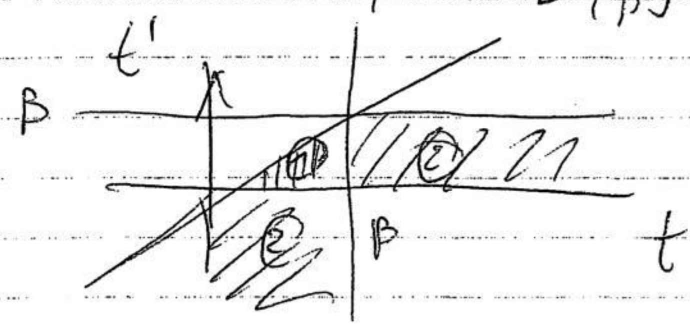
$$\|\chi_I\|_{H^b} = \left(\int_I \int_I \frac{\chi_I(t) - \chi_I(t')}{|t-t'|^{1+2b}} \, dt\,dt' \right)^{1/2} + \|\chi_I\|_{L^2}$$

Assume $t > t'$

$\sim |I|^{1/2}$

$<\!< |I|^{1/2-b}$

Assume $t > t'$ say $I = [\alpha, \beta]$ I small



otherwise dunrect. Function is 0

but on ① $\chi_I(t) = \chi_I(t') = 1$

② $t' \in I$, $t > \beta$

$$\left(\int_I \int_\beta^{+\infty} \frac{1}{|t-t'|^{1+2b}} \, dt\,dt' \right)^{1/2} \sim \left(\int_\alpha^\beta \frac{1}{|t'-\beta|^{2b}} \, dt' \right)^{1/2}$$

Since $b < \frac{1}{2}$

$$\sim (|\alpha - \beta|)^{1/2 - b}$$

$$\|(S(t)\phi_0)\chi_I\|_{4\sigma_0, b} \lesssim N^{4\sigma_0(1-s)} |I|^{1/2-b} \lesssim 1$$

by choosing $\tau = |I| < N^{-10(1-s)}$

prove of N

$$4\sigma_0(1-s) - 5(1-s) + 10b(1-s) =$$

$$= (1-s)\left[4\sigma_0 - 5 + 5 - 5\sigma_0 +\right) = -(\sigma_0+)(1-s)\sigma_0$$

$$b = 1 - \frac{\sigma_0}{2} +$$

u sol of NLS

Thm $\|u\|_{4\sigma_0, b} = o(1)$

$u_0 \in HS$

b/c $u = S(t)f_0 + N(u)$

Take $\|\cdot\|_{4\sigma_0,b} \Rightarrow \|u\|_{4\sigma_0,b} \leq \|S(t)f_0\|_{4\sigma_0,b} + \|u\|_{\sigma,b}\|u\|^4_{4\sigma_0,b}$

$\|S(t)f_0\|_{4\sigma_0,b} = o(1)$, $\|u\|_{\sigma,b} \leq R$

Note: $\|\cdot\|_{X^{4\sigma_0,b}([0,t])}$ is continuous in time and $= 0$ at $t=0$

$\Rightarrow \|u\|_{X^{0,b}[0,0]} = 0 = o(1)$

by continuity $\|u\|_{X^{0,b}([0,t])} = o(1)$

$X \leq o(1) + cX^4$ (for $\|u\|_{X^{4\sigma_0,b}[0,t]} \ll 1$)

Also by LWP $\|u\|_{\sigma, \frac{1}{2}+} \leq 2\|u(0)\|_{H^\sigma}$

For low freq.

$$\|w\|_{s,\frac{1}{2}+} \leq 2 \|\phi_0\|_{H^s} \lesssim N^{\delta(1-s)}$$

$\leftarrow$ (3.1) (3.2) & interpolation $\forall$

(independent of cutoff N_1)

Choose τ small s.t. $\|w\|_{s,b'} < N^{-1}$ $b' = \frac{1+2b}{4} \in \left(b, \frac{1}{2}\right)$

$\because$ $\|w\|_{s,b'} \lesssim N^{\delta(1-s)} \tau^{\frac{1}{2}-b'} < N^{-1}$ for $t \sim N^{-k}$ large

$\log \frac{1}{\tau} \sim \log N$

$$V(t) = S(t)\phi_1 - 3 \int_0^t S(t-\tau)\, A\, V(\tau)\, d\tau \quad (6.32)$$

$$\underbrace{- 2 \int_0^t S(t-\tau)\, B\, \bar{V}(\tau)\, d\tau}_{(6.33)} + \underbrace{\int_0^t S(t-\tau)\, O(|V|^2)\, d\tau}_{6.34}$$

$$\underbrace{- \int_0^t S(t-\tau)\, \bar{\tau}\, \mathbb{P}^c_{N_1}\left(|w|^4 w\right)(\tau)\, d\tau}_{(6.35}$$

$n^{s_0} n^{s_1 - s} = n^{s_1}$

$\land\!\!\mid$

$n^s N_1^{s_1-s}$

$n \geq N_1$

Assume $\|\phi_1\|_{H^s} < \mathfrak{z} < N_1^{\overbrace{s - s_1/10}^{>0}}$

$$\Rightarrow \|\phi_1\|_{H^{s_1}} \leq N_1^{s_1 - s} \|\phi\|_{H^s} < N_1^{\frac{s_1-s}{2}}$$

$$\Rightarrow \|S(t)\phi_1\|_{s_1, \frac{1}{2}+} \lesssim \|\phi_1\|_{H^{s_1}} < N_1^{(s_1+s)/2}$$

$$\| (6.32) \sim (6.34) \|_{S_1, \frac{1}{2}+} \leq \| \ldots \|_{S_1, 1-b}^{>\frac{1}{2}} \lesssim$$

$$\lesssim \|V\|_{S_1,b} \|w\|^4_{4\sigma_0,b} + \|V\|_{4\sigma_0,b} \|w\|_{S_1,b} \|w\|^3_{4\sigma_0,b}$$ $\left. \begin{matrix} AV \\ B\bar{V} \end{matrix} \right.$

use the fact

$$\| \Omega^2 V \|_{L^6_{x,t}} \leq \| \underset{1}{\underset{\shortparallel}{\Omega}} \|^2_{L^\infty} \|V\|_{L^6_{x,t}}$$

$$+ \|V\|_{S_1,b} \|V\|_{4\sigma_0,b} \left(\|w\|_{4\sigma_0,b} + \|V\|_{4\sigma_0,b} \right)^3$$ $O(|V|^2)$

$$+ \|V\|^2_{4\sigma_0,b} \|w\|_{S_1,b} \left(\quad " \quad \right)^2$$

In summary $\lesssim \|V\|_{S_1,b} \left(\|w\|_{S,b'} + \|V\|_{4\sigma_0,b} \right)^4$

$$\lesssim \|V\|_{S_1,\frac{1}{2}+} \left(\underset{\uparrow (6.30)}{N^{-1}} + \|V\|_{S_1,\frac{1}{2}+} \right)^4$$

$$\|(6.35)\|_{S_1,\frac{1}{2}+} \lesssim \| \mathbb{P}^c_{\frac{1}{5}N_1} w \|_{S_1,b} \|w\|^4_{S_1,b} < N^{\overbrace{S_1 - S}^{<0}}$$

$\mathbb{P}^c_{N_1} (|w|^4 w) \Leftarrow$ at least one of the freq $\geq \frac{1}{5} N_1$

$$\sim \mathbb{P}^c_{N_1} \left((\mathbb{P}_{\frac{1}{5}N_1} w) \, \overline{w_1}^2 w^2 \right) + \mathbb{P}^c_{N_1} \left(\mathbb{P}^c_{\frac{1}{5}N_1} \bar{w} \, w^3 \bar{w} \right)$$

~~Hence~~,

$$\|P_{N_1}^c w\|_{s_1,b} \lesssim N_1^{s_1-s}\|w\| \underset{(6.30)}{<} N^{-1} N_1^{s_1-s}$$

$$\sim N_1^{s_1-s}$$ N

Hence,

$$\|v\|_{s_1,\frac{1}{2}+} \lesssim N_1^{\frac{s_1-s}{2}} + \left(N^{-1}+\|v\|_{s_1,\frac{1}{2}+}\right)^4 \|v\|_{s_1,\frac{1}{2}+}$$

(6.35) forcing term even smaller

$$\Rightarrow \|v\|_{s_1,\frac{1}{2}+} \lesssim N_1^{\frac{s_1-s}{2}}$$ again by continuity argument

<u>Improved (6.32)</u>

$$Av = (|w|^4 - \xi)v$$

$$\|(6.32)\|_{s_1,\frac{1}{2}+} < N_1^{(s_1-s)/2}$$

<u>Improved (6.33)</u>

$$B\bar{v} = |w|^2 w^2 \bar{v}$$

$$\|(6.33)\|_{s_1,\frac{1}{2}+} < N \cdot N_1^{(s_1-s)/2}$$

Also, $$\|(6.34)\|_{s,\frac{1}{2}+} \lesssim C\left(\|v\|_{s,b}+\|w\|_{s,b}\right)\|v\|_{4\sigma_0}\left(\|v\|_{4\sigma_0,b}+\|w\|_{4\sigma_0,b}\right)^3$$

$$\le C\Big(\|v\|_{s,b}+\underbrace{\|w\|_{s,b}}_{(6.37)\,<N^{-1}}\Big)\underbrace{\Big(\|v\|_{4\delta,b}+\|w\|_{4\delta,b}\Big)^3}_{\lesssim 1}\underbrace{\|v\|_{4\delta}}_{(6.44)\,<N_1^{\frac{s_1-s}{2}}}$$

$\uparrow\ (\le N_1^{s_1-s}+N^{-1})\le N^{-\frac{3}{8}}\sim N_1^{0-}$

$$< N_1^{(s_1-s)/2}\Big(\|v\|_{s,b}+N^{-1}\Big)$$

$$\|(6.35)\|_{s,\frac{1}{2}+}\lesssim \|w\|_{s,b}^5 < N^{-5}$$

Hence, $\|v\|_{s,\frac{1}{2}+}\lesssim \underbrace{\|\phi_1\|_{H^s}}_{<Z}+\underbrace{N\cdot N_1^{\frac{s_1-s}{2}}}_{\text{small}}+\underbrace{N_1^{\frac{s_1-s}{2}}\|v\|_{s,\frac{1}{2}+}}_{\text{small}}+\underbrace{N^{-5}}_{\text{small}}$

$$\Rightarrow \|v\|_{s,\frac{1}{2}+}\lesssim Z \qquad (6.50)$$

Substitute this in (6.48)

$$\|(6.34)\|_{s,\frac{1}{2}+}\lesssim Z\cdot N_1^{\frac{s_1-s}{2}} < N_1^{\frac{s_1-s}{4}} \quad \left(\because Z< N_1^{\frac{s-s_1}{10}}\right)$$

(6.52) <u>Low</u> $\|P_{N_1}v\|_{s,\frac{1}{2}+} < N_1^{\frac{s_1-s}{4}}$

the linear part in (6.35) came of the high freq projection $\mathbb{P}^c_{N_1}$

(6.53) <u>High</u>: $\|\mathbb{P}^c_{N_1}(v-S(t)\phi_1)\| < N^{-5}$

No linear part

Recall: $\sup \|f(t)\|_{H^s} \lesssim \|f\|_{X^{s,1/2+}}$

$$\Rightarrow \|P_{N_1} V(t)\|_{H^s} \lesssim N^{\frac{s_1 - s}{4}}$$

$$\| P_{N_1}^c (V - S(t)\phi_1)\|_{H^s} \lesssim N^{-s} \quad |t| < t$$

$$u(t) = w(t) + \Omega(t) V(t) = \left[w(t) + \Omega(t) P_{N_1} V(t) \right] +$$

$$+ \Omega(t) (P_{N_1}^c V)(t)$$

Write $u(\tau) = \underset{\text{low}}{\psi_0} + \underset{\text{high}}{\psi_1}$ ← keeping the nonlinear part $\neq$ high-low freq. etc.!

$$P_{N_1} \psi_0 = \psi_0 \qquad P_{N_1}^c \psi_1 = \psi_1$$
$$\psi_0 \overset{\shortparallel}{\ne} P_{N_1}^c u(\tau) \qquad \psi_1 = P_{N_1}^c u(\tau)$$

all the nonlinear part of the high freq. in the low freq. at the next step

$$\|\psi_0 - w(\tau)\|_{H^s} = \| (\Omega(t)) P_{N_1} V(t) \|_{H^s} \lesssim N^{(s_1 - s)/4}$$

$$\|\psi_1 - e^{i\tau\Delta}\phi_1 - \Omega(\tau)\|_{H^s} \lesssim N^{-s}$$

$\psi_1 = \Omega(t) P_N^c V_1(\tau)$ by 6.55

$$\|I\psi_0 - I w(\tau)\|_{H^1} < N^{1-s} N_1^{\frac{s_1-s}{4}}$$

$$|k(\psi_0) - k w(\tau)| < N^{3(1-s)} \|\psi_0 - w(\tau)\|_{H^s} <$$
$$< N^{3(1-s)} N_1^{(s_1-s)/4} < N^{(s_1-s)/5}$$

By triangular inequality:

$$|k(\psi_0) - k(\phi_0)| \lesssim |k(\psi_0) - k(w(\tau))| +$$
$$+ |k(w(\tau)) - k(\phi_0)| \underset{(6.14)}{<} N^{2(1-s)-k}\tau$$
$$< N^{2(1-s)-k}\tau + N_1^{(s_1-s)/5}$$

Iterate by replacing $(\phi_0, \phi_1) = (P_N u(0), P^c_{N_1} u(0))$ by (ψ_0, ψ_1) and $z = \|P^c_{N_1} u(0)\|_{H^s}$ by

$z+1$

Digression: high-low freq. decomposition

$$\phi_0, \phi_1 \xrightarrow{\tau} \psi_0 = w(t) + \underbrace{(v - S(\tau)\phi_1)}_{\text{nonlin. part of high freq.}}$$

$\psi_1 = S(\tau)\phi_1 \Leftarrow$ on $\geq N_1$

supp on all freq. ~~but~~ very SMALL!

Very important the L^2- conservation in the high-low - frequency

nonlinear smoothing ...

Low freq "preserves" L^2 (even under rescal form) $\Leftarrow (3.1)$

$K(\phi_0) \sim \| I \phi_0 \|^2_{H^1} < N^{2(1-s)} \Leftarrow (3.2)$

$\Rightarrow$ Repeat argument until $K(\phi_0) \lesssim N^{2(1-s)}$

High freq:

repeat until $t < N_1^{(s_1-s)/10}$ i.e. until identity

$t+1 \Leftarrow S(t)\phi_1 + \underbrace{\mathbb{P}^c_{N_1} v(\tau) - S(t)\phi_1}_{< N^{-5}}$

· # of steps $= T/\tau$ for $[0,T]$

Low: $N^{2(1-s)-k} T + N_1^{\frac{s_1-s}{5}} \frac{T}{t} < N^{2(1-s)}$

① $\quad \frac{T}{t}$ ✗ $(6.58) \times \frac{T}{t}$

High: $\frac{T}{\tau} < N_1^{(s_1-s)/10} \quad \left(\text{or } \frac{T}{t} N^{-5} < N_1^{\frac{s_1-s}{10}} \right)$

②

By before (6.30)

$t = N^{-c} \Leftarrow$ ~~pred~~ by

$(1) \Rightarrow$

By def. $N_1 = N^{\frac{1-s}{1-2s}}$ (4)

$$\vee$$

$$\left(N^c T\right)^{10/s_1-s} \Leftarrow \text{by } (2) \text{ and } t = N^{-c}$$

Note: (1) $N_1^{\frac{s_1-s}{5}} \frac{T}{t} < N_1^{\frac{s_1-s}{5}} N_1^{\frac{s-s_1}{10}} = N_1^{\frac{s_1-s}{10}}$

$\ll 1$

$\ll N^{2(1-s)}$ i.e. 2nd term in (1) is vacuous

Given T choose $N > N(T)$ as in Lemma 5.23 (Section 5)

so that it satisfies (3) $T^{1/k} < N$

(4) $(N^c T)^{\frac{10}{s_1-s}} < N^{\frac{1-s}{1-2s}}$

$N > N(T) = T^{\frac{1}{4(s-s_*)}}$ large as in Section 5

$N \sim T^{\max\left(\frac{1}{4(s-s_*)}, \frac{1}{k}\right)} \leq N$

RHS of (4) $\left(N^c \cdot N^{\frac{1}{2}}\right)^{\frac{10}{s-s_1}} < N^{\frac{1-s}{1-2s}}$ for fixed $s < \frac{1}{2}$

by choosing s_1 close to s

Obtain solu for $|t| \le T$ s.t. $\| P_{N_1} u(t) \|_{H^s} \lesssim$

$$\left[\begin{array}{ll} \| If \|_{H^1} \le N^{1-s} \| f \|_s & \\ \ \ \rotatebox{90}{$\le$} & m(n)\cdot n = \begin{cases} n & n \le N \\ N^{1-s} & n \ge N \end{cases} \\ \| f \|_s & \end{array} \right]$$

$\le \| P_{N_1} I u(t) \|_{H^1} \lesssim N^{1-s}$

and $\| P^c_{N_1} u(t) \|_{H^s} < N_1^{(s-s_1)/10}$

$\Rightarrow \| u(t) \|_{H^s} < T^{C(s)} \quad \left(\sim N_1^{(s-s_1)/10} \right)$

SUMMARY of the THEOREM

- high - low freq. because otherwise no normal found
- Normal form just for the low frequency part [illegible]
- I method on low freq. part
- High freq. part just do local wellposedness
- ...

$$\begin{tikzcd} \text{Low} \arrow[d] & & \text{High} \arrow[d] \\ \text{N.F.} \arrow[d] & & \text{careful LWP} \arrow[dl] \arrow[d] \\ \text{I-method} \arrow[d] & & \text{high } 2 \arrow[uu, bend right] \\ \text{GWP with control} \lesssim N^{1-s} & \text{low } \tau N^{1-s} \arrow[l] & \end{tikzcd}$$

1. On (4.11) in Lemma 4.1

Define M by

$$M := \int_0^{\frac{2\pi}{D}} \int_{\mathbb{T}} \prod_{j=1}^{3} |e^{it\Delta}\phi_j|^2 dx dt \tag{1}$$

$$= \sum_{\substack{m_j, n_j \\ \sum m_j = \sum n_j}} \int_0^{\frac{2\pi}{D}} e^{iF(m,n)t} dt \prod_{j=1}^{3} \widehat{\phi_j}(m_j)\overline{\widehat{\phi_j}(n_j)},$$

where $F(m,n) = -m_1^2 - m_2^2 - m_3^2 + n_1^2 + n_2^2 + n_3^2$. Also, define A_k by

$$A_k = \big\{(m_j, n_j) : |F(m,n)| \sim [2^{k-1}D, 2^k D]\big\}, \quad k \geq 1,$$

and $A_0 = \big\{(m_j, n_j) : |F(m,n)| \sim [0, D]\big\}$. Then, we have $M = \sum_k M_k$, where M_k is the contribution of the multilinear expression over A_k. By a direct computation, we have

$$\int_0^{\frac{2\pi}{D}} e^{iF(m,n)t}\big|_{A_k} dt \lesssim 2^{-k}D^{-1}. \tag{2}$$

(For k = 0, use Mean Value Theorem, and for $k \geq 1$, use $|e^{2\pi i F(m,n)/D} - 1| \leq 2$.)

Let $\{\sigma_k\}$ be a sequence of positive numbers such that $\sum_k \sigma_k = 1$. e.g. take $\sigma_k = C_\varepsilon 2^{-\varepsilon k}$, where $C_\varepsilon = 1 - 2^{-\varepsilon}$. Then, there exists at least one k such that

$$|M_k| \geq \sigma_k |M|. \tag{3}$$

By Hölder inequality with (2), we have

$$|M_k| \lesssim 2^{-k}D^{-1} \sum_{\substack{m_j, n_j \\ \sum m_j = \sum n_j}} \prod_{j=1}^{3} |\widehat{\phi_j}(m_j)||\widehat{\phi_j}(n_j)|$$

By defining $f_j(x)$ by $\widehat{f_j}(n) = |\widehat{\phi_j}(n)|$,

$$= 2^{-k}D^{-1} \sum_{\substack{m_j, n_j \\ \sum m_j = \sum n_j}} \prod_{j=1}^{3} \widehat{f_j}(m_j)\widehat{f_j}(n_j) = 2^{-k}D^{-1} \int_{\mathbb{T}} \prod_{j=1}^{3} |f_j|^2 dx$$

$$\leq 2^{-k}D^{-1} \|f_1\|_{L^2}^2 \|f_2\|_{L^\infty}^2 \|f_3\|_{L^\infty}^2 \leq 2^{-k}D^{-1}\Delta_2\Delta_3 \tag{4}$$

since $\|f_j\|_{L^2} = \|\phi_j\|_{L^2} = 1$. (Recall $\Delta_j = \mathrm{diam}\,(\mathrm{supp}\,\phi_j)$.)

Now, let $\varepsilon = \frac{1}{2}$ in the definition of σ_k. Then, from (3) and (4), we have

$$2^{\frac{k}{2}} \leq C \frac{\Delta_2\Delta_3}{D|M|}, \tag{5}$$

which clearly can hold *only* for $k \leq K_0$ for some $K_0 > 0$. This in particular shows that the contribution from $k > K_0$ is small:

$$\sum_{k > K_0} M_k \leq \sum_{k > K_0} |M_k| \leq |M| \sum_{k > K_0} \sigma_k \leq 2^{-\frac{1}{2}K_0}|M| \leq \tfrac{1}{2}|M|.$$

Hence, the contribution from $k \leq K_0$, i.e. on $\{\,|F(m,n)| \leq 2^{K_0}D\} = \{\,|F(m,n)| = O(D)\}$ is at least $\frac{1}{2}M$. Therefore, (4.11) is valid...but the choice of K_0 depends on M...

Note that this does not cause a problem for the proof of Lemma 4.1 since the right hand side of (4.8) is *not* M but rather N_1^{0-}. Now, repeat the computation with $\sigma_k = 6/(\pi^2 k^2)$. Hence, the previous computation leads to becomes

$$2^k D \leq C k^2 N_1^{0+} \Delta_2 \Delta_3 \lesssim N_1^{2-} \tag{6}$$

for $k \leq K_0$. Now, we set $O(D)(=$ size of $F) \leq 2^{K_0} D$ in (4.11). Note the following. Trivially, we have $|F| \lesssim N_1^2$. Thus, it is enough to take K_0 such that $2^{K_0} D \lesssim N_1^2$. i.e. $K_0 \lesssim \ln N_1$. Then, from (6), we have

$$\text{size of } F \leq 2^{K_0} D \leq C (\ln N_1)^2 N_1^{\varepsilon} \Delta_2 \Delta_3 \leq C N_1^{\varepsilon+} \Delta_2 \Delta_3,$$

which is basically the condition (4.9). Moreover, it is the condition needed in (4.12).

2. Lemma 4.1 $\Longrightarrow$ Corollary 4.17 with D^{1+}

Now, we show how Corollary 4.17 with D^{1+} follows from Lemma 4.1. We consider

$$\Bigg| \sum_{\substack{\sum (-1)^j n_j = 0 \\ \sum (-1)^j n_j^2 \in I}} \prod_{j=1}^{6} \widehat{\psi}_j(n_j) \Bigg| \tag{7}$$

where $N_1 \geq N_2 \geq \cdots \geq N_6$ with $N_1 > N_6^{1+\delta}$ and $I \subset \mathbb{Z}$ is of size D. Let $I = [a, a+D-1]$. Without loss of generality, assume $\widehat{\psi_j}(n) \geq 0$, $n \in \mathbb{Z}$. We use $(f_j)^{\star_j}$ to denote f_j for odd j and $\overline{f_j}$ for even j. Also, ϕ_j denotes ψ_j or $\overline{\psi_j}$.

By letting $F(n) = \sum (-1)^j n_j^2$, we have

$$\begin{aligned}
(7) &= \sum_{k=0}^{D-1} \sum_{\sum(-1)^j n_j = 0} \prod_{j=1}^{6} \widehat{\psi}_j(n_j) \int_0^{2\pi} e^{i(F(n)-a-k)t} dt \\
&= \sum_{k=0}^{D-1} \sum_{\sum(-1)^j n_j = 0} \prod_{j=1}^{6} \widehat{\psi}_j(n_j) \bigg(\int_0^{\frac{2\pi}{D}} + \int_{\frac{2\pi}{D}}^{\frac{4\pi}{D}} + \cdots + + \int_{\frac{2(D-1)\pi}{D}}^{2\pi} \bigg) e^{i(F(n)-a-k)t} dt \\
&= \sum_{k=0}^{D-1} \sum_{\sum(-1)^j n_j = 0} \sum_{l=0}^{D-1} e^{-i\frac{2\pi l}{D} F(n)} \int_0^{\frac{2\pi}{D}} (S(t) e^{-i(a-k)t} \phi_1)^\wedge(n_1) \prod_{j=2}^{6} (\widehat{S(t)\phi_j})^{\star_j}(n_j) dt \\
&= \sum_{k=0}^{D-1} \sum_{l=0}^{D-1} \underbrace{\sum_{\sum(-1)^j n_j = 0} \int_0^{\frac{2\pi}{D}} (S(t) e^{-i(a-k)t} \phi_1^{(l)})^\wedge(n_1) \prod_{j=2}^{6} (\widehat{S(t)\phi_j^{(l)}})^{\star_j}(n_j) dt}_{=: M_{k,l}}
\end{aligned}$$

where $\widehat{\phi_j^{(l)}}(n_j) = e^{-i\frac{2\pi l}{D} n_j^2} \widehat{\phi}_j(n_j)$. Note $\|\phi_j^{(l)}\|_{L^2_x} = \|\phi_j\|_{L^2_x}$.

By Lemma 4.1 and (4.6), we have

$$|M_{k,l}| \leq \|S(t)\phi_1 S(t)\phi_5 S(t)\phi_6\|_{L^2_{x,t}} \|S(t)\phi_2 S(t)\phi_3 S(t)\phi_4\|_{L^2_{x,t}} \lesssim N_1^{-\delta'} (C_{N_1})^3$$

where $C_N = \exp(c \ln N / \ln\ln N) \ll N^{0+}$. Hence, we have

$$(7) \lesssim D^2 N_1^{-\delta'} (C_{N_1})^3 \tag{8}$$

(We can make it $D^{\frac{3}{2}}$ by dividing the interval $[0, 2\pi)$ into subintervals only in the first factor *after* Cauchy-Schwarz inequality.)

Also, by (5.13), we have

$$(7) \lesssim D(C_{N_1})^6. \tag{9}$$

By interploating (8) and (9), we obtain

$$(7) \lesssim D^{1+\theta} N_1^{-(\theta)\delta'} (C_{N_1})^{6-3\theta} \ll D^{1+\theta} N_1^{-\frac{\theta\delta'}{2}}. \tag{10}$$

Note that (10) is *not* good enough to prove (5.8)... i.e. we can not afford to lose a small power of D...

3. Lemma 4.1 $\implies$ Corollary 4.17 with $D(\ln D)^{\frac{1}{2}}$

We present how Corollary 4.17 with $D(\ln D)^{\frac{1}{2}}$ follows from Lemma 4.1. By repeating a similar computation as before, we have

$$\begin{aligned}
(7) &= \sum_{k=0}^{D-1} \sum_{\sum(-1)^j n_j = 0} \prod_{j=1}^{6} \widehat{\psi}_j(n_j) \int_0^{2\pi} e^{i(F(n)-a-k)t} dt \\
&= \sum_{\sum(-1)^j n_j = 0} \int_0^{2\pi} \underbrace{\Big(S(t) \sum_{k=0}^{D-1} e^{-i(a-k)t} \phi_1\Big)^\wedge (n_1)}_{=:\widehat{G}(n_1,t)} \prod_{j=2}^{6} (\widehat{S(t)\phi_j})^{\star_j}(n_j) dt \\
&\le \|G(t)S(t)\phi_5 S(t)\phi_6\|_{L^2_{x,t}(\mathbb{T}^2)} \|S(t)\phi_2 S(t)\phi_3 S(t)\phi_4\|_{L^2_{x,t}(\mathbb{T}^2)} \\
&\lesssim (C_{N_1})^3 \|G(t)S(t)\phi_5 S(t)\phi_6\|_{L^2_{x,t}(\mathbb{T}^2)}.
\end{aligned}$$

Using

$$\sum_{k=0}^{D-1} \sum_{k'=0}^{D-1} e^{-i(k-k')t} = \frac{1-e^{-iDt}}{1-e^{-it}} \frac{1-e^{iDt}}{1-e^{it}} = \frac{1-\cos Dt}{1-\cos t} =: H(t),$$

we have

$$\begin{aligned}
\|G(t)S(t)\phi_5 S(t)\phi_6\|^2_{L^2_{x,t}(\mathbb{T}^2)} &= \int_{\mathbb{T}} \sum_{k=0}^{D-1} \sum_{k'=0}^{D-1} \int_0^{2\pi} e^{-i(k-k')t} |S(t)\phi_1 S(t)\phi_5 S(t)\phi_6|^2 dt dx \\
&= \int_{\mathbb{T}} \int_{-\pi}^{\pi} H(t) |S(t)\phi_1 S(t)\phi_5 S(t)\phi_6|^2 dt dx =: M.
\end{aligned}$$

Now, we separately estimate the contributions to M for intervals

$$I_0 = \Big\{ |t| \le \frac{2\pi}{D} \Big\}, \quad I_1 = \Big\{ |t| \ge D^{\frac{1}{2}} \frac{2\pi}{D} \Big\}, \quad I_l = \Big\{ D^{\frac{1}{l}} \frac{2\pi}{D} > |t| \ge D^{\frac{1}{l+1}} \frac{2\pi}{D} \Big\}, \quad l = 2, 3, \cdots.$$

Note that the index goes up to at most $\ln D$.

On I_0, it follows from $|H(t)| \le D^2$ and Lemma 4.1 that

$$M_0 = M\big|_{I_0} \lesssim D^2 N_1^{-\delta'}.$$

On I_1, we have $1 - \cos t \gtrsim D^{-1}$. Then, by dividing I_1 into $O(D)$-many subintervals of size $\frac{2\pi}{D}$ (indexed by $\alpha \in \mathcal{A}$, $|\mathcal{A}| \leq D$) and by change of variables as before, we obtain

$$\begin{aligned} M_1 &= D \sum_{\alpha \in \mathcal{A}} \int_{\mathbb{T}} \int_0^{\frac{2\pi}{D}} |S(t)\phi_1^{(\alpha)} S(t)\phi_5^{(\alpha)} S(t)\phi_6^{(\alpha)}|^2 dt dx \\ &\sim D^2 \int_{\mathbb{T}} \int_0^{\frac{2\pi}{D}} |S(t)\phi_1 S(t)\phi_5 S(t)\phi_6|^2 dt dx \lesssim D^2 N_1^{-\delta'}. \end{aligned}$$

In general, on I_l, we have $1 - \cos t \gtrsim D^{-2+\frac{2}{l+1}}$. Then, by dividing I_l into $O(D^{\frac{1}{l}})$-many subintervals of size $\frac{2\pi}{D}$ and by change of variables as before, we obtain

$$M_l \lesssim D^{2-\frac{2}{l+1}+\frac{1}{l}} N_1^{-\delta'} < D^2 N_1^{-\delta'} \tag{11}$$

since we have, for $2 \leq l \lesssim \ln D$,

$$\frac{2}{l+1} + \frac{1}{l} = \frac{2l-1}{l(l+1)} \geq \frac{1}{l+1} \gtrsim \frac{1}{\ln D + 1}. \tag{12}$$

Hence, we obtain

$$(7) \lesssim D(\ln D)^{\frac{1}{2}} N_1^{-\frac{\delta'}{2}} (C_{N_1})^3 \lesssim D(\ln D)^{\frac{1}{2}} N_1^{-\frac{\delta'}{4}}. \tag{13}$$

Note that (13) suffices for the argument in Section 5.

Question: Can we prove Corollary 4.17 with D? It follows from (11) and (12) that Corollary 4.17 with D follows (by the argument above) if we have

$$\sum_{l=2}^{K \ln D} D^{-1/l} \lesssim 1. \tag{14}$$

Let l_0 be the largest integer such that $l_0 \ln l_0 \leq K \ln D$. Then, we have

$$\sum_{l=2}^{l_0} D^{-1/l} \leq l_0 D^{-1/l_0} \leq 1.$$

However, we do not know how to control the sum for $l_0 < l \leq K \ln D$. Indeed, it seems that (14) is false due to the contribution from $l_0 < l \leq K \ln D$.

4. Improvement of L^6-Strichartz needed for (6.25)

Recall the usual L^6-Strichartz estimate:

$$\Big\| \sum_{|n| \leq N} a_n e^{i(nx + n^2 t)} \Big\|_{L^6(\mathbb{T}_x \times \mathbb{T}_t)} \ll N^{\varepsilon} \Big(\sum_{|n| \leq N} |a_n|^2 \Big)^{\frac{1}{2}}.$$

Now, suppose that n is of size N (= large) but in an small interval Q around n_0 with $|n_0| \sim N$. Then, with $n = m + n_0$, $|m| \leq |Q|$, we have

$$nx + n^2 t = (n_0 x + n_0^2 t) + m(x + 2n_0 t) + m^2 t.$$

By change of variable $x' = (x + 2n_0 t)$, we have

$$\Big\| \sum_{n\in Q} a_n e^{i(nx+n^2t)} \Big\|_{L^6(\mathbb{T}_x\times\mathbb{T}_t)} = \Big\| \sum_{m\le|Q|} a_{m+n_0} e^{i(x'+m^2t)} \Big\|_{L^6(\mathbb{T}_{x'}\times\mathbb{T}_t)}$$

$$\ll |Q|^{\varepsilon} \Big(\sum_{n\in Q} |a_n|^2 \Big)^{\frac{1}{2}}. \tag{15}$$

i.e. we did not lose a power of N which may be much larger than Q.

By a similar argument, we can improve the other L^6 estimate. Suppose that $\widehat{u}(n,t)$ is supported on Q for any t. Then, we have

$$\|u\|_{L^6_{x,t}} \ll |Q|^{\frac{1}{3}} \|u\|_{X^{0,\frac{1}{3}+}}. \tag{16}$$

An interpolation between (15) and (16) gives an improvement of (6.24), which is needed for (6.25).

5. Improved estimate on (6.32): $\|(6.32)\|_{s,\frac{1}{2}+} \lesssim N_1^{\frac{s_1-s}{2}}$

Recall that $A = |w|^4 - \int_{\mathbb{T}} |w|^4 dx$. i.e. $\widehat{A}(0,t) = 0$ for any t. (i.e. $n \ne n_1$ below.) By writing

$$\widehat{AV}(n,\lambda) = \sum_{\substack{n=n_1-n_2+n_3-n_4+n_5\\ n\ne n_1\\ \lambda=\lambda_1-\lambda_2+\cdots+\lambda_5}} \int \widehat{V}(n_1,\lambda_1)\overline{\widehat{w}(n_2,\lambda_2)}\cdots\widehat{w}(n_5,\lambda_5)$$

- **Case 1:** $|n_1| = n_1^*$. This correspronds to (6.38): $\|V\|_{s_1,b}\|w\|^4_{4\sigma_0,b}$.

Suppose $|\lambda - n^2| \ge |n|^\gamma$. Then, we have

$$\|(6.32)\|_{s,\frac{1}{2}+} \le \|(6.32)\|_{s_1,1-b} \le \cdots \le N_1^{\frac{s_1-s}{2}} N^{-1} \le N_1^{\frac{s_1-s}{2}}$$

for $\gamma = (s - s_1)/(1 - b - (\frac{1}{2}+))$. i.e. smoothing of order $1 - b - (\frac{1}{2}+)\gamma = (\frac{\sigma_0}{2}-)\gamma$.

Suppose $\max(|n_2|,\cdots,|n_5|) \ge |n|^\gamma$. Then, one of the factors of $\|w\|^4_{4\sigma_0,b}$ in (6.38) can be used to absorb a derivative, giving smoothing of order $(s - 4\sigma_0)\gamma$.

Now, suppose $|\lambda_j - n_j^2| \ge |n|^\gamma$. In view of (6.30), we have smoothing of order $(b' - b)\gamma = (\frac{\sigma_0}{4}-)\gamma$.

Hence, we can assume (6.45), for otherwise we can choose s_1 close to s such that $s - s_1 < (\frac{\sigma_0}{4}-)\gamma$. Now, take $\gamma = \frac{1}{4}$. Then, we have

$$\begin{aligned}
|n|^\gamma > |\lambda_1 - n_1^2| &= |\lambda + \lambda + 2 - \lambda_3 + \lambda_4 - \lambda_5 - n_1^2| + \sum_{j=2}^{5} n_j^2 - 4|n|^{2\gamma} \\
&= |\lambda - n^2 + \sum_{j=2}^{5}(\lambda_j - n_j^2) + n^2 - n_1^2| - 4|n|^{2\gamma} \ge |n^2 - n_1^2| - 4|n|^{2\gamma} - 5|n|^\gamma \\
&\ge |n + n_1||n - n_1| - 9|n|^{2\gamma} \gtrsim |n|.
\end{aligned}$$

We used the fact that (i) n and n_1 have the same sign since $n_2,\cdots,n_5$ are of smaller order, and (ii) $n - n_1 \ne 0$. This leads to a contradiction.

- **Case 2:** $|n_1| \ll n_1^*$. This correspronds to (6.39): $\|V\|_{4\sigma_0,b}\|w\|_{s_1,b}\|w\|^3_{4\sigma_0,b}$.

By (6.3) and (6.44), we have

$$\|(6.32)\|_{s,\frac{1}{2}+} \lesssim \|V\|_{4\sigma_0,b}\|w\|_{s,b}\|w\|_{4\sigma_0,b}^3 \lesssim N_1^{\frac{s_1-s}{2}}N^{-4}.$$

6. Improved estimate on (6.33):: $\|(6.33)\|_{s,\frac{1}{2}+} \lesssim NN_1^{\frac{s_1-s}{2}}$

First, we have

$$|\Omega'| \sim |\zeta| = \|w\|_{L^4}^4 \lesssim \|w\|_{H^{\frac{1}{4}}}^4 \lesssim (N^{\frac{1}{4}(1-s)})^4 = N^{1-s}. \tag{6.21}$$

Then, we have

$$\begin{aligned}\int_{|\mu|>\mu_0} |\widehat{\overline{\Omega}^2}(\mu)|d\mu &\leq \Big(\int_{|\mu|>\mu_0} \mu^{-2}d\mu\Big)^{\frac{1}{2}}\Big(\int \mu^2|\widehat{\overline{\Omega}^2}(\mu)|^2 d\mu\Big)^{\frac{1}{2}} \\ &\leq \mu_0^{-\frac{1}{2}}\|\overline{\Omega}^2\|_{H^1(I)} \leq \mu_0^{-\frac{1}{2}}N^C\end{aligned}$$

by (6.21) and $|\overline{\Omega}| = 1$.

Now, we consider the contribution of $B\overline{V} = |w|^2w^2\overline{\Omega}^2\overline{V}$.

- **Case 1:** $|n_1| = n_1^*$. Then, we can assume (6.45) as before. Moreover, if $|\mu| > \mu_0 = N^{2C}|n|^{2\gamma}$, then we have $\int_{|\mu|>\mu_0}|\widehat{\overline{\Omega}^2}(\mu)|d\mu < |n|^{-\gamma}$. This clearly gives smoothing. Recall that we previously used $\|\overline{\Omega}\|_{L_t^\infty} = 1$ in applying L^6-Hölder inequality. Now, we instead apply Young's inequality on the Fourier side and use $\|\widehat{\overline{\Omega}^2}\|_{L^1_\mu(|\mu|>\mu_0)} \leq |n|^{-\gamma}$.

Hence, we assume (6.45) and $|\mu| < N^{2C}|n|^{2\gamma}$. Then, as before, we have

$$\begin{aligned}|n|^\gamma > |\lambda_1 - n_1^2| &= |-\lambda + \lambda + 2 + \lambda_3 + \lambda_4 - \lambda_5 + \mu - n_1^2| + \sum_{j=2}^5 n_j^2 - 4|n|^{2\gamma} \\ &= |\lambda - n^2 + \sum_{j=2}^5(\lambda_j - n_j^2) + n^2 - n_1^2| - 4|n|^{2\gamma} - N^{2C}|n|^{2\gamma} \gtrsim |n| - N^{2C}|n|^{2\gamma}.\end{aligned}$$

This leads to a contradiction for $|n| \gg N^{\frac{2c}{1-2\gamma}}$. Lastly, note that if $|n| \lesssim N^{\frac{2c}{1-2\gamma}}$, then we can gain a small power in $|n|$ by losing a power of N (which corresponds to an extra N in (6.47).)

- **Case 2:** $|n_1| \ll n_1^*$. Same as before.