R. McCann taught Core PDE in Fall 2006:
He covered Evans [§1 – §5.7] \ §4.
students have read Appendices in Evans.

- Fourier perspective on Sobolev spaces
- Clean examples revealing power of Sobolev spaces in PDE: elliptic regularity
- Littlewood – Paley Theory
- Interpolation
- Fourier Transform.
  clean examples of usefulness of F.T.
  Malgrange – Ehrenpreis Theorem
- paradifferential calculus
- pseudodifferential operators + calculus
- Singular integral operators
- stationary phase
- Hardy's inequality
- Hardy – Littlewood – Sobolev
- optimizers; converse Sobolev.

Function Spaces / Harmonic Analysis

Triebel
Tao's Notes
Chemin's Text
Survey Evans + Syllabus expectations.
Sogge's Blue Book
Saint Raymond ?

Evans

§6 Lax – Milgram; 2nd order elliptic
§8 Calculus of Variations
§9 Nonvariational techniques: monotonicity, fixed point methods

Solution Construction methods

Compactness methods; energy method
Contraction mapping method
Other fixed point methods
Nash – Moser iteration

Inverse Problems
Dirichlet – Neumann map.

Fluids
BKM ?

Optimal Transportation
Best PDE examples.

Hamiltonian PDE

Mathematical Physics

Geometric PDE
GR; Ricci Flow; Harmonic Maps;

PDE + GMT
Hasimoto transformation.
Binormal Curvature flow ⟷ $NLS_3(\pi)$.

Hormander I (overview)

I. Test Functions

- $\exists \; \phi \in C_0^\infty(\mathbb{R}^n)$, $\phi \geq 0$, $\phi(0) > 0$.
- Vanishing Theorems
- convolution; mapping properties
- cutoff functions; partition of unity

II Distributions

- Basic Definitions: limit properties,
  $Nf^\wedge$ as finite sum of derivatives,
  $v \geq 0$ in $\mathcal{D}' \Rightarrow$ $v \geq 0$ as measure
  singular support
- Localization
- Distributions w. compact support

III Diff/Multiplication by Functions

- Def's & Examples
- Homogeneous Distributions
- Fundamental Solutions
  - Laplace, Heat, Schrödinger
- Evaluation of integrals

IV. Convolution

- smooth functions, of distribs.
- support property
- Fundamental solutions
- L^p mapping property

V. Distribs in product space

Schwartz Kernel theorem

VI. Composition w. smooth maps

- Definitions
- Fundamental solutions: $\Box^{-1}$
- Distribs on mflds.
- T^*M, TM.

VII Fourier Transform

- $\mathcal{F}$ in $\mathcal{S}$, $\mathcal{S}'$.
  $\mathcal{F}: \mathcal{S} \xrightarrow{\sim} \mathcal{S}$.
  $C_0^\infty \subset \mathcal{S}$ dense
  $\mathcal{F}: L^2 \rightarrow L^2$
  $\mathcal{F}: L^p \rightarrow L^{p'}$; $1 \leq p \leq 2$.
  Dilation, $\mathcal{F}(uv) = \mathcal{F}(u)$[illegible]$(v)$
  Elliptic Regularity
  $\mathcal{F}$ on desities on submanifolds
  (simple layers)

VIII Spectral analysis of singularities

wave front set.

$\boxed{\text{Tao UCLA 254A.}}$

t physical space

τ frequency space

$f \in \mathcal{S}$ "test functions" if

$\sup_x |x^\alpha \partial^\beta f| < \infty \quad \forall \alpha, \beta.$

$C_0^\infty \subset \mathcal{S} \subset C^\infty$. Fréchet seminorm.

Schwartz class.

Fourier Transform: $\forall f \in L^1(\mathbb{R}_t)$

$$\mathcal{F}(f)(\tau) = \hat{f}(\tau) = \int e^{-2\pi i t \tau} f(t)\,dt.$$

phase plane $(t, \tau) \in \mathbb{R}^2$. Level set decomposition

musical notation ⟷ of f in phase plane

Algebraic properties ~~of~~ the Fourier Transform:

- $(f+g)^\wedge = \hat{f} + \hat{g}$, $\widehat{cf} = c\hat{f}$. (linearity)

- Translations in physical space correspond to modulations in frequency space.

$$T_{t_0}(f) := f(t - t_0).$$

$$(T_{t_0}(f))^\wedge(\tau) = e^{-2\pi i t_0 \tau} \hat{f}(\tau).$$

- Modulations in physical space correspond to translations in frequency space.

$$M_{\tau_0}(f) := e^{2\pi i \tau_0 t} f(t)$$

$$(M_{\tau_0} f)^\wedge(\tau) = \hat{f}(\tau - \tau_0).$$

- Scaling in physical space " " dual scaling in frequency space.

e.g. $(S_\lambda f)(t) := f(t/\lambda) \quad \forall \lambda > 0.$ ← preserves L^∞_t

$$\Rightarrow (S_\lambda f)^\wedge(\tau) = \lambda \hat{f}(\tau\lambda).$$ ← preserves L^1_τ.

This can be adapted with a prefactor λ^γ.

- Conjugation in physical space corresponds to conjugation + reflection in freq. sp.

$$(\overline{f(t)})^\wedge(\tau) = \overline{\hat{f}(-\tau)}.$$

- Multiplication in physical space corresponds to convolution in frequency space

$$(fg)^\wedge(\tau) = \int_{\tau_1 + \tau_2 = \tau} \hat{f}(\tau_1)\,\hat{g}(\tau_2) := \hat{f} * \hat{g}.$$

- Convolution in physical space " " multiplication in freq space

$$(f * g)^\wedge(\tau) = \hat{f}\hat{g}(\tau).$$

$$\begin{tikzcd}
\boxed{\text{physical space}} & \boxed{\text{frequency space}} \\
\underline{\text{Translation}} \quad (T_{t_0}f)(t) = f(t-t_0) \arrow[r] \arrow[d, bend right, leftrightarrow] & \underline{\text{Modulation}} \quad (T_{t_0}f)^\wedge(\tau) = e^{2\pi i t_0 \tau}\hat{f}(\tau) \\
\underline{\text{Modulation}} \quad (M_{\tau_0}f)(t) = e^{2\pi i \tau_0 t} f(t) \arrow[r] & \underline{\text{Translation}} \quad (M_{\tau_0}f)^\wedge(\tau) = \hat{f}(\tau-\tau_0). \\
\underline{\text{Scaling}} \quad (\delta_\lambda f)(t) = f(t/\lambda) \arrow[r] & \underline{\text{Dual Scaling}} \quad (\delta_\lambda f)^\wedge(\tau) = \lambda\hat{f}(\tau\lambda). \\
\underline{\text{Multiplication}} \quad f(t)g(t) \arrow[r] \arrow[d, bend right, leftrightarrow] & \underline{\text{Convolution}} \quad (fg)^\wedge(\tau) = \int_{\tau=\tau_1+\tau_2} \hat{f}(\tau_1)\hat{g}(\tau_2) \\
\underline{\text{Convolution}} \quad (f*g)(t) \arrow[r] & \underline{\text{Multiplication}} \quad (f*g)^\wedge(\tau) = (\hat{f}\hat{g})(\tau).
\end{tikzcd}$$

$(\rho_\lambda f)(t) = \lambda^{-1/2} f(t/\lambda)$.

spawns procedures for localization.

⊛

Example:

$$g(t) = e^{-\pi t^2}.$$

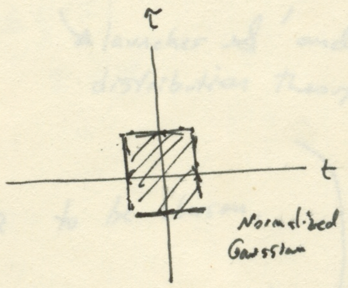
$$\hat{g}(\tau) = \int e^{-2\pi i \tau t} e^{-\pi t^2} dt$$

$$= e^{-\pi\tau^2} \int e^{-\pi(t - i\tau)^2} dt$$

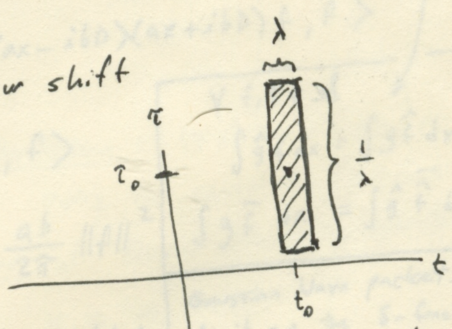
justify

$$= e^{-\pi\tau^2} \int e^{-\pi t^2} dt \qquad \text{contour shift}$$

$$= e^{-\pi\tau^2}$$



Normalized Gaussian



Jack this up into Gaussian wave packets.

$$f(t) = M_{\tau_0}\left[T_{t_0} \left[\rho_\lambda g \right]\right](t)$$

$$= \lambda^{-1/2} \exp(2\pi i \tau_0 t)\, e^{-\pi\left(\frac{t-t_0}{\lambda}\right)^2}.$$

play with these.

☆

Consider the extremes. Dirac Mass; Plane wave

Musical Notation

$$\hat{f}(\tau) = \lambda^{\frac{1}{2}} \underbrace{e^{2\pi i t_0 \tau_0}}_{\text{ugly}} e^{-2\pi i t_0 \tau} e^{-\pi(\lambda(\tau-\tau_0))^2}.$$

☆ ↓

The shaded areas have area $\gtrsim O(1)$. This is the uncertainty principle.

↘ pursue this a bit then return to ⊛

- Define the operators $Xf(t) = t f(t)$, $Df(t) = \frac{1}{2\pi i} f'(t)$. We can calculate $(Df)^{\wedge}(\tau) = \tau \hat{f}(\tau)$ so D is like X on the Fourier side.

– The expression $\frac{\|Xf\|_{L^2}}{\|f\|_{L^2}} = \left(\frac{\int |t|^2 |f|^2 dt}{\int |f|^2} \right)^{\frac{1}{2}}$ measures the avg. value of $|t|$ for f.

– Similarly, $\frac{\|Df\|_{L^2}}{\|f\|_{L^2}} = \left(\frac{\int |\tau|^2 |\hat{f}(\tau)|^2 d\tau}{\int |\hat{f}|^2 d\tau} \right)^{\frac{1}{2}}$ measures $|\tau|$ on average for $\hat{f}$.

- We used <u>Planchevel Identity</u>:

$$\langle f, g \rangle = \int f(t) \overline{g(t)} \, dt = \hat{f} * \hat{g} (0)$$
$$= \langle \hat{f}, \hat{g} \rangle .$$

Thus, $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$ and $\mathcal{F} : L^2 \to L^2$ is an isometry.

↘ launches $\mathcal{S}'$ and distribution theory...

- Both X and D are self-adjoint:

$$\langle Xf, g \rangle = \langle f, Xg \rangle$$
$$\langle Df, g \rangle = \langle \tau \hat{f}, \hat{g} \rangle = \langle f, Dg \rangle .$$

- Note that we have <u>nontrivial commutator</u>

$$[D, X] f = DXf - XDf = \frac{1}{2\pi i} f .$$

$\boxed{\text{Heisenberg Uncertainty Principle}}$ $\frac{\|Xf\|_2}{\|f\|_2} \frac{\|Df\|_2}{\|f\|_2} \geq \frac{1}{4\pi}$.

<u>proof</u>: Consider $\|(aX + ibD)f\|_{L^2}^2$, with $a, b \in \mathbb{R}$ to be chosen.

$$\|aX + ibDf\|_{L^2}^2 = \langle (ax + ibD)f , (ax + ibD)f \rangle = \langle (ax - ibD)(ax + ibD) f , f \rangle$$

$$= \langle [a^2 X^2 + b^2 D^2 + abi(XD - DX)] f, f \rangle$$

$$= a^2 \|Xf\|^2 + b^2 \|Df\|^2 - \frac{ab}{2\pi} \|f\|^2$$

justify

$$\Longrightarrow a^2 \|xf\|^2 + b^2 \|Df\|^2 \geq \frac{ab}{2\pi} \|f\|^2 .$$

Choose $a = \|Df\|$, $b = \|xf\|$ to obtain

$$2 \|Df\|^2 \|xf\|^2 \geq \frac{\|Df\| \|xf\|}{2\pi} \|f\|^2 .$$

<u>RK.</u> Using modulation and translation we can recenter preceding analysis using t, τ with $t - t_0$, $\tau - \tau_0$.

$\forall f, g \in \mathcal{S}$

$$\int \hat{g} f \, dx = \int g \hat{f} \, dx$$

$$\int g \bar{f} \, dx = \int \hat{g} \hat{\bar{f}} \, dx$$

Gaussian Wave packets limit on the δ-function → Fourier Inversion Formula.

$$g * \delta^{-1} e^{-\pi \left(\frac{\cdot}{\delta}\right)^2} .$$

$$f(t) = \int e^{2\pi i t \tau} \hat{f}(\tau) \, d\tau$$

Phase Space Localization

Gaussians are exponentially localized. Their transforms are also Gaussians, with exponential tails.
The tails can not be avoided!

Proposition There does not exist a non-zero integrable f such that both f and $\hat{f}$ are compactly supported.

why? $\hat{f}(\tau) = \int e^{-2\pi i t \tau} f(t)\, dt$

extends to entire function of exponential type

$$\hat{f}(z) = \int e^{-2\pi i t z} f(t)\, dt.$$

But $\{z \in \mathbb{C} : \hat{f}(z) = 0\}$ is discrete.

We can investigate localization using The convolution + product property.

How can we localize $f : \mathbb{R}_t \to \mathbb{C}$ to a subinterval $I \subset \mathbb{R}_t$? physical space localize

- via rough cutoff function $\chi_I(t) = \begin{cases} 1 & t \in I \\ 0 & t \notin I \end{cases}.$ spatial multiplier

$$f(t)\,\chi_I(t).$$

- via smooth cutoff function $\tilde{\chi}_I \in C_0^\infty(\mathbb{R}_t)$ supported on I. with derivative bounds

$$f(t)\,\tilde{\chi}_I(t). \qquad |\tilde{\chi}_I^{(j)}(t)| \lesssim c_j |I|^{-j}.$$

We can also frequency space localize functions: Suppose $I \subset \mathbb{R}_\tau$.

- via rough cutoff function

$$\widehat{\pi_I f}(\tau) = \chi_I(\tau)\,\hat{f}(\tau)$$

- via smooth cutoff

$$\widehat{(\tilde{\pi}_I f)}(\tau) = \tilde{\chi}_I(\tau)\,\hat{f}(\tau).$$

The operators π_I, $\tilde{\pi}_I$ are examples of Fourier Multipliers: $(Tf)\hat{}(\tau) = \hat{m}(\tau)\hat{f}(\tau)$. m is the symbol of the Fourier multiplier T. Using the convolution/product property, we obtain the convolution representation of Fourier multiplier operators

$$(Tf)(t) = (f * K)(t) = \int f(t-s)\,K(s)\,ds \quad \text{where } K = m^\vee \text{ e.g.}$$

$$K(s) = \int e^{2\pi i s \tau} m(\tau)\, d\tau. \quad K \text{ is the kernel associated to } T.$$

How do Fourier multiplier operators act in physical space? How do spatial multiplier operators act in frequency space? As convolution operators with kernel K.

Rough Frequency cutoff

Example: $I = [-1,1]$. $K(s) = \int e^{2\pi i s \tau} \chi_{[-1,1]}(\tau)\, d\tau = \frac{\sin 2\pi s}{s}$. slow decay of rough cutoff.

$$\Longrightarrow |\pi_I f(s)| \lesssim \|f\|_{L^1}\, \text{dist}(s,\Omega)^{-1} \quad \text{if } \text{spt} f \subset \Omega.$$

whenever $\text{dist}(x,\Omega) \geq 1$.

Frequency cutoffs applied to compactly supported functions leak.

Now, consider the <u>smooth frequency cutoff</u> to $I = [-1,1]$. $\tilde{\pi}_I f$ is a convolution operator with kernel $K(s) = \int e^{2\pi i s\tau} \tilde{\chi}_I(\tau)\,d\tau$. Since $\tilde{\chi}_{[-1,1]}$ is not explicit we don't have a closed form expression. But we can still estimate things: If $f \in L^1$ and $\operatorname{spt} f \subset \Omega$ then $\forall j$

$$|\tilde{\pi}_I f(s)| \lesssim C_j \|f\|_{L^1} \operatorname{dist}(s, \Omega)^{-j} \text{ if } \operatorname{dist}(s,\Omega) \geq 1.$$

Why? Clearly $|K(s)| \lesssim 1$ for all s. For $|s| >> 1$ we can write using integration by parts $K(s) = -\frac{1}{2\pi i s} \int e^{2\pi i s\tau} \frac{d}{d\tau}\tilde{\chi}_{[-1,1]}(\tau)\,d\tau$. and this step can be iterated. Since $\tilde{\chi}_{[-1,1]}$ has all derivatives bounded, we get $K(s) \lesssim C_j |s|^{-j}$.

<u>Proposition</u> Let f have frequency support in $[-1,1]$. Then f contains no sudden spikes compared to nearby values: $\forall\ j, k > 0\ \exists\ C_{j,k}$ s.t.

$$\oplus \quad |f^{(k)}(t)| \lesssim C_{jk} \int |f(t-s)| \frac{ds}{(1+|s|)^j}$$

<u>proof</u>: $f(t) = \tilde{\pi}_{[-3,3]} f(t) = \int f(t-s)\, K(s)\, ds$ for some $K \in \mathcal{S}$.

$f^{(k)}(t) = \int f(t-s)\, K^{(k)}(s)\, ds$. Use $K \in \mathcal{S}$ to get $C_{k,j}$ and the decay w.r.t. s in claim.

<u>Uncertainty Principle</u> Suppose f has Fourier support in $[-1,1]$. Then f is essentially constant at unit scales. For $|x-y| >> 1$, $f(x)$ and $f(y)$ are essentially unrelated. More generally, if f has Fourier support in I centered at the origin then f is essentially constant on scale $|I|^{-1}$. For I centered around $\tau_0 \neq 0$, $e^{-2\pi i t\tau_0} f(t)$ is essentially constant on scale $|I|^{-1}$.

<u>Bernstein's Inequality</u> If $1 \leq p \leq q \leq \infty$ and f has Fourier support in an interval I then $\|f\|_{L^q} \lesssim |I|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^p}$.

Suppose f is a gaussian wave packet w. Fourier support on I. Then f is supported on interval of size $|I|^{-1}$. $|I|^{-\frac{1}{q}} \leq C\, |I|^{\alpha}\, |I|^{-\frac{1}{p}}$?

Lower L^p norms with frequency localization upgrade into higher L^q control.

<u>proof</u>: $p = q$ ✓ $p = 1, q = \infty$; use $\oplus$ with centered I and $|I| = 1$. Then rescale and interpolate.

L^1-Theory

Fourier Transform Mapping Properties

Recall that $\mathcal{F}(f)(\tau) = \hat{f}(\tau) = \int e^{-2\pi i t\tau} f(\tau)\, d\tau$.

The mapping $f \mapsto \hat{f}$ is bounded for $L^1 \to L^\infty$: $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$.

Exercise Prove a) If $f \in L^1(\mathbb{R}_t)$ then $\hat{f}$ is uniformly cts.

b) (Riemann-Lebesgue) If $f \in L^1(\mathbb{R}_t)$ then $|\hat{f}(\tau)| \to 0$ as $|\tau| \to \infty$.

Thus, $\mathcal{F} : L^1 \longrightarrow \mathcal{F}(L^1) \overset{?}{\dot{\subset}} C_0 \dot{\subsetneq} L^\infty$ The set $\mathcal{F}(L^1)$ is not C_0 and has not been characterized.

... following [Stein-Weiss p16].

L^2 Theory and Plancherel Theorem

Theorem (i) If $f \in L^1 \cap L^2$ then $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$.

(ii) The Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$.

(Thus $\mathcal{F}$ preserves inner products and is onto.) isometry.

proof of (i): Let $g(x) = \overline{f(-x)}$. Form $h = f * g$. Then $\hat{h} = \hat{f}\hat{g}$.

A calculation shows $\hat{g} = \overline{\hat{f}}$ so $\hat{h} = |\hat{f}|^2$. A convolution estimate we will soon consider establishes that $h \in L^1$. Moreover h is uniformly cts. so it may be restricted to a point:

$$\int_{\mathbb{R}^n} |\hat{f}|^2\, dx = \int_{\mathbb{R}^n} \hat{h}\, dx = h(0) = \int f(x)\, g(0-x)\, dx = \int |f|^2\, dx.$$

For (ii): $L^1 \cap L^2$ is dense in L^2. (i) asserts $\mathcal{F}$ is a bounded linear operator (isometry in fact) on $L^1 \cap L^2$. For $f \in L^2$ we can define $\hat{f}$ as the L^2-limit of $\{h_K\}$ where $L^1 \cap L^2 \ni h_K \to f \in L^2$. Often it is convenient to use $h_K(t) = \chi_{|t| \leq K}(t)\, f(t)$. This defines $\mathcal{F}$ on L^2.

Since $\mathcal{F}$ is an isometry, its range is a closed subspace of $L^2(\mathbb{R}^n)$.

If this subspace is not all of $L^2(\mathbb{R}^n)$ then $\exists\, g$ s.t. $\int \hat{f} g\, dx = 0$ $\forall f \in L^2$ with $\|g\|_2 \neq 0$. $\int f \hat{g}\, dx = \int \hat{f} g\, dx = 0$ $\forall f \in L^2 \Longrightarrow$ $\hat{g} = 0$ a.e. $\Rightarrow \|g\|_{L^2} = \|\hat{g}\|_{L^2} = 0$ (C!).

$$\mathcal{F}^{-1}(f) = \mathcal{F}^*(f)(x) = \int e^{2\pi i x \cdot \xi} f(\xi)\, d\xi$$

Thus, $\mathcal{F} : L^2 \xrightarrow{\sim} L^2 : \mathcal{F}^{-1}$ $\longleftarrow$

Fourier Transform Mapping Properties

Convolution Estimate If $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$ and $g \in L^1(\mathbb{R}^d)$ then $h = f * g$ is well-defined and belongs to $L^p(\mathbb{R}^d)$. Moreover $\|h\|_p \leq \|g\|_1 \|f\|_p$.

proof: clearly $|h(x)| \leq \int_{\mathbb{R}^n} |f(x-y)||g(y)|\,dy$. By Minkowski's integral inequality

$$\|h\|_{L^p} \leq \int_{\mathbb{R}^n} \|f(x-y)\|_{L^p} |g(y)| \, dy = \|f\|_{L^p} \|g\|_{L^1}.$$

Young's Inequality For Convolutions: If $f \in L^p(\mathbb{R}^d)$, $g \in L^r(\mathbb{R}^d)$, $1 \leq p, r$ and $\frac{1}{p} + \frac{1}{r} \geq 1$ then $h = f * g \in L^q(\mathbb{R}^d)$ where $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$. and

$$\|f * g\|_{L^q} \leq \|f\|_{L^p} \|g\|_{L^r}.$$

why?

$$\|g * f\|_{L^p} \leq \|g\|_{L^1} \|f\|_{L^p}$$

so $T : g \longmapsto f * g$ is of type $(p_0, q_0) = (1, p)$

interpolate

$$\|g * f\|_{L^\infty} \leq \|g\|_{L^{p'}} \|f\|_{L^p}.$$

so $T : g \longmapsto f * g$ is of type $(p_1, q_1) = (p', \infty)$

Hausdorff - Young Theorem

$\mathfrak{F}$ is defined on $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ $\Rightarrow$ also on $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ and this space is dense in $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$. We have the estimates

$$\|\mathfrak{F} f\|_{L^\infty} \leq \|f\|_{L^1}$$

interpolate

$$\|\mathfrak{F} f\|_{L^2} = \|f\|_{L^2}.$$

$$\|\mathfrak{F} f\|_{L^{p'}} \leq \|f\|_{L^p} \quad \forall \; 1 \leq p \leq 2.$$

$$\mathfrak{F} : L^1 \longrightarrow \mathfrak{F}(L^1) \subset C_0 \subset L^\infty$$

interpolate

$$\mathfrak{F} : L^p \longrightarrow \mathfrak{F}(L^p) \overset{?}{\subset} L^{p'}$$

$$\mathfrak{F} : L^2 \longrightarrow \mathfrak{F}(L^2) = L^2.$$

What have we accomplished? Quite a bit! Initially, $\mathfrak{F} : L^1 \longrightarrow L^\infty$. We then extended it to $\mathfrak{F} : L^2 \longrightarrow L^2$ then to $\mathfrak{F} : L^1 + L^2 \longrightarrow L^\infty + L^2$, and also $\mathfrak{F} : L^p \longrightarrow L^{p'}$ for $1 \leq p \leq 2$.

Tempered Distributions

Motivational Remaks (Hörmander Introduction vol. I)

Consider classical solutions of Laplace's equation ($\partial_x^2 u + \partial_y^2 u = 0$) and the wave equation $\partial_x^2 u - \partial_y^2 u = 0$ in 2 space dimensions. Uniform limits of classical solutions of Laplace's equation are classical solutions. classical solutions of wave eq. are of the form $u(x,y) = f(x+y) + g(x-y)$, with $f, g \in C^2$. Uniform limits of wave solutions include all $f, g \in C$. Should we interpret $u(x,y) = f(x+y) + g(x-y)$, $f, g \in C$ as a "solution" of $\partial_x^2 u - \partial_y^2 u = 0$?

Next, consider inhomogeneous analogs: $\partial_x^2 u + \partial_y^2 u = F$, $\partial_x^2 u - \partial_y^2 u = F$. Suppose F is cts function with compact support. If $F \in C^1$ then

$$u(x,y) = \iint_{\eta - y + |x-\xi| < 0} - F(\xi,\eta)\,\frac{d\xi\, d\eta}{2}$$

is a C^2 solution of inhomog. wave eq. But this formula defines a C^2 solution for merely cts. or even worse F. Similarly, Poisson's equation has a solution of the form $u(x,y) = c \iint F(\xi,\eta) \log (x-\xi)^2 + (y-\eta)^2)\, d\xi\, d\eta$. The solution formula "make sense" for beyond The "classical" solution setting.

We thus question the notion of solution.

Thought Experiment Suppose you are presented with a laboratory apparatus involving a field u which satisfies $\partial_x^2 u - \partial_y^2 u = F$. How would you experimentally validate that u does indeed satisfy the equation? You'd measure it and measurement involves some kind of spatial averaging procedure. In other words, you'd take some function ϕ supported near $(x_0, y_0) \in \mathbb{R}^2$ and measure $\iint \phi \left[\partial_x^2 u - \partial_y^2 u\right] dx\, dy$ and $\iint \phi F\, dx\, dy$ and check to see you get the same number. Thus, the way you touch the equation experimentally is through interaction against the "testing functions" ϕ you have at your disposal in your laboratory.

We have two "perspectives" on, say, the wave equation: $\partial_x^2 u - \partial_y^2 u = F$ (W) and $\iint \phi \left[\partial_x^2 u - \partial_y^2 u\right] dx\, dy = \iint \phi F\, dx\, dy$ (W′). The LHS of (W′) may be reexpressed $\iint u \left(\partial_x^2 \phi - \partial_y^2 \phi\right) dx\, dy$ via IBP, for $u \in C^2$.

Suppose u is merely cts. The mapping $\phi \longmapsto L(\phi) = \iint u\,(\partial_x^2\phi - \partial_y^2\phi)\,dx\,dy$ can not be written in the form $L(\phi) = \iint \phi F\,dxdy$ for a cts. function F. The "laboratory perspective" on the PDE forces us to accept the "measurement" $L(\phi)$ even when it is not of the form $\iint \phi F\,dx\,dy$.

Distributions

Distributions are the things you can measure by testing against "test functions." Thus, distributions are defined as continuous linear transformations acting on the testing functions. Which distribution family you get depends upon which family of testing functions you measure with. ... Folland & Taylor I.

Example $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$. Let $C_0^\infty(\mathbb{R}^n)$ denote the "test functions". A sequence $\{\phi_j\} \subset C_0^\infty(\mathbb{R}^n)$ converges to $\phi \in C_0^\infty(\mathbb{R}^n)$ if the ϕ_j's are supported in a common compact set and $\|\partial^\alpha(\phi_j - \phi)\|_\infty \to 0$ for all multiindices α. A **distribution** is a cts. linear functional on $C_0^\infty(\mathbb{R}^n)$. ~~Which~~ Topology on $\mathcal{D}'$ is convergent in the weak topology: $\{u_j\} \subset \mathcal{D}'$. $u_j \longrightarrow u$ in $\mathcal{D}'$ if $\langle u_j, \phi\rangle \longrightarrow \langle u, \phi\rangle$ $\forall \phi \in C_0^\infty(\mathbb{R}^n)$.

Example $\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty : \|\phi\|_{\alpha,\beta} := \|x^\alpha \partial^\beta \phi\|_\infty < \infty \quad \forall \alpha,\beta\}$. (Schwartz class of testing functions.) A **tempered distribution** is a linear functional on $\mathcal{S}$. Weak topology is also placed on $\mathcal{S}'$.

Remark $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ and is dense. Thus, every tempered distribution is also a distribution. However, not every distribution is a tempered distribution. (e.g. e^x is not a tempered distribution.)

Examples
- $L^p(\mathbb{R}^n) \ni f \longmapsto \langle f, \phi\rangle = \int f(x)\,\phi(x)\,dx \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$.
- $\{\text{Finite Measures on } \mathbb{R}^n\} \ni \mu \longmapsto \langle \mu, \phi\rangle = \int \phi\,d\mu \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$.
  e.g. $\langle \delta, \phi\rangle = \phi(0)$. Dirac Delta. $\left[\varepsilon^{-n} f(\varepsilon^{-1}x) \xrightarrow[\mathcal{D}']{} \left(\int f\,dx\right)\delta_0(x)\right]$ $\cdot\ \forall f \in L^1$
- $D_j = +\frac{1}{i}\partial_{x_j}$: $\langle u, D_j\phi\rangle = -\langle D_j u, \phi\rangle \quad \forall u \in \mathcal{S}',\ \phi \in \mathcal{S}$.
  $\langle u, D^\alpha\phi\rangle = (-1)^{|\alpha|}\langle D^\alpha u, \phi\rangle$

$\langle \delta', \phi \rangle = - \langle \delta, \phi' \rangle = - \phi'(0)$.

Fourier Transform and Distributions

$\langle \mathcal{F} u, \phi \rangle = \langle u, \mathcal{F}\phi \rangle$; $\langle \mathcal{F}^* u, \phi \rangle = \langle u, \mathcal{F}^*\phi \rangle$. We now have

$\mathcal{F} : \mathcal{S}' \longrightarrow \mathcal{S}'$; $\mathcal{F}^* : \mathcal{S}' \longrightarrow \mathcal{S}'$; $\mathcal{F}\mathcal{F}^* = I$ on $\mathcal{S}'(\mathbb{R}^n)$.

$\mathcal{F}\delta = ?$

Example :

$$\langle \mathcal{F}\delta, \phi \rangle = \langle \delta, \mathcal{F}\phi \rangle = \int \delta_0(\xi) \int e^{-2\pi i x\cdot\xi} \phi(x)\, dx\, d\xi = \iint \delta_0(\xi) e^{-2\pi i x\xi}\, d\xi\, \phi(x)\, dx$$

$$= \langle \langle \delta_0, e^{-2\pi i x \cdot} \rangle, \phi \rangle = \langle 1, \phi \rangle.$$

$$\boxed{\mathcal{F}\delta = 1}$$

Application of Distributions to PDE (Taylor I)

⊛ **Proposition** If $w \in \mathcal{S}'(\mathbb{R}^n)$ is supported by $\{0\}$, $\exists\ k$, $a_\alpha \in \mathbb{C}$ s.t. $w = \sum_{|\alpha| \le k} a_\alpha D^\alpha \delta$.

proof: weak topology on $\mathcal{S}' \Rightarrow \exists\ k$ s.t.

$$|\langle w, \phi \rangle| \le C\, p_k(\phi) = \sum_{|\alpha| \le k} \sup_{x \in \mathbb{R}^n} \langle x \rangle^k |D^\alpha u(x)|.$$

Support property of $w \Rightarrow w$ kills all functions in $\mathcal{E}_0$, the linear space in $C_0^\infty(\mathbb{R}^n)$ consisting of functions vanishing on a neighborhood of 0. Also, w extends to linear space B_k of C^k consisting of those functions for which the p_k seminorm is finite. Thus, w annihilates the closure of $\mathcal{E}_0$ inside B_k, which we will call $\mathcal{E}_k = \{ u \in B_k : D^\alpha u(0)\ \forall |\alpha| \le k \}$.

$\forall\ u \in B_k$, we have Taylor expansion

$$u(x) = \chi \left[\sum_{|\alpha| \le k} \frac{u^{(\alpha)}(0)}{\alpha!} x^\alpha \right] + u^b(x), \quad \chi \in C_0^\infty(\mathbb{R}^n),$$

$\chi(x) = 1$ for $|x| < 1$, $u^b \in \mathcal{E}_k$. Apply w and observe

⊛.

Proposition: Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies $\Delta u = 0$ in $\mathbb{R}^n$. Then u is a polynomial in $(x_1, \dots, x_n)$.

proof: $\Delta u = f \in \mathcal{S}'(\mathbb{R}^n) \Longleftrightarrow -|\xi|^2 \hat{u}(\xi) = \hat{f}(\xi)$ in $\mathcal{S}'$.

For $f = 0$, we learn $\hat{u}(\xi) |\xi|^2 = 0$ so supp $\hat{u} \subset \{0\}$.

Thus $\hat{u} = \sum_{|\alpha| \le k} a_\alpha D^\alpha \delta$ and this proves the claim.

Proposition If u is harmonic on $\mathbb{R}^n$ + bdd. then u is constant.

Why? Any bounded harmonic function must be a bounded polynom

(Corollary If $p(z)$ is a plynomial on $\mathbb{C}$, and if it has no zeros, then $g(z) = \frac{1}{p(z)}$ is holomorphic ($\Longrightarrow$ harmonic) on $\mathbb{C}$. Clearly $|g(z)| \longrightarrow 0$ as $|z| \longrightarrow \infty$ if $\deg p \geq 1$. Thus, $g = c$ but this is a contradiction $\Longrightarrow$ Fundamental Theorem of Algebra.)

... Folland.

Local Solvability

A linear differentiable operator $L = \sum a_\alpha \partial^\alpha$ w. C^∞ coefficients is locally solvable at $x_0 \in \mathbb{R}^n$ if $\forall f \in C_0^\infty$ $\exists$ function (or distribution) u such that $Lu = f$ in some Nbhd of x_0. This is a local issue since the ~~problem~~ issue does not change if we replace f by ϕf for $\phi = 1$ near x_0 and $\phi \in C_0^\infty$.

Which differential operators are locally solvable?

Malgrange - Ehrenpreiss Theorem (Constant coefficient linear differential operators have constant coefficients. are locally solvable.) Let $L = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha$
$\forall f \in C_0^\infty$ $\exists$ $u \in C_0^\infty$ s.t. $Lu = f$.

Lewy Example Consider $\mathbb{R}^3 = \{(x,y,t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}\}$. Set

$$L = \partial_x + i\,\partial_y - 2i(x+iy)\,\partial_t .$$

Theorem (Lewy). Let $f \in C$ with f taking real values. If $\exists$ $u: \mathbb{R}^3 \longrightarrow \mathbb{C}$ and f depends only upon t. satisfying $Lu = f$ in some Nbhd of the origin Then f is analytic at $t = 0$.

Consider $L = \sum_{|\alpha| \le k} a_\alpha(x)\,\partial^\alpha$. This is a linear differential operator of order k on $\Omega \subset \mathbb{R}^n_x$. The *characteristic form* or *principal symbol* *at $x \in \Omega$* of L is a homogeneous polynomial of degree k on $\mathbb{R}^n_\xi$, denoted by, $\sigma_x(L, \cdot)$ and defined by $\sigma_x(L,\xi) = \sum_{|\alpha| = k} a_\alpha(x)\,\xi^\alpha$. A vector $\xi \in \mathbb{R}^n$ is called *characteristic for L at x* if $\sigma_x(L,\xi) = 0$. The set of all $\{\xi$ such that $\sigma_x(L,\xi) = 0$ is$\}$ called the *characteristic variety of L at x*.

proof of Malgrange-Ehrenpreiss Theorem: We are considering a constant coefficient differential operator of order k. We want to solve $Lu = f$ so we rewrite this equation via Fourier transformation as $p(\xi)\,\hat{u}(\xi) = \hat{f}(\xi)$ where $p(\xi) = \sum_{|\alpha| \le k} a_\alpha (2\pi i \xi)^\alpha$. To obtain the solution u, the obvious thing to try is to write $\hat{u}(\xi) = \frac{1}{p(\xi)} \hat{f}(\xi)$. However, $p(\xi)$ will have a bunch of zeros in general so we will have trouble applying the inverse Fourier transform to obtain u. What to do? Since $f \in C_0^\infty(\mathbb{R}^n)$, we know that $\hat{f}$ is an entire function of $\xi \in \mathbb{C}^n$. Of course p is a polynomial in ξ so it also extends into $\mathbb{C}^n$ as an entire function. We will now deform the contour to obtain a formula for u which avoids the zeros of $p(\xi)$.

Since L is really of order k, we can assume that the vector $n = (0, 0, \dots, 1)$ is non-characteristic for L. By multiplying L by a constant we can further assume that $\sigma(L, n) = (2\pi i)^{-k}$. Then $p(\xi) = \xi_n^k +$ lower order terms in ξ_n. What we are saying here is that $p(\xi) = 1\,\xi_n^k + a_{k-1}(\xi')\,\xi_n^{k-1} + \cdots + a_1(\xi')\,\xi_n + a_0(\xi')$. Let us consider with $(\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ as coordinates on $\mathbb{R}^n$. Then $p(\xi', \xi_n)$ as a polynomial in ξ_n, one complex variable, with ξ'-dependent coefficients. This polynomial factors over ξ_n so there are zeros $\lambda_1(\xi'), \dots, \lambda_k(\xi')$: $p(\xi', \lambda_j(\xi')) = 0$ $\forall\, j = 1, \dots, k$. We can order the zeros by the conditions: $i \le j$ if $\operatorname{Im} \lambda_i(\xi') \le \operatorname{Im}_j$ and, when $\operatorname{Im}(\lambda_i(\xi')) = \operatorname{Im}(\lambda_j(\xi'))$ use $i \le j$ when $\operatorname{Re}(\lambda_i(\xi')) \le \operatorname{Re}(\lambda_j)$

By Rouché's theorem from complex variables, small perturbation in ζ' produces a small perturbation in $\lambda_j(\zeta')$ so one can see that $\text{Im}\,\lambda_j(\zeta')$ is cts. w.r.t. ζ'.

Claim: $\exists$ measurable function $\phi: \mathbb{R}^{n-1} \longrightarrow [-K-1, K+1]$ s.t. $\forall \zeta' \in \mathbb{R}^{n-1}$

$$\min \left\{ |\phi(\zeta') - \text{Im}(\lambda_j(\zeta'))| : 1 \leq j \leq K \right\} \geq 1.$$

Assume the claim for the moment. Then we define u by

$$u(x) = \int_{\mathbb{R}^{n-1}} \int_{\text{Im}\,\zeta_n = \phi(\zeta')} e^{2\pi i x\cdot\zeta} \frac{1}{P(\zeta)} \hat{f}(\zeta)\, d\zeta_n\, d\zeta'.$$

The function $\hat{f}$ is rapidly decreasing as $\text{Re}\,\zeta \longrightarrow \infty$, (Why?) when $\text{Im}\,\zeta$ remains bounded. By construction, $\text{Im}\,\zeta_n = \phi(\zeta')$ stays at least unit distance in the ζ_n plane from any zero of $P(\zeta)$, and also at most $K+1$ distance from the real axis. Therefore, the integrand is bounded and rapidly decreasing at infinity, so the integral is absolutely convergent. Therefore we can differentiate under the integral sign and observe that $u \in C^\infty$. OK fine, but is u a solution to our PDE $Lu = f$. Apply L to u to observed

$$Lu(x) = \int_{\mathbb{R}^{n-1}} \int_{\text{Im}\,\zeta_n = \phi(\zeta')} L\left(e^{2\pi i x\cdot\zeta}\right) \frac{1}{P(\zeta)} \hat{f}(\zeta)\, d\zeta_n\, d\zeta' = \int_{\mathbb{R}^{n-1}} \int_{\text{Im}\,\zeta_n = \phi(\zeta')} e^{2\pi i x\cdot\zeta} \hat{f}(\zeta)\, d\zeta_n\, d\zeta'.$$

Now, we have no zeros to worry about and can deform the contour down to $\text{Im}\,\zeta_n = 0$ (since the vertical contour portions make zero contribution). Thus, $Lu = f$.

proof of claim: (pigeon hole principle + measure theory). The interval $[-K-1, K+1]$ has length $2K+2$. $\forall \zeta'$ there must be a subinterval of length 2 which contains none of the K points $\text{Im}\,\lambda_j(\zeta')$. For this ζ' we can take $\phi(\zeta')$ to be the midpoint of the subinterval. All that remains is to show that this can be done measurably w.r.t. ζ'.

Set $\mu_0(\zeta') = -K-1$. $\mu_{K+1}(\zeta') = K+1$. For $1 \le j \le K$,

set $\mu_j(\zeta') = \max\left(\min\{\operatorname{Im}\lambda_j(\zeta'), K+1\}, -K-1\right)$.

The functions $\mu_j(\cdot)$ are cts. so the sets

$$V_j = \{\zeta' : \mu_{j+1}(\zeta') - \mu_j(\zeta) \ge 2\} \qquad j = 0, \dots, K$$

are measurable. By the ordering of the λ_j's, the sets V_j cover $\mathbb{R}^{n-1}$ so we can construct disjoint measurable sets $W_j \subset V_j$ which still cover $\mathbb{R}^{n-1}$. Now set

$$\phi(\zeta') = \frac{1}{2}\left[\mu_{j+1}(\zeta') + \mu_j(\zeta')\right] \quad \text{when } \zeta' \in W_j.$$

This is measurable and proves the claim.

<u>proof of Lewy's Theorem</u> suppose $Lu = f$ on the set $\{(x,y,t) : x^2+y^2 < R^2, |t| < R\}$ for some small $R > 0$. Set $z = x+iy = re^{i\theta}$. Set $s = r^2$. Consider then the quantity

$$V(t,r) = \int_{|z|=r} u(x,y,t)\,dz = ir\int_0^{2\pi} u(r\cos\theta, r\sin\theta, t)\,e^{i\theta}\,d\theta.$$

By Stokes' Theorem, this boundary integral around $\partial B(0,r)$ may be reexpressed (?)

$$V(t,r) = i\iint_{|z|\le r}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)(x,y,t)\,dx\,dy$$

$$= i\int_0^r\int_0^{2\pi}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)(\rho\cos\theta, \rho\sin\theta, t)\,\rho\,d\rho\,d\theta.$$

We calculate

$$\frac{\partial V}{\partial r} = i\int_0^{2\pi}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)(r\cos\theta, r\sin\theta, t)\,r\,d\theta$$

$$= \int_{|z|=r}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)(x,y,t)\,r\,\frac{dz}{z}.$$

For $0 \le s < R^2$, $|t| < R$ with $Lu = f$ we have

$$\frac{\partial V}{\partial s} = \frac{1}{2r}\frac{\partial V}{\partial r} = \int_{|z|=r} \left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)(x,y,t)\,\frac{dz}{2z}$$

$$= i\int_{|z|=r} \frac{\partial u}{\partial t}(x,y,t)\,dz + f(t)\int_{|z|=r}\frac{dz}{2z}$$

$$= i\frac{\partial V}{\partial t} + \pi i f(t).$$

Set $F(t) = \int_0^t f(\tau)\,d\tau$. Define $U(t,s) = V(t,s) + \pi F(t)$.

Observe that $\frac{\partial U}{\partial t} = \frac{\partial V}{\partial t} + f\pi$, $\frac{\partial U}{\partial s} = \frac{\partial V}{\partial s}$.

Therefore $\frac{\partial U}{\partial t} + i\frac{\partial U}{\partial s} = 0$ so U is holomorphic function of $w = t + is$, inside the region $|t| < R$, ~~and~~ $s < R^2$ and U is cts up to the line $s = 0$. When $s = 0$, we have $V = 0$ so $U(t,0) = \pi F(t)$ is real valued. By Schwarz reflection principle, $U(t,-s) = \overline{U(t,s)}$ gives a holomorphic continuation of U to a full Nbhd. of the origin. Therefore $U(t,0) = \pi F(t)$ is analytic in t, hence so is $f = F'$. ∎

ⓘ The application of Stokes' theorem troubled me for a while. I learned the following from Young-Hoon Kim in class.

$$\int_{\partial B} F\,dz = \int_B d(F\,dz) = \int_B \partial F \underbrace{dz \wedge dz}_{=0} + \int_B \bar\partial F\,d\bar z \wedge dz$$

Thus,

$$\int_{|z|=r} u\,dz = \int_{|z|\le r} \bar\partial u\,dx\,dy$$ (up to inessential constants.)

Further Remarks on local solvability

I distributed in class 3 articles:

- Lewy, H '57
- Treves, F '70
- Greiner, P. ; Kohn, J ; Stein E. '78.

It is historically interesting (to me at least) that Lewy's paper makes no mention of the geometrical context in which the Lewy operator arises. This is the thrust of the [GKS] article which provides a complex analytic interpretation of the Lewy operator in the setting of $\{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_2 > |z_1|^2\}$. The more general question of when local solvability holds is surveyed in the article by Treves.

Lewy's operator and the boundary behavior of holomorphic functions

$\{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_2 > |z_1|^2\}$ is holomorphically equivalent to $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$. In this sense $\{\operatorname{Im} z_2 > |z_1|^2\}$ is the "upper half space" model of the unit ball in $\mathbb{C}^2$.

(Note: The Riemann mapping theorem fails to hold in $\mathbb{C}^2$.)

The domain $\mathcal{D} = \{\operatorname{Im} z_2 > |z_1|^2\}$ has a boundary which can be identified with $\mathbb{R}^3$ according to the mapping $(z, t) \longmapsto (z, t + i|z|^2)$.

$b\mathcal{D} = \{\operatorname{Im} z_2 = |z_1|^2\}$. Next, consider the complex vector field on $\mathbb{C}^2$. We calculate A.

$$A = + \overline{\partial}_{z_1} - 2i z_1 \overline{\partial}_{z_2}$$

$$A\left[\frac{z_2 - \overline{z}_2}{2i} - z_1\overline{z}_1\right] = \left(+\overline{\partial}_{z_1} - 2iz_1\overline{\partial}_{z_2}\right)\left[\frac{1}{2i}(z_2 - \overline{z}_2) - z_1\overline{z}_1\right]$$

$$= -z_1 + z_1 = 0.$$

Therefore, the vector field A is tangential to $b\mathcal{D}$. The $L^2(\mathbb{R}^3)$ adjoint of the v.f. A is essentially Lewy's operator so we see now a geometric setting in which Lewy's operator appears.

Cauchy - Szegö Integral

$\forall f \in L^2(\mathbb{R}^3) \simeq L^2(b\mathcal{D})$ the formula $C(f)(z_1, z_2) = \int_{b\mathcal{D}} S(z,w) f(w)\, d\sigma_w$ defines a holomorphic extension of f from $b\mathcal{D}$ into $\mathcal{D}$. Here

$$S(z,w) = \frac{1}{\pi^2} \frac{1}{[i(\bar{w}_2 - z_2) - 2\bar{w}_1 z_1]^2}, \quad (w_1, w_2) = (z, t + i|z|^2)$$

and $d\sigma_w$ is the measure on $b\mathcal{D}$ arising from Lebesgue measure on $\mathbb{R}^3$. $C_b(f) = C(f)\big|_{b\mathcal{D}}$. Thus $C_b : L^2(b\mathcal{D}) \longrightarrow L^2(b\mathcal{D})$.

(This requires proof.) If $f = g\big|_{b\mathcal{D}}$ for some g which is holomorphic on an open Nhbd of $\mathcal{D}$ then it turns out that $C(f) = f$.

Theorem [GKS] Given f, the equation $(\partial_z + i\bar{z}\partial_t)\, u = f$ has a solution in a neighborhood of a point $p \in \mathbb{R}^3$ if and only if the Cauchy-Szegö integral $C_b(f)$ is real-analytic on a Nhbd of p.

Beyond Lewy's example; toward necessary + sufficient conditions for local solvability

Consider $Pu = f$ where $P = P(x,D) = \sum_{|\alpha| \le m} c_\alpha(x) D^\alpha$ and define its *principal symbol* $P_m(x,\xi) = \sum_{\alpha = m} c_\alpha(x) \xi^\alpha$. Similarly, define $P_m(x,D)$ to be the *principal part* of P. Next, consider the commutator $[P_m(x,D), \overline{P_m(x,D)}] = P_m(x,D)\overline{P_m(x,D)} - \overline{P_m(x,D)}P_m(x,D)$. This will be a linear differential operator of order $2m-1$. We can then define its principal part $C_{2m-1}(x,D)$ and its principal symbol $C_{2m-1}(x,\xi)$.

Hörmander's Necessary Condition: If P is locally solvable at x then $\forall \xi \in \mathbb{R}^n \setminus \{0\}$

$$P_m(x,\xi) = 0 \Longrightarrow C_{2m-1}(x,\xi) = 0.$$

Bicharacteristics

Suppose $A(x,\xi)$ is given. We can then consider Hamilton's equations $\frac{dx}{dt} = \nabla_\xi A(x,\xi)$, $\frac{d\xi}{dt} = -\nabla_x A(x,\xi)$. These ODEs induce a flow $t \longmapsto (x(t),\xi(t))$ inside the level sets of the function A. Indeed consider the map $t \longmapsto A(x(t),\xi(t))$ and apply $\frac{d}{dt}$. $\frac{d}{dt} A(X(t),\xi(t)) = \nabla_x A \cdot \dot{x}(t) + \nabla_\xi A \cdot \dot{\xi}(t) = 0.$

Given a point (x_0,ξ_0) where $A(x,\xi) = 0$ the integral curve solving Hamilton's equations passing through (x_0,ξ_0) is called the **null bicharacteristic of A through (x_0,ξ_0)**.

Condition: $\text{Re } P_m(x,\xi)$ does not change sign along any null bicharacteristic of $\text{Im } P_m(x,\xi)$ and $\text{Im } P_m(x,\xi)$ does not change sign over any null bicharacteristic of $\text{Re } P_m(x,\xi)$.

Conjecture The equation $Pu = f$ is locally solvable at every point (x,ξ) where **condition** holds.

This is the Nirenberg-Treves conjecture. Incremental steps toward this result were made by various authors, especially N-T, and the conjecture was eventually established as a Theorem by Beals + Fefferman in 1973. A pseudodifferential extension has recently been established by Denker

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Sobolev Spaces on $\mathbb{R}^d$

$\forall\ s \in \mathbb{R}$ we define the <u>inhomogeneous Sobolev space</u> $H^s(\mathbb{R}^d) = \{ u \in \mathcal{S}' : \|u\|_{H^s(\mathbb{R}^d)} = \{ \int_{\mathbb{R}^d} (1+|\xi|^2)^s\, |\hat{u}(\xi)|^2 d\xi \}^{\frac{1}{2}} < \infty \}$. Similarly, the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d) = \{ u \in \mathcal{S}' \text{ s.t. } \hat{u} \in L^1_{loc} : \|u\|_{\dot{H}^s(\mathbb{R}^d)} = (\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi)^{\frac{1}{2}} < \infty \}$.

$(1+|\cdot|^2)^{\frac{1}{2}} = \langle \cdot \rangle$

"The" <u>Sobolev Embedding Estimate</u>

$f(x) = \int e^{2\pi i x\cdot\xi} \hat{f}(\xi)\, d\xi$. Therefore $|f(x)| \leq \int |\hat{f}(\xi)|\, d\xi = \int \langle\cdot\rangle^{-s} \langle\cdot\rangle^{s} |\hat{f}|\, d\xi$ $\leq \left(\int (1+|\xi|^2)^{-2s} d\xi \right)^{\frac{1}{2}} \|f\|_{H^s(\mathbb{R}^d)}$. The first integral converges provided that $s > \frac{d}{2}$. Since the right side is independent of x we obtain that $\|f\|_{L^\infty(\mathbb{R}^d)} \leq C_{d,s} \|f\|_{H^s(\mathbb{R}^d)}$ provided $s > \frac{d}{2}$.

<u>Scaling invariance</u> When can we possibly expect that $\|f\|_{L^p(\mathbb{R}^d)} \leq c\|f\|_{\dot{H}^s(\mathbb{R}^d)}$?

There is a <u>homogeneity requirement</u> that must hold among the parameters p, s, d for this to possibly hold. Here is the idea... Introduce the dilation $f_\sigma(x) = f(\frac{x}{\sigma})$. We then study the estimate under consideration with the one parameter family f_σ. We calculate via change of variables that $\|f_\sigma\|_{L^p(\mathbb{R}^d)} \sim \sigma^{\frac{d}{p}} \|f\|_{L^p(\mathbb{R}^d)}$. Similarly, $\|D_x^s f_\sigma\|_{L^2(\mathbb{R}^d)} \sim \sigma^{\frac{d}{2}-s} \|f\|_{L^2(\mathbb{R}^d)}$. Thus, the inequality requires $\frac{d}{p} = \frac{d}{2} - s$. It is useful to develop your own efficient mnemonic to quickly reproduce these rescaling calculations.

Does the inequality $\|u\|_{L^p(\mathbb{R}^d)} \leq \|u\|_{\dot{H}^s(\mathbb{R}^d)}$ actually hold true when the homogeneity condition $\frac{d}{p} = \frac{d}{2} - s$ holds?

... following [Chemin - Desjardins - Gallagher - Grenier]

Theorem (Sobolev Embedding) If $0 < s < \frac{d}{2}$ then $\dot{H}^s(\mathbb{R}^d)$ is continuously embedded in $L^p(\mathbb{R}^d)$ (sometimes this is written $\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$) and

$$\|f\|_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \leq C \|f\|_{\dot{H}^s(\mathbb{R}^d)}.$$

proof By Fubini, $\|f\|_{L^p}^p = \int_{\mathbb{R}^d} |f|^p \, dx = \int_{\mathbb{R}^d} \int_0^{|f(x)|} p \Lambda^{p-1} \, d\Lambda \, dx$

$\circledast$ $= p \int_0^\infty \Lambda^{p-1} \text{ meas } \{x \in \mathbb{R}^d : |f(x)| > \Lambda\} \, d\Lambda.$ Next, we make a hi/low frequency decomposition: $f_{<A} = \mathcal{F}^{-1}(\chi_{\overline{B(0,A)}} \hat{f})$, $f_{>A} = \mathcal{F}^{-1}(\chi_{\mathbb{R}^d \setminus \overline{B(0,A)}} \hat{f})$ where $A > 0$ is a parameter to be determined.

We control the low frequency piece using "the" Sobolev embedding estimate.

Lemma Let $s \in (-\infty, \frac{d}{2})$ and $K \subset\subset \mathbb{R}^d$. If $f \in \dot{H}^s(\mathbb{R}^d)$ and spt $\hat{f} \subset K$ then

$$\|f\|_{L^\infty} \lesssim \left(\int_K \frac{d\xi}{|\xi|^{2s}} \right)^{\frac{1}{2}} \|f\|_{\dot{H}^s}$$

proof: **Exercise** (a) Prove this lemma. (b) Prove the Besov refinement of the Sobolev embedding estimate: $\|f\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{\dot{B}^{\frac{d}{2}}_{2,q}}$ for an appropriate choice of q. ■

Since $\overline{B(0,A)} \subset\subset \mathbb{R}^d$, we can apply the lemma to $f_{<A}$ to learn

$$\|f_{<A}\|_{L^\infty} \leq C_{s,d} A^{\frac{d}{2} - s} \|f_{<A}\|_{\dot{H}^s}.$$

By the triangle inequality, $\{x \in \mathbb{R}^d : |f(x)| > \Lambda\} \subset \{x \in \mathbb{R}^d : 2|f_{<A}(x)| > \Lambda\} \cup \{x \in \mathbb{R}^d : 2|f_{>A}(x)| > \Lambda\}$. For fixed Λ, we will choose $A = A_\Lambda$ by the condition $\Lambda = C_{s,d} A^{\frac{d}{2} - s} \|f\|_{\dot{H}^s}$. Thus,

$A_\Lambda = c\,\Lambda^{\frac{2}{d-2s}} = \Lambda^{\frac{p}{d}}$ if $\frac{d}{p} = \frac{d}{2} - s$. With this choice of Λ and the L^∞ bound we have on $|f_{<A}|$ we observe that $\{x \in \mathbb{R}^d : 2|f_{<A}(x)| > \Lambda\} = \phi$. So, all the action inside ⊛ takes place in the high frequency part of f:

$$\|f\|^p_{L^p(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} |f|^p \, dx \leq p \int_0^\infty \Lambda^{p-1} \operatorname{meas}\{x \in \mathbb{R}^d : |f_{>A}(x)| > \tfrac{\Lambda}{2}\} \, d\Lambda.$$

By Tchebychev's inequality,

$$\operatorname{meas}\{x \in \mathbb{R}^d : |f_{>A_\Lambda}(x)| > \tfrac{\Lambda}{2}\} = \int_{\{x \in \mathbb{R}^d : |f_{>A_\Lambda}(x)| > \frac{\Lambda}{2}\}} dx$$

$$\leq \int_{\{\dots\}} \frac{4\,|f_{>A_\Lambda}(x)|^2}{\Lambda^2} \, dx$$

$$\leq \frac{4}{\Lambda^2} \|f_{>A_\Lambda}\|^2_{L^2(\mathbb{R}^d)}.$$

Thus, we have

$$\|f\|^p_{L^p(\mathbb{R}^d)} \leq 4p \int_0^\infty \Lambda^{p-3} \|f_{>A_\Lambda}\|^2_{L^2(\mathbb{R}^d)} \, d\Lambda.$$

By Plancherel's theorem, we have

$$\|f\|^p_{L^p(\mathbb{R}^d)} \leq 4p \int_0^\infty \Lambda^{p-3} \left(\int_{|\xi| > A_\Lambda} |\hat{f}(\xi)|^2 \, d\xi \right) d\Lambda.$$

Recall our choice of A_Λ: $|\xi| > A_\Lambda \iff |\xi| > c\Lambda^{\frac{p}{d}} \iff \Lambda < c|\xi|^{\frac{d}{p}}$.

By Fubini, $\|f\|^p_{L^p} \leq 4p \int_{\mathbb{R}^d} \left(\int_0^{c|\xi|^{\frac{d}{p}}} \Lambda^{p-3} \, d\Lambda \right) |\hat{f}(\xi)|^2 \, d\xi$

$\leq c\frac{4p}{p-2} \int_{\mathbb{R}^d} |\xi|^{\frac{d}{p}(p-2)} |\hat{f}(\xi)|^2 \, d\xi \leq C_{p,d,s} \|f\|^2_{\dot{H}^s}$.

Corollary (Gagliardo - Nirenberg Estimate)

If $p \in [2, \infty)$ satisfies $\frac{1}{p} > \frac{1}{2} - \frac{1}{d}$ then $\exists\ C$ s.t. $\forall$ domain $\Omega \subset \mathbb{R}^d$ we have $\forall\ u \in H^1_0($[illegible]$) = \overline{\{ f \in C_0^\infty(\text{[illegible]}) : \|f\|_{H^1} < \infty \}}$ that

$$\|u\|_{L^p(\Omega)} \leq C \|u\|^{1-\sigma}_{L^2(\Omega)} \|\nabla u\|^{\sigma}_{L^2(\Omega)} \text{ with } \sigma = \frac{d(p-2)}{2p}.$$

proof: By density, we may assume $u \in C_0^\infty(\mathbb{R}^d)$. By Sobolev embedding we have that $\|u\|_{L^p(\Omega)} \leq C \|u\|_{\dot{H}^{\frac{d(p-2)}{2p}}(\mathbb{R}^d)}$.

We then conclude using the convexity of the Sobolev norms

$$\text{\#} \qquad \|u\|_{\dot{H}^\sigma(\mathbb{R}^d)} \leq \|u\|^{1-\sigma}_{L^2(\mathbb{R}^d)} \|u\|^{\sigma}_{\dot{H}^1(\mathbb{R}^d)} \quad \forall\ \sigma \in [0,1].$$

Exercise: Prove #. Hint: use Hölder's inequality on the Fourier transform side.

Exercise: Suppose $u \in H^{\frac{d}{2}+\mu}(\mathbb{R}^d)$ for some $1 > \mu > 0$. Prove that $u \in C^\alpha(\mathbb{R}^d)$ for some $\alpha = \alpha(\mu, d)$.

~~Littlewood - Paley Theory~~

~~Let $\phi \in C_0^\infty(B_2(0))$ where $B_r(x) = \{ y \in \mathbb{R}^d : |y - x| < r \}$ with $\phi \equiv 1$ on $B_1(0)$, $\phi \geq 0$.~~

~~Form $\psi(\xi) = \phi(\xi) - \phi(2\xi)$. Then $\psi \in C_0^\infty$~~

Littlewood Paley Theory .. following [Tao 254A] ..

Let $0 \le \phi \in C_0^\infty(B_2(0))$ with $\phi \equiv 1$ on $B_1(0)$, ϕ radial and monotone.

Form $\psi(\xi) = \phi(\xi) - \phi(2\xi)$. Then ψ is a bump function supported on $\{\xi : 1 \le |\xi| < 2\}$. By construction we have $\sum_{k \in \mathbb{Z}} \psi(\xi/2^k) = 1$.

Let's check that... Fix $\xi \in \mathbb{R}^d$. $\exists!\ j \in \mathbb{Z}$ such that $2^j \le |\xi| < 2^{j+1}$.

Expand out the sum near $k = j$

$$\dots + \phi\left(\frac{\xi}{2^{j-1}}\right) - \phi\left(\frac{\xi}{2^{j-2}}\right) + \phi\left(\frac{\xi}{2^j}\right) - \phi\left(\frac{\xi}{2^{j-1}}\right) + \phi\left(\frac{\xi}{2^{j+1}}\right) - \phi\left(\frac{\xi}{2^j}\right) + \left(\phi\left(\frac{\xi}{2^{j+2}}\right) - \phi\left(\frac{\xi}{2^{j+1}}\right)\right) + \cdots$$

Observe that $\frac{|\xi|}{2^{j-1}} > 2$, $\frac{|\xi|}{2^{j+1}} < 1$ and recall the support properties of ϕ to see the term by term contributions are

$$\dots + 0 - 0 + \phi\left(\frac{\xi}{2^j}\right) - 0 + 1 - \phi\left(\frac{\xi}{2^j}\right) + (1-1) + \cdots$$

Thus, we can decompose unity into the functions $\psi(\xi/2^k)$ for integers k, and each function ψ is supported on the annulus $|\xi| \sim 2^k$.

We define the <u>Littlewood-Paly "projection" operators</u> P_k, $P_{\le k}$ by

$$\widehat{P_k f}(\xi) = \psi(\xi/2^k)\,\hat{f}(\xi), \quad \widehat{P_{\le k} f}(\xi) = \phi(\xi/2^k)\,\hat{f}(\xi).$$

Informally, P_k projects $\hat{f}$ into the annulus $|\xi| \sim 2^k$ while $P_{\le k}$ projects $\hat{f}$ onto the ball $|\xi| \le 2^k$. We sometimes write $P_{<k}$ for $P_{\le k-1}$.

Observe that $P_k = P_{\le k} - P_{<k}$. If $f \in L^2$ then $P_{\le k} f \xrightarrow{L^2} 0$ as $k \to -\infty$ and $P_{\le k} f \xrightarrow{L^2} f$ as $k \to +\infty$.

We thus have the <u>Littlewood - Paley decomposition</u>: $\forall f \in L^2$

$$f = \sum_{k \in \mathbb{Z}} P_k f.$$

This formula decomposes the function f into countably many pieces $P_k f$, each of which contains frequencies of magnitude roughly 2^k.

What does one Littlewood-Paley piece $P_K f$ of f look like? To probe this, we calculate

$$P_{\leq K} f(x) = \left(\phi\left(\tfrac{\xi}{2^K}\right)\hat{f}(\xi)\right)^{\vee}(x) = f * \left(2^{dK}\hat{\phi}(2^K \cdot)\right)(x) = \int f(x + 2^{-K}y)\,\hat{\phi}(y)\,dy.$$

Note that $\phi \in \mathcal{S}$ so $\hat{\phi} \in \mathcal{S}$ and $\int \hat{\phi}(y)\,dy = \phi(0) = 1$. So, we see that $P_{\leq K} f$ is an average of f which removes oscillations at physical scales $\lesssim 2^{-K}$. So, $P_{\leq K} f$ is essentially constant at physical scales $\ll 2^{-K}$.

Since $P_K f = P_{\leq K+2} P_K f$ we observe the self-reproducing formula

$$P_K f(x) = \int P_K f(x + 2^{-K-2} y)\,\hat{\phi}(y)\,dy.$$

Thus, $P_K f$ is essentially constant at scales $\ll 2^{-K}$. We also have that $P_{\leq K-2} P_K f = 0$ so

$$\int P_K f(x + 2^{-K+2} y)\,\hat{\phi}(y)\,dy = 0 \quad \forall x \in \mathbb{R}^d.$$

This reveals that $P_K f$ has mean zero at scales $\lesssim 2^{-K+2}$.

Thus, $P_K f$ is smooth at scales $\ll 2^{-K}$ and on each ball of radius 2^{-K} $P_K f$ contains $O(1)$ oscillations.

How do littlewood-Paley pieces behave under differentiation?

Lemma Let $K \in \mathbb{Z}$, $\operatorname{spt} \hat{f} \subset \{2^{K-1} \leq |\xi| < 2^{K+1}\}$. Then $\forall\, 1 \leq p \leq \infty$ $\|\nabla f\|_{L^p} \sim 2^K \|f\|_{L^p}$. In particular $\forall f$ we have $\|\nabla P_K f\|_{L^p} \sim 2^K \|P_K f\|_{L^p}$.

Exercise: Prove this.

Heuristically, the lemma suggests $\nabla \sim \sum_K 2^K P_K$.

How do the sizes of the Littlewood-Paley pieces relate to the size of f ?

Recall the formula $P_{\le k} f(x) = \int f(x + 2^{-k} y)\,\hat{\phi}(y)\,dy$. Taking L^p_x norm and using Minkowski's integral inequality reveals that $\forall\ 1 \le p \le \infty$

$$\|P_{\le k} f\|_{L^p} \le \int \|f(x+2^{-k}y)\|_{L^p_x}\, |\hat{\phi}(y)|\,dy \lesssim \|f\|_{L^p}.$$

So $P_{\le k} f$ does not get any bigger than f in the L^p sense. Similarly, $\|P_k f\|_{L^p} \lesssim \|f\|_{L^p}$. Meanwhile, we have $f = \sum_k P_k f$ so by the triangle inequality we obtain the cheap Littlewood-Paley inequality

$$\sup_k \|P_k f\|_{L^p} \lesssim \|f\|_{L^p} \le \sum_k \|P_k f\|_{L^p}.$$

This is an easy inequality which is remarkably powerful, especially in problems where you have an extra "ε derivative" to use up.

Non-endpoint Sobolev Embedding

$1 \le p < q \le \infty$ such that $\frac{d}{q} > \frac{d}{p} - 1$. Then $\forall f$ for which the right side is finite we have

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} + \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

proof By density, we may assume $f \in \mathcal{S}$. Denote the right side by X. From cheap Littlewood-Paley inequality, we have $\|P_k f\|_{L^p} \lesssim X$, for all k. We also have $\|\nabla P_k f\|_{L^p} \le X$. By the lemma, we also have $\|P_k f\|_{L^p} \lesssim 2^{-k} X$ so we obtain $\|P_k f\|_{L^p} \lesssim \min(1, 2^{-k})\, X$. For low frequencies, $k \le 0$, we use 1 while for high frequencies we use 2^{-k}.

The function $P_k f$ has Fourier support in the annulus $|\xi| \sim 2^k$. This annulus has volume 2^{dk} so we can use Bernstein's inequality to obtain quantitative control on $\|P_k f\|_{L^q}$ in terms of $\|P_k f\|_{L^p}$. Indeed we have

$$\|P_k f\|_{L^q} \lesssim (2^{dk})^{(\frac{1}{p} - \frac{1}{q})} \|P_k f\|_{L^p} \lesssim 2^{dk(\frac{1}{p} - \frac{1}{q})} \min(1, 2^{-k})\, X.$$

By cheap littlewood-paley again, we get

$$\|f\|_{L^q} \leq \sum_k \|P_k f\|_{L^q} \leq \sum_k 2^{(\frac{1}{p}-\frac{1}{q})dk} \min(1, 2^{-k}) \, X$$

Since $p < q$, $\frac{1}{p} - \frac{1}{q} > 0$ so for $k < 0$ we have geometric convergence. For $k > 0$ the min is 2^{-k} so we encounter the summand $2^{(\frac{1}{p}-\frac{1}{q})dk} 2^{-k} = 2^{(\frac{1}{p}-\frac{1}{q})dk} 2^{(-\frac{1}{d})dk} = 2^{(\frac{1}{p}-\frac{1}{q}-\frac{1}{d})dk}$.

But our scaling hypothesis that $\frac{d}{q} \geq \frac{d}{p} - 1 \Longrightarrow \frac{1}{p} - \frac{1}{q} - \frac{1}{d} < 0$ so the sum converges in the $k \to \infty$ direction as well. $\blacksquare$

The cheap Littlewood-Paley inequality is not the shapest statement linking the size of the Littlewood-Paley pieces to the function. For example, in case $p=2$ we have that $\|f\|_{L^2_x} = (\sum_k \|P_k f\|^2_{L^2_x})^{\frac{1}{2}}$. This follows from Plancherel.

This can be rewritten by interchanging the ℓ^2_k and L^2_x norms to read

(#) $\|f\|_{L^2_x} = \| (\sum_k |P_k f(x)|^2)^{\frac{1}{2}} \|_{L^2_x}$. The quantity $(\sum_k |P_k f(x)|^2)^{\frac{1}{2}}$ is known as the <u>Littlewood-Paley Square Function</u> and is sometimes denoted $|Sf(x)|$. We will use $Sf(x) = (P_k f(x))_{k \in \mathbb{Z}}$ to denote the vector valued function consisting of all the Littlewood-Paley pieces.

<u>Littlewood-Paley inequality</u> $\forall \; 1 < p < \infty$ $\| |Sf| \|_{L^p} \sim \|f\|_{L^p}$, with the implicit constant depending upon p.

The proof of the Littlewood-Paley inequality we will describe relies upon the Calderón-Zygmund theory of singular integral operators. This is a delicate subject and we will only make a quick tour.

Calderón - Zygmund Theory ... still following [Tao 254A]

Definition A Calderón - Zygmund operator T is a linear operator on $\mathbb{R}^d$ of the form

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)\, f(y)\, dy$$

with kernel $K(x,y)$ which satisfies: $\forall\ x \neq y$

$$|K(x,y)| \lesssim \frac{1}{|x-y|^d}$$

K could be matrix valued.

$$|\nabla K(x,y)| \lesssim \frac{1}{|x-y|^{d+1}}.$$

(We also require $T: L^2 \longrightarrow L^2$ bounded.)

Remark: $|x|^{-d}$ is just barely non-integrable on $\mathbb{R}^d$. If $|x|^{-d} \in L^1$ then by Young's inequality for convolutions the operator $Tf(x) = f * |\cdot|^{-d} = \int f(x-y)|y|^{-d}\, dy$ would satisfy

$$\|Tf\|_{L^p_x} \lesssim \left\| \int f(x-y)|y|^{-d}\, dy \right\|_{L^p_x} \lesssim \int \|f(x-y)\|_{L^p_x} |y|^{-d}\, dy \leq \|f\|_{L^p}.$$

Because $|x|^{-d}$ is just barely singular, an operator like T is often described as a singular integral.

Example The standard example of a Calderón - Zygmund operator is the Hilbert transform $Hf(x) := \mathrm{pv} \int \frac{1}{x-y} f(y)\, dy$ on $\mathbb{R}^1$. This operator may be studied on the Fourier transform side and reveals itself to be a Fourier multiplier operator with symbol $-\pi i \operatorname{sgn}(\xi)$. Since $|\operatorname{sgn}(\xi)| \leq 1$, we see that $H: L^2 \longrightarrow L^2$ is bounded.

Calderón - Zygmund Theorem: A Calderón - Zygmund operator T maps $L^p \longrightarrow L^p$ for all $1 < p < \infty$.

We postpone the proof of this theorem...

Let's return to the square function operator $f \longmapsto Sf = \{P_K f\}_{K \in \mathbb{Z}}$. Recall that $P_K f(x) = \int 2^{dK} \hat{\psi}(2^K(x-y)) f(y)\, dy$ so P_K is a convolution operator with kernel $K_K(x,y) = 2^{dK}\hat{\psi}(2^K(x-y))$. Since ψ (and hence $\hat{\psi}$) is Schwartz, we certainly have the decay properties required to observe that P_K is a Calderón - Zygmund operator. Moreover, we have that $f \mapsto Sf$ is bounded from L^2_x to $\ell^2_K L^2_x = L^2_x \ell^2_K$ by the trivial orthogonality argument ⊕ appearing two pages back. Therefore, we see that $L^2_x \ni f \longmapsto |Sf| \in L^2_x$ and $f \longmapsto Sf$ is a Calderón - Zygmund operator. By the Calderón - Zygmund theorem, we have one side of the Littlewood - Paley inequality: $\|Sf\|_{L^p} \lesssim \|f\|_{L^p}$ $\forall\ 1<p<\infty$.

In fact, since $1<p<\infty$ includes all choices of $p, p' \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{p'}=1$, a simple duality argument establishes the opposite inequality.

Duality: If $T: A \to B$ with $\|Ta\|_B \le C\|a\|_A$ then $\exists$ dual operator $T^* : B^* \to A^*$ s.t. $\|T^* b^*\|_{A^*} \le C\|b^*\|_{B^*}$.

$\langle b^*, Ta \rangle = \langle T^* b^*, a \rangle$. $\|T^* b^*\|_{A^*} = \sup_{\|a\|_A = 1} \langle T^* b^*, a\rangle$

why? $\|Ta\|_B = \sup_{\|b^*\|_{B^*}=1}$ $a \mapsto \langle b^*, Ta\rangle \in A^*$ $C\|a\|_A \le C\|b^*\|_{B^*}$.

$= \sup_a \langle b^*, Ta \rangle \le \sup \|b^*\|_{B^*} \|Ta\|_B \le \|b^*\|_{B^*}$

what is S^*? $\|Sf\|_{L^p} = \| \|P_K f(x)\|_{\ell^2_K} \|_{L^p_x} = \sup_{g_K} \int \sum_K g_K(x) P_K f(x)\, dx$ where

sup is over $g_K(x)$ s.t. $\| \|g_K(x)\|_{\ell^2_K} \|_{L^{p'}} \le 1$. Thus, $\|S^* f\|_{L^{p'}} \lesssim \|f\|_{L^{p'}}$ unravels as

$\| \sum P_K g_K(x) \|_{L^{p'}} \lesssim \| (\sum |g_K|^2)^{\frac{1}{2}} \|_{L^{p'}}$. Then choose $g_K = P_K f$ to get $\|f\|_{L^{p'}} \lesssim \|Sf\|_{L^{p'}}$.

The Littlewood - Paley Inequality may be viewed as an assertion of the form $|\sum_K P_K f| \approx (\sum_K |P_K f|^2)^{\frac{1}{2}}$. This is clearly false if the $P_K f$ are not somehow pairwise orthogonal.

Some consequences of the Littlewood - Paley inequality: If $\tilde{P}_K$ looks like P_K in the sense that $\tilde{P}_K$ is given by a bump function adapted to the annulus $\{|\xi| \sim 2^K\}$ then

smooth freq. localization operator

$$\Big\| \Big(\sum_K |\tilde{P}_K f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \lesssim \|f\|_{L^p} \qquad \forall\ 1<p<\infty. \qquad *$$

The dual statement is that for arbitrary functions f_K we have

$$\Big\| \sum_K \tilde{P}_K f_K \Big\|_{L^p} \lesssim \Big\| \Big(\sum_K |f_K|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p}. \qquad *'$$

Hörmander - Mikhlin Multiplier Theorem Let $m(\xi)$ be a Fourier multiplier (symbol) such that $|\nabla^j m(\xi)| \lesssim |\xi|^j$ for all $j \geq 0$ (with implicit constant depending upon j). Let T_m be the associated Fourier multiplier operator: $(T_m f)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$. Then T_m is bounded on L^p for all $1 < p < \infty$.

proof: Let $\tilde{P}_K = P_K T_m$, $\tilde{\tilde{P}}_K = \sum_{K-2 \leq K' \leq K+2} P_K$. Then $T_m = \sum_{K,K'} P_K T_m P_{K'} = \sum_K \tilde{P}_K \tilde{\tilde{P}}_K$.

Observe that $\tilde{P}_K$ and $\tilde{\tilde{P}}_K$ are both smooth frequency localization operators to annulus $\{|\xi| \sim 2^m\}$.

We now estimate

$$\| T_m f \|_{L^p} = \| \sum_K \tilde{P}_K (\tilde{\tilde{P}}_K f) \|_{L^p} \overset{*'}{\lesssim} \| (\sum_K |\tilde{\tilde{P}}_K f|^2)^{\frac{1}{2}} \|_{L^p} \overset{*}{\lesssim} \|f\|_{L^p}.$$

Application: $(\Delta f)^\wedge(\xi) = -|\xi|^2 \hat{f}(\xi)$; $(\partial_{x_j}\partial_{x_k} f)^\wedge(\xi) = -\xi_j \xi_k \hat{f}(\xi)$. We can relate the off-diagonal second derivatives to the trace of second derivative matrix (the Laplacian) with the formula $(\partial_{x_j}\partial_{x_k} f)^\wedge(\xi) = \frac{\xi_j \xi_k}{|\xi|^2}(\Delta f)^\wedge(\xi)$.

Consider the multiplier $m_{jk}(\xi) = \frac{\xi_j \xi_k}{|\xi|^2}$. This is homogeneous of degree zero and satisfies $|\nabla^K m_{jk}(\xi)| \lesssim |\xi|^{|K|}$. By Hörmander - Mikhlin multiplier theorem, we see that $\| \partial_{x_j}\partial_{x_k} f \|_{L^p(\mathbb{R}^d)} \lesssim \| \Delta f \|_{L^p(\mathbb{R}^d)}$. Thus, the control on the Laplacian implies control on all the second derivatives. This is an example of elliptic regularity theory.

Function Spaces since $\|\nabla P_K f\|_{L^p} \sim 2^K \|P_K f\|$ we can embellish our new way to represent the L^p norm via Littlewood-Paley inequality to define various scales of function space. For example,

$$\| \left(\sum_K |(1+2^K)^s |P_K f(x)||^q \right)^{\frac{1}{q}} \|_{L^p_x} = \|u\|_{F^{s,p}_q} \qquad \text{Triebel-Lizorkin}$$

$$\left\{ \sum_K |(1+2^K)^s \|P_K f\|_{L^p_x}|^q \right\}^{\frac{1}{q}} = \|u\|_{B^{s,p}_q} \qquad \text{Besov.}$$

Littlewood-Paley inequality $\Rightarrow$ $F^{0,p}_2 = L^p$ when $1 < p < \infty$.

$$W^{s,p} = F^{s,p}_2 \quad \text{when } 1 < p < \infty.$$

Regularity of Laplace's Equation

Let $\Omega \subset\subset \mathbb{R}^d$ be a nice domain and suppose $u : \Omega \longrightarrow \mathbb{R}$ satisfies $\Delta u = 0$ in Ω. This is the Euler-Lagrange equation characterizing minimizers of the energy functional $E[u] = \int_\Omega |\nabla u|^2 \, dx$, often posed with an auxiliary boundary condition. Thus, the harmonic function we are considering is in $W^{1,2}_{loc}$ by definition.

Next, we observe from Littlewood-Paley theory that the elliptic regularity estimate $\|f\|_{W^{2,p}} \leq \|f\|_{L^p} + \|\Delta f\|_{L^p}$ holds true. Why? The issue is to control $\|\partial_j \partial_k f\|_{L^p}$ for all choices of j, k and we saw how to do this following the Hörmander-Mikhlin multiplier theorem. Similarly, $\|f\|_{W^{s+2,p}} \leq \|f\|_{W^{s,p}} + \|\Delta f\|_{W^{s,p}}$. Next, let's choose η to be a C^∞ bump function supported inside Ω. Let's apply the elliptic regularity estimate to $f = \eta u$. We obtain

$$\|\eta u\|_{W^{s+2,p}} \leq \|\eta u\|_{W^{s,p}} + \|\Delta(\eta u)\|_{W^{s,p}}.$$

A simple calculation shows

$$\Delta(\eta u) = \Delta\eta \, u + 2\nabla\eta \cdot \nabla u + \eta \underbrace{\Delta u}_{=0} = (-\Delta\eta)\, u + 2\nabla\cdot(\nabla\eta \, u).$$

Thus, $\|\Delta(\eta u)\|_{W^{s,p}} \leq \|\Delta\eta \, u\|_{W^{s,p}} + 2\|\nabla\eta \cdot \nabla u\|_{W^{s+1,p}}$. So, we have

$$\|\eta u\|_{W^{s+2,p}} \leq \|\eta u\|_{W^{s,p}} + \|\Delta\eta \, u\|_{W^{s,p}} + 2\|\nabla\eta \, u\|_{W^{s+1,p}}.$$

Since $u \in W^{1,2}_{loc}$, all terms on the right side are finite when $s = 1$, $p = 2$. We therefore obtain that $u \in W^{2,2}_{loc}$. Reinserting this into the right side yields that $u \in W^{4,2}_{loc}$ so $u \in W^{k,2}_{loc}$ for all k.

Product Estimates; Littlewood-Paley Trichotomy

For which s,p is $W^{s,p}(\mathbb{R}^d)$ a Banach algebra? A Banach space A is called a **Banach algebra** if it is closed under multiplication: $\|fg\|_A \leq \|f\|_A \|g\|_A$. Simple considerations show that L^p is not a Banach algebra for any $p<\infty$ but L^∞ is an algebra. Heuristically, we thus expect that $W^{s,p}$ will be an algebra provided s,p are arranged so that $W^{s,p}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$. Recall that $s > \frac{p}{d} \Longrightarrow W^{s,p}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$.

proposition If $1<p<\infty$, $s > p/d$ then $\|fg\|_{W^{s,p}(\mathbb{R}^d)} \leq \|f\|_{W^{s,p}(\mathbb{R}^d)} \|g\|_{W^{s,p}(\mathbb{R}^d)}$.

The proof we present introduces some general ideas about decomposing frequency interactions in the product. These ideas are the basis of **paradifferential calculus** and are very important in study of nonlinearity.

proof: we decompose the left side into Littlewood-Paley pieces: $\|fg\|_{W^{s,p}} \sim \|P_{\leq 0}(fg)\|_p + \left\|\left(\sum_{k\geq 0} |2^{sk} P_k(fg)|^2\right)^{\frac12}\right\|_p$

The very low frequency part is fine since $\|P_{\leq 0}(fg)\|_{W^{s,p}} \leq \|fg\|_{L^p} \leq \|f\|_{L^p}\|g\|_{L^\infty} \leq \|f\|_{W^{s,p}}\|g\|_{W^{s,p}}$

To analyze the high freq. part, we consider one of its Littlewood-Paley pieces and then decompose f,g into their Littlewood-Paley pieces.

$$P_k(fg) = \sum_{k',k''\in\mathbb{Z}} P_k\left((P_{k'}f)(P_{k''}g)\right) = P_k\left(\left(\sum_{k'\in\mathbb{Z}} P_{k'}f\right)\left(\sum_{k''} P_{k''}g\right)\right)$$

$$= P_k\left(\left(\sum_{k'\ll k} P_{k'}f + \sum_{k'\sim k} P_{k'}f + \sum_{k''\gg k} P_{k'}f\right)\left(\sum_{k''\ll k} P_{k''}g + \sum_{k''\sim k} P_{k''}g + \sum_{k''\gg k} P_{k''}g\right)\right)$$

$$= P_k\Big(\underbrace{P_{\ll k}f \; P_{\sim k}g}_{\text{low} \quad \text{medium}} + \underbrace{P_{\sim k}f \; P_{\ll k}g}_{\text{medium} \quad \text{low}} + \underbrace{P_{\sim k}f \; P_{\sim k}g}_{\text{medium} \quad \text{medium}} + \underbrace{P_{\gg k}f \; P_{\gg k}g}_{\text{very high} \quad \text{very high}}\Big)$$

Note that the low-low interaction will not contribute anything to $|\xi| \sim 2^k$ so P_k kills that term. The low-medium and medium-low interactions will involve some work. Note that the high-high interaction will have to involve a delicate frequency balance in order to contribute to $|\xi| \sim 2^k$ since generally high-high will spread out over $|\xi| \lesssim 2^{\gg k}$ and we dismiss all the interaction outputs not contributing to $|\xi| \sim 2^k$. The same remark applies to medium-medium since interaction outputs with freq $\ll 2^k$ will be killed off by P_k. We have thus decomposed the interactions contributing to $P_k(fg)$ into low-medium, medium-medium, high-high frequency interaction contributions.

(medium-low similar)

low-medium contribution $\quad \left\| \left\{ \sum_{k>0} \left|2^{ks} P_k(P_{\ll k}f \; P_{\sim k}g)\right|^2 \right\}^{\frac12} \right\|_{L^p_x}$

medium-medium contribution $\quad \left\| \left\{ \sum_{k>0} \left|2^{ks} P_k(P_{\sim k}f \; P_{\sim k}g)\right|^2 \right\}^{\frac12} \right\|_{L^p_x}$

high-high contribution $\quad \left\| \left\{ \sum_{k>0} \left|2^{ks} P_k(P_{\gg k}f \; P_{\gg k}g)\right|^2 \right\}^{\frac12} \right\|_{L^p_x}$

By ideas like we discussed when proving Littlewood-Paley inequality, we have that

$$\left\| \left(\sum_{k\geq 0} |P_k f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p_x} \leq \left\| \left(\sum_{k>0} |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p_x}.$$ (Think of P_k as a projection.)

Therefore, low-medium contribution is bounded by $\left\| \left(\sum_{k>0} \left| 2^{sk} (P_{\ll k} f)(P_{\sim k} g) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p_x}$.

We have a pointwise upper bound on $|P_{\ll k} f(x)|$ in terms of the <u>Hardy-Littlewood maximal operator</u> $Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$. For example, $P_{\ll k} f(x) = P_{\leq k-5} f(x)$ and, recalling the convolution representation of $P_{\leq k-5}$, we have

$$|P_{\leq k-5} f(x)| = \left| \int f(y)\, 2^{d(k-5)} \hat{\phi}(2^{k-5}(x-y))\, dy \right| \lesssim 2^{dk} \int |f(y)| (1 + 2^k |x-y|)^{-100d}\, dy$$

$$\lesssim 2^{dk} \int_{B(x,2^{-k})} |f(y)|\, dy + \sum_{j>0} 2^{dk} 2^{-100dj} \int_{B(x,2^{-k+j})} |f(y)|\, dy$$

$$\lesssim Mf(x) + \sum_{j>0} 2^{-\overset{99}{\cancel{100}}dj}\, 2^{dk} 2^{-dj} \int_{B(x,2^{-k+j})} |f(y)|\, dy$$

$$\lesssim Mf(x) \left[1 + \sum_{j>0} 2^{-99dj} \right] \lesssim Mf(x).$$

$$\boxed{\begin{array}{l} \|Mf\|_{L^\infty} \leq C \|f\|_{L^\infty}. \\ (\text{also } \|Mf\|_{L^p} \lesssim \|f\|_{L^p} \\ \text{for } 1<p<\infty.) \end{array}}$$

low-medium contribution is less than

$$\lesssim \left\| \left(\sum_{k>0} \left| 2^{sk}\, Mf(x)\, P_{\sim k} g(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p_x}$$

$$\lesssim \|Mf\|_{L^\infty} \left\| \left(\sum_{k>0} \left| 2^{sk} P_{\sim k} g(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p_x} \leq \|f\|_{L^\infty} \|g\|_{W^{s,p}}$$

and by Sobolev embedding we have

$$\lesssim \|f\|_{W^{s,p}} \|g\|_{W^{s,p}}.$$

Similar arguments take care of the medium-low and medium-medium contributions.

high-high contribution

$$\Big\| \Big(\sum_{k>0} \Big| 2^{sk} P_k \Big\{ \sum_{k' \gg k} P_{k'} f \Big\} \Big\{ \sum_{k'' \gg k} P_{k''} g \Big\} \Big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p_x}$$

The only way to get a contribution is to have $|k'-k''| < 3$ so essentially $k' = k''$. By the triangle inequality we control by

$$\sum_{\substack{k'', k' \gg k \\ |k'-k''| \leq 3}} \Big\| \Big(\sum_{k>0} \big| 2^{sk} P_k \big(P_{k'} f \, P_{k''} g \big) \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p_x}.$$

We can drop P_k. We control $|P_{k'} f(x)| \leq Mf(x)$. What remains is

$$\sum_{k'' \gg k} \|Mf\|_{L^\infty} \Big\| \Big(\sum_{k>0} \big| 2^{sk} P_{k''} g \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p_x}.$$

We reindex the sums

$$\sum_{b \geq 5} \|Mf\|_{L^\infty} \Big\| \Big(\sum_{k>0} \big| 2^{sk} (P_{k+b} g) \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p_x}.$$

Next, we adjust the derivative weight using $2^{sk} = 2^{s(k+b)} 2^{-sb}$ to obtain

$$\sum_{b \geq 5} \|Mf\|_{L^\infty} \Big\| \Big(\sum_{k>0} \big| 2^{s(k+b)} (P_{k+b} g) \big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p_x} 2^{-sb}$$

$$\lesssim \|f\|_{W^{s,p}} \|g\|_{W^{s,p}}.$$

$$\leq \Big(\sum_{b \geq 5} 2^{-sb} \Big) \|Mf\|_{L^\infty} \|g\|_{W^{s,p}}$$

Multilinear Operators ... [Tao 254A]

The pointwise product operator $(f,g) \longmapsto f(x)g(x)$ is a bilinear operator which may be reexpressed on the Fourier side as $(fg)^\wedge(\xi) = \int_{\xi=\xi_1+\xi_2} \hat{f}(\xi_1)\hat{g}(\xi_2)$. As another example, consider the quadrilinear operator $(f_1,f_2,f_3,f_4) \longmapsto \int f_1(x)f_2(x)f_3(x)f_4(x)\,dx$. Reexpressing the output, we have $\int \left(\int \hat{f}_1(\xi)e^{ix\xi_1}d\xi_1\right)\left(\int \hat{f}_2(\xi_2)e^{ix\xi_2}d\xi_2\right)(\quad_3)(\quad_4)\,dx$

$$= \int_{\xi_1}\int_{\xi_2}\int_{\xi_3}\int_{\xi_4} \left(\int e^{ix(\xi_1+\xi_2+\xi_3+\xi_4)}dx\right)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)\hat{f}_4(\xi_4)\,d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Since $\left(\int e^{ix\cdot\eta}dx\right)$ is the inverse Fourier transform of 1, we recognize this expression as $\delta_{\{\eta=0\}}$. Therefore, we observe that

$$\int f_1(x)f_2(x)f_3(x)f_4(x)\,dx = \int_{\xi_1+\xi_2+\xi_3+\xi_4=0 \leftarrow \Sigma_4(0)} \hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)\hat{f}_4(\xi_4).$$

The first example $(f,g) \longmapsto f(x)g(x) = \int\left[\int_{\xi=\xi_1+\xi_2}\hat{f}(\xi_1)\hat{g}(\xi_2)\right]e^{ix\cdot\xi}d\xi$ maps a pair of functions to a function. The second example $(f_1,f_2,f_3,f_4) \longmapsto \int_{\Sigma_4(0)}\hat{f}_1(\xi_1)\cdots\hat{f}_4(\xi_4)$ maps a 4-tuple of functions to a number. As a special case of the second example, suppose $f_1 = f_3 = f$, $f_2 = f_4 = \bar{f}$. We then observe that $\|f\|_{L^4}^4 = \int_{\Sigma_4}\hat{f}(\xi_1)\hat{\bar{f}}(\xi_2)\hat{f}(\xi_3)\hat{\bar{f}}(\xi_4)$.

We envelop these two examples into a general framework with the following:

Definition: A multilinear Fourier multiplier operator with symbol m of order n is an operator of the form

$$T_m(f_1,\dots,f_n)^\wedge(\xi) = \int_{\xi=\xi_1+\dots+\xi_n} m(\xi_1,\dots,\xi_n)\hat{f}_1(\xi_1)\cdots\hat{f}_n(\xi_n).$$

Examples: ① $T_1(f,g)(x) = \int e^{ix\cdot\xi}T_1(f,g)^\wedge(\xi)\,d\xi = \int e^{ix\cdot\xi}\left(\int_{\xi=\xi_1+\xi_2}\hat{f}(\xi_1)\hat{g}(\xi_2)\right)d\xi = f(x)g(x)$. ② $T_1(f_1,f_2,f_3,f_4)^\wedge(0) = \int_{0=\xi_1+\xi_2+\xi_3+\xi_4}\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)\hat{f}_4(\xi_4) = \int f_1f_2f_3f_4\,dx$. In particular, $T_1(f,\bar{f},f,\bar{f})^\wedge(0) = \|f\|_{L^4}^4$. ③ $(f,g,h) \longmapsto f''gh + f'g'h'$ is a 3-linear Fourier multiplier operator with symbol $\xi_1^2 + \xi_1\xi_2\xi_3$.

Any combination of derivatives and products may be represented using a multilinear Fourier operator so these operators are ubiquotous in nonlinear PDE. Not all such operators are represented in terms of derivatives and products: $M(\xi_1,\xi_2,\xi_3) = \mathrm{sgn}(\xi_2-\xi_3)$.

Many PDE issues boil down to a core question: Does the multilinear Fourier multiplier operator $T_m : L^{p_1} \times \cdots \times L^{p_n} \longrightarrow L^p$? That is, do we have the estimate $\| T_m (f_1, f_2, \dots, f_n) \|_{L^p} \lesssim \|f_1\|_{L^{p_1}} \cdots \|f_n\|_{L^{p_n}}$? For example, consider the issue of whether $\|(fg)\|_{W^{s,p}} \leq \|f\|_{W^{s_1,p_1}} \|g\|_{W^{s_2,p_2}}$? (Last week, we considered this question in the special case when $s = s_1 = s_2$ and $p = p_1 = p_2$ when we showed that $W^{s,p}(\mathbb{R}^d)$ is a Banach algebra when $s > p/d$ using Littlewood-Paley trichotomy.) Consider the 2-symbol $m(\xi_1, \xi_2) = \frac{\langle \xi_1 + \xi_2 \rangle^s}{\langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2}}$. The estimate under consideration is equivalent to the boundedness of the 2-multiplier operator $T_m : L^{p_1} \times L^{p_2} \longrightarrow L^p$.

Multilinear Fourier multipliers and almost conservation laws

The general theory of multilinear Fourier multiplier operators emerges naturally in the study of certain Hamiltonian PDE. We discuss some aspects in the setting of the Korteweg-de Vries equation (KdV). The KdV initial value problem is

$$\text{KdV} \quad \begin{cases} \partial_t u + \partial_x^3 u + \frac{1}{2} \partial_x u^2 = 0, & u : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R} \\ u(0,x) = u_0(x). \end{cases}$$

The main question here is: what happens as u_0 evolves to $u(t)$? Before answering this, however, there are various "pre-questions" to consider such as whether a solution exists at all and, if so, for which data? Does the solution maintain the smoothness properties of the data? Is the solution unique among some large family of possible solutions? What is meant by the term "solution"? We sidestep many of these interesting questions with the goal of showing how certain multilinear operators have arisen in the study of KdV.

Conservation Laws

Multiply the KdV equation by u and rearrange to observe:

$$0 = u \partial_t u + u \partial_x^3 u + u \tfrac{1}{2} \partial_x u^2 = \partial_t \left(\tfrac{1}{2} u^2\right) + \partial_x \left\{ u \partial_x^2 u - \tfrac{1}{2} (\partial_x u)^2 \right\}.$$

Now integrate over $-\infty < x < \infty$ and (formally) assume u and its first few derivatives decay as $|x| \to \infty$ to kill the boundary term to observe that

$$\partial_t \int_{-\infty}^{\infty} \tfrac{1}{2} u^2 \, dx = - \int_{-\infty}^{\infty} \partial_x \{ \ \} \, dx = \{ \ \} \Big|_{-\infty}^{+\infty} = 0$$ so $\int u^2(t,x) \, dx = \int u_0^2(x) \, dx$.

Thus, the L^2_x norm of $u(t,x)$ is conserved during the KdV evolution if it is initially finite.

Similarly, we can rewrite the KdV equation as $\partial_t u + \partial_x(\partial_x^2 u + \frac{1}{2}u^2) = 0$ to observe that $\partial_t \int_{-\infty}^{\infty} u(t,x)\,dx = 0$ so the mean value of $u(t,x)$ is conserved during the evolution.

Exercise: $H[u] = \int \frac{1}{2}u_x^2 - \frac{1}{6}u^3\,dx$ is conserved under the KdV flow. Prove it.

In fact, KdV is a completely integrable infinite dimensional Hamiltonian system and there exist infinitely many conserved quantities.

(aside on IST)

(aside on solitons)

Low regularity global well-posedness: In the early 90s, J. Bourgain introduced new ideas into the study of the KdV initial value problem and proved that KdV is locally well-posed for initial data $u_0 \in L^2$. Suppressing certain details, Bourgain showed that for any $u_0 \in L^2$ there is a time $T > 0$ such that $T = T(\|u_0\|_{L^2})$ and there exists a function $u : [0,T]\times\mathbb{R} \to \mathbb{R}$ which "solves" KdV and is unique in a certain Banach space of functions of spacetime $X_T \subset C([0,T]; L^2(\mathbb{R}))$. Thus, we can advance the data u_0 forward under KdV flow to $u(t)$ for $t \in [0,T]$. We then apply this same result to the KdV initial value problem with initial data $u(T)$. Since T is determined by $\|u_0\|_{L^2}$ and $\|u(t)\|_{L^2}$ is conserved, the lifetime of existence of $u(T) \mapsto u(t)$ is the same as before. By iteration, the solution may be extended to an arbitrary time interval so no finite time singularity can occur and the problem is globally well-posed for L^2 initial data.

Bourgain's local well-posedness result relied upon a delicate bilinear analysis of the nonlinearity $\partial_x(uv)$ in the "X_T" function space. This analysis is another example of the relevance of multilinear Fourier multipliers in the study of PDEs but we forge on with a different target in mind. After Bourgain's work, C. Kenig, G. Ponce and L. Vega optimized the bilinear analysis and proved that KdV is in fact locally well-posed for initial data $u_0 \in H^s$ for $s > -\frac{3}{4}$: $\forall\, u_0 \in H^s(\mathbb{R})$ $\exists\, T = T(\|u_0\|_{H^s})$ and $\exists\, u : [0,T]\times\mathbb{R} \to \mathbb{R}$ which "solves" KdV and is unique in $X_T^s \subset C([0,T]; H^s(\mathbb{R}))$.

Low regularity global well-posedness issue: What is the fate of the local-in-time solutions obtained by [KPV]? A multilinear Fourier analysis building certain "almost conserved" quantities related to the L^2 conservation property proved those solutions are in fact global-in-time. [C-Keel-Staffilani-Takaoka-Tao].

Fourier-side proof of L^2 conservation

The KdV equation $\partial_t u = -\partial_x^3 u - \partial_x(\frac{1}{2}u^2)$ may be Fourier transformed w.r.t x to read $\partial_t \hat{u}(\xi) = +i\xi^3\hat{u}(\xi) - i\xi\frac{1}{2}(u^2)^\wedge(\xi)$. The L^2-norm of an $\mathbb{R}$-valued function u may be expressed $\|u\|_{L^2_x}^2 = \|\hat{u}\|_{L^2_x}^2 = \int \hat{u}(\xi)\overline{\hat{u}(\xi)}\,d\xi = \int \hat{u}(\xi)\hat{u}(-\xi)\,d\xi = \int \hat{u}(\xi)\hat{u}(-\xi)\,d\xi$ $= \int_{\xi_1+\xi_2=0}\hat{u}(\xi_1)\hat{u}(\xi_2)$. We can now calculate, assuming $t \mapsto u(t)$ solves KdV, using symmetry

$$\partial_t \|u(t)\|_{L^2}^2 = 2\int_{\xi_1+\xi_2=0}\widehat{u(t)}(\xi_1)\,\partial_t\widehat{u(t)}(\xi_2)$$

$$= 2i\int_{\xi_1+\xi_2=0}\widehat{u(t)}(\xi_1)\,\xi_1^3\,\widehat{u(t)}(\xi_2) - i\int_{\xi_1+\xi_2=0}\xi_2(u^2)^\wedge(\xi_2)\,\widehat{u(t)}(\xi_1)$$

$$= i\int_{\xi_1+\xi_2=0}\widehat{u(t)}(\xi_1)\cancelto{0}{(\xi_1^3+\xi_2^3)}\,\widehat{u(t)}(\xi_2) - i\int_{\xi_1+\xi_2+\xi_3=0}\widehat{u(t)}\,\underbrace{(\xi_2+\xi_3)}\,\hat{u}\,\cancel{(t)}(\xi_2)\,\widehat{u(t)}(\xi_3)$$

$$-\xi_1 \to -\frac{1}{3}\cancelto{0}{(\xi_1+\xi_2+\xi_3)}$$

$$= 0.$$

In particular, the L^2-conservation property is seen to hold on the Fourier transform side pointwise in ξ. Therefore, we may be able to localize the calculation to obtain new dynamical insights into the motion of L^2-mass or to consider situations where there is infinite mass.

Let's imitate the preceding calculation but for a Fourier multiplier transformed u. Let $\widehat{Iu}(\xi) = m(\xi)\hat{u}(\xi)$ where m is some (undetermined at this point) Fourier multiplier.

$$\|Iu(t)\|_{L^2}^2 = \|\widehat{Iu(t)}\|_{L^2_\xi}^2 = \int_{\xi_1+\xi_2=0} m(\xi_1)\hat{u}(\xi)\,\overline{m}(-\xi_2)\,\hat{u}(\xi_2).$$

$$\partial_t \|Iu(t)\|_{L^2}^2 = i\int_{\xi_1+\xi_2=0} m(\xi_1)\overline{m(-\xi_2)}\left\{\cancelto{0}{\xi_1^3+\xi_2^3}\right\}\widehat{u(t)}(\xi_1)\,\widehat{u(t)}(\xi_2)$$

$$+ i\int_{\xi_1+\xi_2=0}\xi_2\, m(\xi_1)\overline{m(-\xi_2)}\,\hat{u}(\xi_1)\,(u^2)(t)^\wedge(\xi_2)$$

$$= i\int_{\xi_1+\xi_2+\xi_3=0}(\xi_2+\xi_3)m(\xi_1)\,\overline{m}(-(\xi_2+\xi_3))\,\hat{u}(\xi_1)\,\hat{u}(\xi_2)\,\hat{u}(\xi_3)$$

$$= i\int_{\xi_1+\xi_2+\xi_3=0}(-\xi_1)\,m(\xi_1)\,\overline{m}(+\xi_1)\,\hat{u}(\xi_1)\,\hat{u}(\xi_2)\,\hat{u}(\xi_3)$$

$$= -\frac{i}{3}\int_{\ldots}\sum_1^3 m(\xi_j)\overline{m}(\xi_j)\,\xi_j\;\hat{u}(\xi_1)\,\hat{u}(\xi_2)\,\hat{u}(\xi_3).$$

For the low regularity global well-posedness issue, let's consider the following choice of Fourier multiplier:

$$m(\xi) = \begin{cases} 1 & |\xi| < N \\ \left(\frac{|\xi|}{N}\right)^s & |\xi| > 2N \end{cases}$$

with m smooth, monotone, even, $\mathbb{R}$-valued

Here N is a parameter. When $N = \infty$, m is the identity operator. When $N = 1$, m is essentially $(1 + |\xi|^2)^{s/2}$. The parameter N separates low and high frequencies. If $\phi \in H^s$, for $s < 0$, then $\| I \phi \|_{L^2}$ is bounded. Let's time integrate the identity we obtained from $0 \leq t \leq T_{lwp}$. We find

$$\| I u(T_{lwp}) \|_{L^2}^2 - \| I u_0 \|_{L^2}^2 = -\frac{i}{3} \int_0^{T_{lwp}} \int_{\xi_1+\xi_2+\xi_3=0} \left[m^2(\xi_1)\xi_1 + m^2(\xi_2)\xi_2 + m^2(\xi_3)\xi_3 \right] \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3)$$

Using Littlewood-Paley theory and local well-posedness ideas, let's imagine we could prove that the right side in absolute value can be shown to be of the size $N^{-\beta} \| I u_0 \|_{L^2}^3$. We would then obtain an "almost" conservation law:

$$(\text{Aclaw}) \qquad \| I u(T_{lwp}) \|_{L^2}^2 = \| I u_0 \|_{L^2}^2 + N^{-\beta} \| I u_0 \|_{L^2}^3 .$$

For "unit sized" data satisfying $\| I u_0 \|_{L^2}^3$, there is a tiny $(N^{-\beta})$ increment in the size of the solution as it moves from time 0 to time T_{lwp}. This should be compared with what we know from the local theory alone:

$$(\text{local}) \qquad \| I u(T_{lwp}) \|_{L^2} \leq 2 \| I u_0 \|_{L^2}^2 = \| I u_0 \|_{L^2}^2 + \| I u_0 \|_{L^2}^2 .$$

The (local) estimate iterates after L steps of the local theory to give an exponential upper bound on $\| I u(t_j) \|_{L^2}$ along the sequence of subsequent local well-posedness times. In contrast, (Aclaw) iterates gives

$$\| I u(L T_{lwp}) \|_{L^2}^2 = \| I u_0 \|_{L^2}^2 + L N^{-\beta} \| I u_0 \|_{L^2}^3$$

provided $L N^{-\beta} \lesssim 1$ so we postpone the doubling of $\| I u(t) \|_{L^2}$ until far into $(L T_{lwp}(\| I u_0 \|_{L^2}))$ the future. Quantifying these ideas depends upon the exact value of β in our imagined (Aclaw). Bigger β's correspond to better control so we'd also like to improve things by making β larger. We automate the preceding calculation and describe a multilinear algorithm for producing "better" almost conserved quantities that have larger β's.

Multilinear Correction Terms [CKSTT ; JAMS]

Definition A k-multiplier is a function $m: \mathbb{R}^k \longrightarrow \mathbb{C}$. A k-multiplier is symmetric if $m(\xi_1, \dots, \xi_k) = m(\sigma(\xi_1, \dots, \xi_k))$ $\forall \sigma \in S_k$. The symmetrization of a k-multiplier is

$$[M]_{sym}(\xi_1, \dots, \xi_k) = \frac{1}{k!} \sum_{\sigma \in S_k} m(\sigma(\xi_1, \dots, \xi_k)).$$

The domain of a k-multiplier is $\mathbb{R}^k$ but we will only be interested in it on the convolution hypersurface

$$\Sigma_k = \{ (\xi_1, \dots, \xi_k) \in \mathbb{R}^k : \xi_1 + \dots + \xi_k = 0 \}.$$

Definition A k-multiplier generates a k-linear functional or k-form acting on k functions $u_1, u_2, \dots, u_k$ by the formula

$$\Lambda_k(M_k ; u_1, \dots, u_k) = \int_{\xi_1 + \dots + \xi_k = 0} m(\xi_1, \dots, \xi_k) \, \hat{u}_1(\xi_1) \cdots \hat{u}_k(\xi_k).$$

We will often apply Λ_k to k copies of the same function u. When this occurs, we will simplify the notation: $\Lambda_k(m_k ; u, u, \dots, u) = \Lambda_k(m)$. If m is symmetric, we will say that $\Lambda_k(m)$ is a symmetric k-linear functional.

Examples $\Lambda_2(1) = \int_{\xi_1 + \xi_2 = 0} \hat{u}(\xi_1) \hat{u}(\xi_2) = \|u\|^2_{L^2}$.

Proposition: Suppose u satisfies the KdV equation and that m is a symmetric k-multiplier. Then

$$\frac{d}{dt} \Lambda_k(m) = \Lambda_k(m \, \alpha_k) - i \frac{k}{2} \Lambda_{k+1}(m(\xi_1, \dots, \xi_{k-1}, \xi_k + \xi_{k+1}) \{ \xi_k + \xi_{k+1} \})$$

where $\alpha_k = i(\xi_1^3 + \xi_2^3 + \dots + \xi_k^3)$. Note that the second term may be symmetrized.

The proof follows by a direct calculation.

Example:

$$\frac{d}{dt} \Lambda_2(1) = \Lambda_2(1 \, i(\xi_1^3 + \xi_2^3)) - i \Lambda_3(\xi_2 + \xi_3)$$

$$= 0 - \frac{i}{3} \Lambda_3(-\xi_1 - \xi_2 - \xi_3) = 0.$$

For an arbitrary even $\mathbb{R}$-valued $\underline{1}$-multiplier m, set $\widehat{Iu}(\xi) = m(\xi)\hat{u}(\xi)$, and define the <u>modified energy</u> $E_I^2(t) = \|Iu(t)\|_{L^2}^2$. By Plancherel and since m and u are $\mathbb{R}$-valued, we have $E_I^2(t) = \Lambda_2(m(\xi_1)m(\xi_2))$. By the Proposition, we can calculate $\partial_t E_I^2 = \Lambda_2(m(\xi_1)m(\xi_2)\cancelto{0}{\alpha_2}) - i\Lambda_3(m(\xi_1)m(\xi_2+\xi_3)\{\xi_2+\xi_3\})$ which we can symmetrize to get $\partial_t E_I^2(t) = \Lambda_3(-i[m(\xi_1)m(\xi_2+\xi_3)\{\xi_2+\xi_3\}]_{sym})$. If $m=1$, the symmetrization process produces $M_3 = c(\xi_1+\xi_2+\xi_3)$ and we have reproduced the proof of L^2 mass conservation. Let's define for later use the (explicit) 3-multiplier that is given to us by the Proposition:
$M_3(\xi_1,\xi_2,\xi_3) = -i[m(\xi_1)m(\xi_2+\xi_3)\{\xi_2+\xi_3\}]_{sym}$.

Now, form the new modified energy $E_I^3(t) = E_I^2(t) + \Lambda_3(\sigma_3)$ with σ_3 a 3-multiplier to be chosen. Using the preceding calculation and the proposition again, we calculate:

$$\partial_t E_I^3(t) = \partial_t E_I^2(t) + \partial_t\Lambda_3(\sigma_3) = \partial_t\Lambda_2(m(\xi_1)m(\xi_2)) + \partial_t\Lambda_3(\sigma_3)$$

$$= \Lambda_2(m(\xi_1)m(\xi_2)\, \cancelto{0}{i(\xi_1^3+\xi_2^3)}) + \Lambda_3(M_3) + \Lambda_3(\sigma_3\alpha_3) + \Lambda_4(M_4)$$

where M_4 is explicitly expressed in terms of σ_3. In particular, we have that $M_4(\xi_1,\xi_2,\xi_3,\xi_4) = [\frac{3}{2}\sigma_3(\xi_1,\xi_2,\xi_3+\xi_4)\{\xi_3+\xi_4\}]_{sym}$. We choose σ_3 to acheive cancellation of the 3-linear terms by writing $\sigma_3 = -\frac{M_3}{\alpha_3}$. Of course, there is the possibility that this is ill-defined but let's ignore that for the moment and suppose it is fine.

Next, we form the newer modified energy $E_I^4(t) = E_I^3(t) + \Lambda_4(\sigma_4)$ with σ_4 to be chosen. We then calculate

$$\partial_t E_I^4(t) = \Lambda_4(M_4) + \Lambda_4(\sigma_4\alpha_4) + \Lambda_5(M_5)$$

with M_5 explicit once we choose σ_4. We choose $\sigma_4 = -\frac{M_4}{\alpha_4}$ to acheive a cancellation of 4-linear terms. Presuming that the σ's can be so defined, we obtain a sequence of modified energies $E_I^n(t)$ satisfying $\partial_t E_n^I(t) = \Lambda_{n+1}(M_{n+1})$.

Exercise Let $E_D^2(t) = \|\partial_x u(t)\|_{L_x^2}^2 = \Lambda_2((i\xi_1)(i\xi_2))$. Apply the multilinear correction term algorithm to $E_D^3(t) = E_D^2(t) + \Lambda_3(\sigma_3)$ to prove, for the appropriately chosen σ_3, that $\partial_t E_D^3 = 0$.

[CKSTT: JAMS] showed that σ_3, σ_4 are well-defined by explicitly showing that M_3 contains α_3 as a factor, at least on the surface Σ_3. Similarly $M_4|_{\Sigma_4}$ contains α_4 as a factor. Therefore, the algorithm is well-defined up to $E_I^4(t)$ and we obtain that $\partial_t E_I^4(t) = \Lambda_5(M_5)$ with an explicit $M5$. We then showed that

$$\left| \int_0^{t_{lwp}} \Lambda_5(M_5; v_1, \dots, v_5)\, dt \right| \lesssim N^{-\beta} \prod_{j=1}^{5} \| I v_j \|_{X^{0,\frac{1}{2}+}}$$

by using Bourgain / Kenig-Ponce-Vega local well-posedness arguments with $\beta = 3 + \frac{3}{4}-$. This decay was sufficient to prove that the Kenig-Ponce-Vega solutions extend to global-in-time solutions.

Research Project Use computer algebra tools (or perform very organized calculations perhaps based upon induction) to prove that σ_n is well-defined for all $n \in \mathbb{N}$. Apply the preceding to $E_{D^j}^2(t) = \|\partial_x^j u(t)\|_{L^2}^2$ using the multilinear correction terms algorithm to prove there exists $K(j)$ finite (probably $K(j) < Cj$) such that $E_{D^j}^{K(j)}(t)$ is exactly conserved. The same results should also be available for 1d cubic NLS.

Bootstrap and Continuity Arguments [Tao: CBMS]

For evolution problems, there is a procedure for proving bounds on the evolution that is "of almost magical power". The idea is called a "bootstrap argument" or an application of the "continuity method". These types of arguments are ubiquitous in the literature but the discussion in research articles is often so short that it is difficult to grasp what is going on. T. Tao has written an abstract version of the "bootstrap principle" which opens up the literature to a more clear perspective. We'll describe the abstract idea and then apply it to an ODE problem for illustration.

Abstract Bootstrap Principle: Let I be a time interval and for each $t \in I$ suppose we have two statements, a "hypothesis" $H(t)$ and a "conclusion" $C(t)$. Suppose we can verify the following four assertions:

(a) (Hypothesis $\Longrightarrow$ Conclusion) If $H(t)$ is true for some $t \in I$ then $C(t)$ is also true for that time t.

(b) (Conclusion is stronger than hypothesis) If $C(t)$ is true for some $t \in I$ then $H(t')$ is true for all $t' \in I$ in a neighborhood of t.

(c) (Conclusion is closed) If $t_1, t_2, \ldots$ is a sequence of times in I which converges to another time $t \in I$ and $C(t_n)$ is true for all t_n then $C(t)$ is also true.

(d) (Base case) $H(t)$ is true for at least one time $t \in I$.

Then $C(t)$ is true for all $t \in I$.

Remark: When this principle is applied, (b), (c), (d) are usually obvious and (a) involves a nonlinear step which requires proof. What does this accomplish? The task "Prove that $C(t)$ holds $\forall\, t \in I$" may be replaced by the seemingly easier task "Prove that $H(t) \Longrightarrow C(t)$ holds". The hypothesis $H(t)$ provides a toe hold toward the stronger statement $C(t)$.

<u>proof</u>: Let $\Omega = \{t \in I : C(t) \text{ holds}\}$. Property (d) + Property (a) $\Longrightarrow$ Ω is nonempty. Property (b) $\Longrightarrow$ $H(t')$ holds for t' in a Nbhd so Property (a) $\Longrightarrow$ Ω is an open set. Property (c) $\Longrightarrow$ Ω is closed. Thus, Ω is both open and closed and Ω is nonempty $\Longrightarrow$ $\Omega = I$.

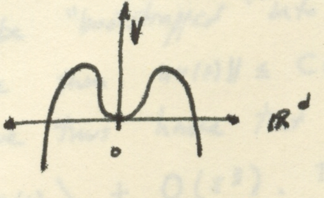
Feedback loop picture.

$$\begin{tikzcd}
 & \text{Conclusion } C(t)\ \text{(closed)} \arrow[d, bend right, "\text{(continuity)}"'] \\
\text{Base case } H(t_0) \arrow[r] & \text{Hypothesis } H(t) \arrow[u, bend right, "\text{nonlinear implication}"']
\end{tikzcd}$$

Informal Bootstrap Principle: If a quantity u can be bounded in a nontrivial way in terms of itself, then under reasonable conditions, one can conclude that u is bounded unconditionally.

Next, we implement the abstract bootstrap principle in the study of a (Hamiltonian) ODE. This is an example illustrating the bootstrap principle in action.

Proposition: Let $V: \mathbb{R}^d \longrightarrow \mathbb{R}$ be such that $V(0) = 0$, $\underline{\nabla} V(0) = \underline{0}$ and $\nabla^2 V(0) > 0$. Let $u_0, u_1 \in \mathbb{R}^d$ with $\|u_0\| + \|u_1\|$ small enough. Then there is a unique classical global-in-time solution $u \in C^2_{loc}(\mathbb{R}_t \to \mathbb{R}^d)$ to the Cauchy problem



$$\begin{cases} \partial_t^2 u(t) = -\nabla V(u(t)) \\ u(0) = u_0 \ ; \ u_t(0) = u_1 . \end{cases}$$

<u>proof</u>: By standard ODE theory based upon Picard iteration we know $\exists$ maximal interval of existence $I = (T_-, T_+) \ni 0$ which supports a unique classical solution $u \in C^2_{loc}(I \to \mathbb{R}^d)$. If $T_+ < \infty$ we know that

$\lim_{t \nearrow T^+} \|u(t)\| + \|u_t(t)\| = \infty$, similarly for T_-.

For any $t \in I$, define $E(t) = \frac{1}{2}\|\partial_t u(t)\|^2 + V(u(t))$. Let's calculate
$\partial_t E(t) = \langle \partial_t u(t), \partial_t^2 u(t)\rangle + \langle \partial_t u(t), \nabla V(t)\rangle = \langle \partial_t u(t), \partial_t^2 u(t) + \nabla V(u(t))\rangle \overset{=}{0}$.
Thus the quantity $E(t)$ does not change with time and we call it the energy.
We thus have the conservation law: $E(t) = \frac{1}{2}\|u_1\|^2 + V(u_0)$. By
choosing the initial data u_0, u_1 sufficiently close to 0 we can ensure
that $E(t) = E(0)$ is as small as we please.

However, we cannot conclude from the smallness of $E(t)$ that we have
smallness of $u(t)$ for all times since V can be arbitrarily
negative. In order for $u(t)$ to feel the negative values of V,
it must be then that $u(t)$ has gotten large. Thus, we need
to assume that $u(t)$ is small in order to prove that $u(t)$ is
small. This seems circular but the bootstrap principle allows us to
justify the argument.

Let $\varepsilon > 0$ be a small parameter to be chosen later. Let $H(t)$
denote the statement that $\|\partial_t u(t)\|^2 + \|u(t)\|^2 \le (2\varepsilon)^2$. Let $C(t)$
denote the (stronger) statement that $\|\partial_t u(t)\|^2 + \|u(t)\|^2 \le \varepsilon^2$. Since
$t \longmapsto u(t)$ is C^2_{loc} we can verify properties (b) and (c) of the
bootstrap principle. If u_0, u_1 are sufficiently close to $\underline{0}$ (depending
upon ε) we certainly have statement (d) for time $t_0 = 0$.

Next, we verify that the hypothesis $H(t)$ may be "bootstrapped" into
the stronger statement $C(t)$. If $H(t)$ is true then $\|u(t)\| \le C\varepsilon$.
From the hypotheses on V and Taylor expansion we thus have that
$V(u(t)) = V(\underline{0}) + \underline{\nabla} V(0) \cdot \underline{u}(t) + \langle (\text{Hess } V)\, u(t), u(t)\rangle + O(\varepsilon^3)$. Thus
$V(u(t)) \ge c\,\|u(t)\|^2 + O(\varepsilon^3)$. We insert this into the energy:

$$V(u(t)) + \frac{1}{2}\|u_t(t)\|^2 = E(t) = E(0) \quad \text{so}$$

$$c\|u(t)\|^2 + \frac{1}{2}\|u_t(t)\|^2 \le E(0) + O(\varepsilon^3).$$

Thus, if we choose u_0, u_1 small enough we will have $E(0)$ small
and we obtain the conclusion $C(t)$. This closes the bootstrap and
allows us to conclude $C(t)$ holds $\forall t \in I$. Thus, I must be

infinite since we know that $\|\partial_t u(t)\|^2 + \|u(t)\|^2$ would blow up at any finite endpoint of I and is cts. w.r.t. t. This completes the proof of the proposition.

<u>Remark</u> The bootstrap argument may be thought of as placing an "impenetrable barrier" $\varepsilon^2 < \|\partial_t u(t)\|^2 + \|u(t)\|^2 \le (2\varepsilon)^2$ in phase space. Property (a) asserts that the system $t \mapsto u(t)$ cannot venture into this barrier. Properties (b), (c) guarantee the system $t \mapsto u(t)$ cannot jump from one side of the barrier to the other side. Property (d) tells us the system starts out on the "good" side of the barrier and we then conclude the evolution stays forever on the good side.

<u>Exercise</u> Show by examples that the abstract bootstrap principle fails if any of the four hypotheses are removed.

<u>Exercise</u> Let I be a time interval and $u \in C^0_{loc}(I \to \mathbb{R}^+)$ be a nonnegative function satisfying the inequality $0 \le u(t) \le A + \varepsilon F(u(t))$ for some $A, \varepsilon > 0$ and some function $F: \mathbb{R}^+ \to \mathbb{R}^+$ which is locally bounded. Suppose $\exists\, t_0 \in I$ s.t. $u(t_0) \le 2A$. If ε is sufficiently small depending upon A and F, show that $u(t) \le 2A$ for all $t \in I$. (Informally, this means that whenever you can show $u(t) \le A + \varepsilon F(u(t))$ then we can drop $\varepsilon F(u(t))$ at the expense of another A so $u(t) \le 2A$ $\forall\, t$.)

Calculus of Variations Evans Chapter 8.

Given $U \subset \mathbb{R}^n$ closed, bounded with smooth ∂U. Consider a function $L: \mathbb{R}^n \times \mathbb{R} \times U \longrightarrow \mathbb{R}$ where the independent variables are labelled $p \in \mathbb{R}^n$, $z \in \mathbb{R}$, $x \in U$. With this L, which we call the Lagrangian density, we can form the associated Action Functional $I[w] = \int_U L(Dw, w, x)\, dx$ for any reasonable function $w: U \longrightarrow \mathbb{R}$. Many problems in physics and PDE concern minimization of the action functional over some class of competitors w.

For example, find the function $u: U \to \mathbb{R}$ among all competitors in the set $\mathcal{A}_g = \{ w: U \to \mathbb{R} \mid w|_{\partial\Omega} = g \}$ which minimizes $I[\cdot]$. Toward this goal, we first observe that: If a minimizer u exists then u solves a PDE called the Euler-Lagrange equation.

Why? What is the PDE? If u is a minimizer then $I[u] \le I[u + \tau v]$ $\forall\ v \in C_0^\infty(U)$ $(\Longrightarrow u + \tau v \in \mathcal{A}_g)$. Therefore, $i(\tau) := I[u + \tau v]$ must satisfy $i'(0) = 0$ $\forall$ $v \in C_0^\infty(U)$. A standard calculation then shows $\forall\ v \in C_0^\infty(U)$

$$0 = i'(0) = \lim_{\tau \to 0} \frac{I[u + \tau v] - I[u]}{\tau} = \int_U \{ -D_x \cdot D_p L(Du, u, x) + D_z L(Du, u, x) \} v\, dx.$$

Therefore, since v is essentially arbitrary, we observe the Euler-Lagrange PDE for u must hold: $-D_x \cdot D_p L(Du, u, x) + D_z L(Du, u, x) = 0$.

Examples

- $L(p, z, x) = \frac{1}{2}|p|^2 \longrightarrow -\Delta u = 0$ Laplace's eq.
- $L(p, z, x) = \frac{1}{2} a^{ij} p_i p_j - z f(x) \longrightarrow -(a^{ij} u_{x_i})_{x_j} = f$ $(a^{ij} = a^{ji})$ poisson
- $L(p, z, x) = \frac{1}{2}|p|^2 - F(z) \longrightarrow -\Delta u = F'(u)$. nonlinear poisson
- $L(p, z, x) = (1 + |p|^2)^{1/2} \longrightarrow \left\{ \frac{u_{x_i}}{(1 + |Du|^2)^{1/2}} \right\}_{x_i} = 0$ minimal surface eq.

Remark There exist interesting action functionals not of the form $\int_U L\, dx$. For example, $I[f] = \sup |f(z)|$ with $f \in \{\text{Schlicht}\} = \{ f : G \to \mathbb{C}$ analytic, one-one, $f(z_0) = 0$ for some $z_0 \in G$, $f'(z_0) = 1 \}$ where G is a simply connected domain in $\mathbb{C}$, arises in proof of Riemann Mapping Theorem.

Example $I[w] = \int_0^1 (w_x^2 - 1)^2 + w^2 \, dx$ among all $w \in A =$
$\{ w : [0,1] \longrightarrow \mathbb{R} \, ; \, w'(0) = +1 , \, w'(-1) = +1 \}$. This functional is
bounded from below but minimizing sequences appear to explode into
sawtooth oscillations so no minimizer exists. Thus, if $I[w] = 0$
but the infimum is not realized for any reasonable "function" w.

subject to $u|_{\partial D} = g$

Example Recall that minimizers of the Dirichlet energy on $D \subset \mathbb{R}^2$ are
harmonic functions which satisfy the boundary condition. Harmonic functions on
the disk may be represented using Fourier series as $u(r, \theta) =$
$\frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$ with a_n, b_n certain constants.
The Dirichlet energy may be calculated using this representation to
equal $\int_D |Du|^2 \, dx \, dy = \pi \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2)$, provided the sum converges.
Suppose we choose the boundary data by specifying the Fourier
coefficients a_n, b_n as follows: $b_n = 0$, $a_n = \begin{cases} 0 & n \neq k! \\ k^{-2} & n = k! \end{cases}$.
So the boundary data is $g(\theta) = \sum_{k=1}^{\infty} k^{-2} \cos((k!)\theta)$.
The harmonic extension of $g(\theta)$ into the disk is given by the formula
$u(r, \theta) = \sum_{k=1}^{\infty} k^{-2} r^{k!} \cos((k!)\theta)$ which is well-defined and
satisfies $\lim_{r \to 1} u(r, \theta) = g(\theta)$ and $g(\theta)$ is the uniform limit
of continuous functions so it is continuous. Meanwhile, the Dirichlet
energy is $\pi \sum_{k=1}^{\infty} \frac{(k!)^2}{k^4}$, (BOOM). Thus, "minimizers" may
have infinite action functional.

First Variation Calculation extends to systems: Let $L : M^{m \times n} \times \mathbb{R}^m \times \bar{U} \longrightarrow \mathbb{R}$
and we use $\underline{p} = \begin{pmatrix} p_1^1 & \cdots & p_n^1 \\ \vdots & \ddots & \vdots \\ p_1^m & \cdots & p_n^m \end{pmatrix}$ to denote the $M^{m \times n}$ independent variables
and z, x to denote the $\mathbb{R}^m \times \bar{U}$ variables. We can then form the
action functional $I[\vec{w}] = \int_U L(D\vec{w}(x), \vec{w}(x), x) \, dx$ where

$$D\bar{w} = \begin{pmatrix} Dw^1 \\ \vdots \\ Dw^m \end{pmatrix} = \begin{pmatrix} w^1_{x_1} & \cdots & w^1_{x_n} \\ \vdots & & \\ w^m_{x_1} & \cdots & w^m_{x_n} \end{pmatrix}.$$

Next, let $\bar{v}^T = (v^1, \dots, v^m) \in C_0^\infty(U \to \mathbb{R}^m)$

and we form $i(\tau) = I[\bar{u} + \tau \bar{v}]$ where we assume that $\bar{u}$ is a smooth minimizer of $I[\cdot]$ over some class of competitors $\mathcal{A}$. By assumption, $i'(0) = 0$ so we calculate that $\forall\ \bar{v} \in C_0^\infty(U \rightarrow \mathbb{R}^m)$

$$0 = \int_U \Big[\sum_{i=1}^{n} \sum_{k=1}^{m} L_{p_i^k}(D\bar{u}(x), \bar{u}(x), x)\, v^k_{x_i} + \sum_{k=1}^{m} L_{z^k}(D\bar{u}(x), \bar{u}(x), x)\, v^k \Big] dx$$

$$\Longrightarrow -\sum_{i=1}^{n} \Big\{ L_{p_i^k}(D\bar{u}(x), \bar{u}(x), x) \Big\}_{x_i} + L_{z^k}(D\bar{u}(x), \bar{u}(x), x) = 0 \quad \text{in } U$$

(system of Euler-Lagrange Eqs.)

$\forall\ k \in \{1, \dots, m\}$.

Null Lagrangians The function L is called a <u>null Lagrangian</u> if the system of Euler-Lagrange equations is automatically satisfied by all smooth functions $u: U \rightarrow \mathbb{R}^m$.

For Null Lagrangians, the value of the action functional depends only upon the boundary values.

Proposition Let L be a null Lagrangian. Assume $u, \tilde{u} \in C^2(U \rightarrow \mathbb{R}^m)$ such that $u = \tilde{u}$ on ∂U. Then $I[u] = I[\tilde{u}]$.

proof: Define $i(\tau) = I[\tau u + (1-\tau)\hat{u}]$, $0 \le \tau \le 1$. We calculate

$$i'(\tau) = \int_U \sum_{i=1}^{n} \sum_{k=1}^{m} L_{p_i^k}(\tau Du + (1-\tau)D\tilde{u}, \tau u + (1-\tau)\hat{u}, x)(u^k_{x_i} - \tilde{u}^k_{x_i})$$

IBP IBP Boundary Term ?

$$+ \sum_{k=1}^{m} L_{z^k}(\tau Du + (1-\tau)D\tilde{u}, \tau u + (1-\tau)\hat{u}, x)\, v^k \quad dx$$

We encounter the Euler-Lagrange equation for L evaluated on $w = \tau u + (1-\tau)\tilde{u}$. Since L is a null Lagrangian, the E-L equation is satisfied so we learn that $i'(\tau) = 0$ and we learn $I[u] = I[\tilde{u}]$.

Exercise ⓐ For $m = 1$ (scalar case), identify the Null Lagrangians. ⓑ Read Evans Chapter 8 and write up a short discussion explaining why the determinant function $L(P, z, x) = \det P$, $P \in M^{n \times n}$, is a null Lagrangian.

Brouwer's Fixed Point Theorem

If $u \in C(\overline{B(0,1)} \longrightarrow \overline{B(0,1)})$ then $\exists\, x \in B(0,1)$ such that $u(x) = x$.

proof write $B = \overline{B(0,1)}$. **Claim**: There does not exist a smooth function $w: B \longrightarrow \partial B$ such that $w(x) = x \;\; \forall x \in \partial B$. Thus, the claim assets that there is no smooth extension of the unit radial vector field taking values in the sphere.

proof of claim: Suppose not. Suppose such a function w does exist. Let's write $\tilde{w}(x) = x$, so $\tilde{w}$ is another name for the identity function: $\tilde{w}(x) = x \;\; \forall x \in B$ so in particular $\forall x \in \partial B$. Since the determinant is a null Lagrangian and $w = \tilde{w}$ on ∂B we must have $\int_B \det Dw \, dx = \int_B \det D\tilde{w} \, dx = |B| \neq 0$. Since w takes values in ∂B we have $|w|^2 = 1$. Differentiating, we find $(Dw)^T w = 0$. Since $|w| = 1$, this says that w is an eigenvector of $(Dw)^T$ with zero eigenvalue. So, 0 is an eigenvalue of $Dw \Longrightarrow \det Dw = 0$ in B and we obtain a contradiction. This proves the claim.

Finally, suppose $v: B \longrightarrow B$ is cts. but has no fixed point. Define $w: B \longrightarrow \partial B$ by setting $w(x)$ to be the point of ∂B hit by the ray emanating from $v(x)$ and passing through x. This is well-defined since $x \neq v(x) \;\; \forall x \in B$ under our assumption. Certainly w is cts. and satisfies the conditions of the claim including that $w(x) = x \;\; \forall x \in \partial B$. By the proof of the claim, we know that such a w cannot possibly exist so v must have a fixed point. ∎

Existence of Minimizers

Q: Under what conditions on L do we expect $I[\cdot]$ to have a minimizer?

Note: $f: \mathbb{R} \rightarrow \mathbb{R}$ bounded below need not attain its infimum, e.g. $f(x) = e^{-x^2}$.
To rule out this type of behavior we make a coercivity hypothesis on L.

Assume $1<q<\infty$ is fixed and suppose $\exists\ \alpha>0,\ \beta\geq 0$ such that $L(p,z,x) \geq \alpha|p|^q - \beta\ \ \forall\ p\in\mathbb{R}^n,\ z\in\mathbb{R},\ x\in U.$ $\Longrightarrow$
$I[w] \geq \alpha\|Dw\|^q_{L^q(U)} - \gamma$ with $\gamma = \beta|U|$ so $I[w]\nearrow\infty$ as $\|Dw\|_{L^q}\nearrow\infty$. With this assumption, we naturally consider the set of competitors $\mathcal{A} = \{w\in W^{1,q}(U) : w = g$ on ∂U in trace sense.$\}$

Of course, $I[w]$ is defined $\forall w\in\mathcal{A}$ but its value may be $+\infty$.
We have also seen an example where the inf $I[\cdot]=0$ but the infimum is not attained on $\mathcal{A}$ due to the emergence of wild sawtooth oscillations. This lower bound on L is called a coercivity estimate.

Let's set $m = \inf_{w\in\mathcal{A}} I[w]$ and choose functions $u_k\in\mathcal{A}$ $(k=1,\dots)$ such that $I[u_k]\longrightarrow m$ as $k\to\infty$. The sequence $\{u_k\}$ is called a minimizing sequence. We'd like to show that some subsequence $\{u_{k'}\}$ converges to an actual minimizer but all we know is boundedness of the sequence in $W^{1,q}(U)$. Since $W^{1,q}(U)$ is an infinite dimensional space, boundedness does not imply compactness so we need to impose some condition(s) on L to ensure compactness of the minimizing sequence $\{u_k\}$. By shifting to the weak topology, we overcome the compactness issue: since $1<q<\infty$, $L^q(U)$ is reflexive so we can conclude that $\exists$ subsequences $\{u_{k_j}\}_{j=1}^{\infty}\subseteq\{U\}_{k=1}^{\infty}$ and $u\in W^{1,q}(U)$ s.t. $u_{k_j}\longrightarrow u$ in $W^{1,q}(U)$ (so that $u_{k_j}\longrightarrow u$ in $L^q(U)$ and $Du_{k_j}\longrightarrow Du$ in $L^q(U)$).

But, the weak topology introduces a new difficulty: we cannot conclude that $I[u] = \lim_{j\to\infty} I[u_j]$ since $Du_{k_j}\longrightarrow Du$ does not imply $Du_{k_j}\to Du$ a.e.
We might have weak convergence but a wild explosion of oscillations.
To conclude that a minimizer exists, we need to prove that
$m\leq I[u]\leq \liminf_{j\to\infty} I[u_j]\longrightarrow m.$

Definition The functional $I[\cdot]$ is (sequentially) weakly lower semicontinuous on $W^{1,q}(U)$ provided $I[u]\leq\liminf_{j\to\infty} I[u_j]$ whenever $u_j\longrightarrow u$ weakly in $W^{1,q}(U)$.

We'd like to identify reasonable conditions on the Lagrangian L that ensure that $I[\cdot]$ is weakly lower semicontinuous. Taking inspiration from the second derivative test from calculus, a second variation calculation shows that at a minimum of $I[\cdot]$ (smooth enough), $\sum_{i,j=1}^{n} L_{p_ip_j}(Du(x),u(x),x)\,\xi_i\xi_j\geq 0\ \ \forall\,\xi\in\mathbb{R}^n, x\in U$ must hold.

Theorem Assume L is bounded below and $p \mapsto L(p,z,x)$ is convex $\forall z \in \mathbb{R}$ $\forall x \in U$. Then $I[\cdot]$ is weakly lower semicontinuous on $W^{1,q}(U)$.

proof ① Choose any sequence $\{u_k\}_{k=1}^{\infty}$ with $u_k \to u$ weakly in $W^{1,q}(U)$ and set $l = \liminf_{k\to\infty} I[u_k]$. Our task is to show $I[u] \leq l$.

② We know that $\sup_k \|u_k\|_{W^{1,q}(U)} < \infty$ and we may assume (pass to a subsequence if necessary) that $l = \lim_{k\to\infty} I[u_k]$. By Rellich-Kondrachov Compactness, the bound in $W^{1,q}(U)$ guarantees $\{u_k\}$ is compact in $L^q(U)$ so we may assume (subsequence again) that $u_k \to u$ strongly in $L^q(U)$ and that $u_k \to u$ a.e. U.

③ Fix $\varepsilon > 0$. The a.e. convergence and Egoroff's theorem (If $\{f_n\}$ is a sequence of measurable functions that converge to a real-valued function f a.e. on a measurable set E of finite measure then, given $\eta > 0$, $\exists$ subset $A \subset E$ with $|A| < \eta$ s.t. $f_n \to f$ uniformly on E.) provides a set E_ε with $|U - E_\varepsilon| < \varepsilon$ and $u_k \to u$ uniformly on E_ε. Next, define $F_\varepsilon = \{x \in U : |u(x)| + |Du(x)| < \frac{1}{\varepsilon}\}$ and, since $u \in W^{1,q}(U)$, $|U - F_\varepsilon| \to 0$ as $\varepsilon \to 0$. Finally, set $G_\varepsilon = E_\varepsilon \cap F_\varepsilon$ and observe that $|U - G_\varepsilon| \to 0$ as $\varepsilon \to 0$.

④ (Main Step) Since L is bounded below, we may assume $L \geq 0$. We approximate an convex function from below using a supporting hyperplane:

$$I[u_k] = \int_U L(Du_k(x), u_k(x), x)\,dx \underset{\text{lower bound}}{\geq} \int_{G_\varepsilon} L(Du_k(x), u_k(x), x)$$

$$\geq \int_{G_\varepsilon} L(Du(x), u_k(x), x)\,dx + \int_{G_\varepsilon} D_p L(Du(x), u_k(x), x) \cdot (Du_k(x) - Du(x))\,dx.$$

$\lim_{k\to\infty} \int_{G_\varepsilon} L(Du(x), u_k(x), x)\,dx = \int_{G_\varepsilon} L(Du(x), u(x), x)\,dx$ since $u_k \to u$ uniformly on E_ε, $|u| < \frac{1}{\varepsilon}$ on F_ε and $G_\varepsilon = F_\varepsilon \cap E_\varepsilon$. Similarly, $D_p L(Du, u_k, x) \to D_p L(Du, u, x)$ uniformly on G_ε. Since $Du_k \to Du$ weakly in $L^q(U; \mathbb{R}^n)$ we have then that $\lim_{k\to\infty} \int_{G_\varepsilon} D_p L(Du(x), u_k(x), x)\,dx \cdot (Du_k - Du)(x)\,dx = 0$.

Thus, $l = \liminf_{k\to\infty} I[u_k] \geq \int_{G_\varepsilon} L(Du(x), u(x), x)\,dx$ $\forall \varepsilon > 0$.

As $\varepsilon \to 0$, RHS converges by the monotone convergence theorem and

$$l \geq \int_U L(Dv(x), v(x), x)\,dx = I[v]$$

which completes our task. $\blacksquare$

Remark: Convexity allows us to exploit weak convergence of $Du_k \to Du$.

__Remark:__ There are interesting generalizations of "convexity" as it appears in the preceding theorem for carrying out the corresponding analysis for the case of systems. In particular, the Hadamard-Legendre condition (a.k.a. rank-one convexity), quasiconvexity and polyconvexity have been considered for systems. There are major long-standing open problems in trying to understand the interrelationships among these convexity notions.

Existence of Minimizers

__Theorem__ Assume $\exists\ \alpha > 0,\ \beta \geq 0$ s.t. $\forall\ p \in \mathbb{R}^n,\ z \in \mathbb{R},\ x \in U$ $\quad L(p,z,x) \geq \alpha |p|^q - \beta$ $(1 < q < \infty)$. Assume $p \longmapsto L(p,z,x)$ is convex $\forall\ z \in \mathbb{R},\ x \in U$. Suppose the set of competitors $\mathcal{A}$ is nonempty. Then there exists at least one function $u \in \mathcal{A}$ solving $I[u] = \inf\limits_{w \in \mathcal{A}} I[w]$.

__proof:__ ① Set $m = \inf\limits_{w \in \mathcal{A}} I[w]$. If $m = +\infty$ we are done so assume $m < \infty$. Select a minimizing sequence $\{u_k\}_{k=1}^{\infty}$ such that $I[u_k] \longrightarrow m$ as $k \longrightarrow \infty$.

② We can take $\beta = 0$ so $L \geq \alpha |p|^q$ and $I[w] \geq \alpha \int_U |Dw(x)|^q\, dx$. Since $m < \infty$ we have $\sup\limits_k \|Du_k\|_{L^q} < \infty$.

③ Fix any $w \in \mathcal{A}$. Since u_k and w are both in $\mathcal{A}$, both equal g in the trace sense along ∂U. This means that $u_k - w \in W_0^{1,q}(U)$. By Poincaré's inequality, we have $\|u_k\|_{L^q(U)} \leq \|u_k - w\|_{L^q(U)} + \|w\|_{L^q(U)} \leq \|Du_k - Dw\|_{L^q(U)} + C \leq C$. Therefore, we also have $\sup\limits_k \|u_k\|_{L^q(U)} < \infty$ and $\{u_k\}_{k=1}^{\infty}$ is bounded in $W^{1,q}(U)$.

④ $\exists$ subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and $u \in W^{1,q}(U)$ s.t. $u_{k_j} \longrightarrow u$ weakly in $W^{1,q}(U)$. Moreover, $u \in \mathcal{A}$. Why? $W_0^{1,q}(U)$ is a weakly closed subspace of $W^{1,q}(U)$ and $u_k - w \in W_0^{1,q}(U)$ for any $w \in \mathcal{A}$. Therefore $u - w \in W_0^{1,q}(U)$ so the trace of u along ∂U must equal the trace of $w \in \mathcal{A}$ along ∂U.

By the convexity hypothesis, we know that $I[u] \leq \liminf\limits_{j \to \infty} I[u_{k_j}] = m$. Since $u \in \mathcal{A}$ we also have that $I[u] = m = \min\limits_{w \in \mathcal{A}} I[w]$. ∎

This is a first fundamental result in the "direct methods" in the calculus of variations. To understand the minimization problem for $I[\cdot]$ over the set of competitors $\mathcal{A}$ we proceed directly by constructing a minimizing sequence and study the limit behavior.

__Summary__ $\mathcal{A} = \{ w : U \to \mathbb{R} \mid w \in W^{1,q}(U),\ w = g \text{ on } \partial U \text{ in trace sense} \}$. $L : \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$.

$I[w] = \int_U L(Dw(x), w(x), x)\, dx$. __Minimization Problem:__ Minimize $I[\cdot]$ over $\mathcal{A}$.

- L bounded below, $p \longmapsto L(p,z,x)$ convex $\Longrightarrow$ $I[\cdot]$ weakly lower semicontinuous
- L coercive, $p \longmapsto L(p,z,x)$ convex $\Longrightarrow$ $\exists\ u \in \mathcal{A}$ s.t. $I[u] = \min\limits_{w \in \mathcal{A}} I[w]$.

Minimizers are weak solutions of the Euler–Lagrange Equation.

Under coercivity and convexity hypotheses on L, we have seen that there exist minimizers of the action functional $I[w] = \int_U L(Du(x), u(x), x)\,dx$, $w \in \mathcal{A}$ and that the minimizer $u \in W^{1,q}(U)$. If we knew that $u \in C^2$ then the 1st variation analysis would shows that u classically satisfies the EL equation $-\sum_{i=1}^{n} (L_{p_i}(Du(x), u(x), x))_{x_i} + L_z(Du(x), u(x), x) = 0$. Question: Does the constructed minimizer $u \in W^{1,q}(U)$ satisfy EL in some weak sense? Yes...

Suppose L also satisfies the following growth conditions: $\forall\ p \in \mathbb{R}^n, z \in \mathbb{R}, x \in U$
$|L(p,z,x)| \leq C(|p|^q + |z|^q + 1)$, $|D_p L(p,z,x)| \leq C(|p|^{q-1} + |z|^{q-1} + 1)$,
$|D_z L(p,z,x)| \leq C(|p|^{q-1} + |z|^{q-1} + 1)$.

Motivation for definition of weak solution: Suppose u solves EL with $u = g$ on ∂U. Let's refer to this boundary value problem as (#). Suppose also that u is a smooth minimizer. Multiply the EL equation by $v \in C_c^\infty(U)$ and integrate by parts to get

$$(*) \quad \int_U \sum_{i=1}^{n} L_{p_i}(Du, u, x) v_{x_i} + L_z(Du,u,x)\, v\, dx = 0.$$

Now, assume $u \in W^{1,q}$. By the growth conditions, observe that $|D_p L(Du, u, x)| \leq C(|Du|^{q-1} + |u|^{q-1} + 1) \in L^{q'}(U)$ for $q' = \frac{q}{q-1}$. Same is true for $|D_z L(Du, u, x)|$. By an approximation argument, we see that (*) will remain valid when $u \in W^{1,q}(U)$ $\forall v \in W_0^{1,q}(U)$.

Definition We say $u \in \mathcal{A}$ is a weak solution of the boundary value problem (#) for the EL equation provided $\int_U \sum_{i=1}^{n} L_{p_i}(Du, u, x) v_{x_i} + L_z(Du,u,x)\, v\, dx = 0$ $\forall\ v \in W_0^{1,q}(U)$.

Theorem Assume L satisfies the growth conditions above an $u \in \mathcal{A}$ satisfies $I[u] = \min_{w \in \mathcal{A}} I[w]$. Then u is a weak solution of the b.v.p. (#).

proof: Fix any $v \in W_0^{1,q}(U)$, set $i(\tau) = I[u + \tau v]$. By the growth conditions, we know $i(\tau)$ is finite $\forall\ \tau \in \mathbb{R}$. Form the difference quotient:

$$\frac{i(\tau) - i(0)}{\tau} = \int_U \frac{L(Du + \tau Dv, u + \tau v, x) - L(Du, u, x)}{\tau}\, dx = \int_U L^\tau(x)\, dx.$$

By familiar calculations, using the smoothness properties of L, we know that

$$L^\tau(x) \longrightarrow \sum_{i=1}^{n} L_{p_i}(Du, u, x) v_{x_i} + L_z(Du, u, x)\, v \quad \text{a.e. as } \tau \to 0.$$

By FTOC, we also have $L^\tau(x) = \frac{1}{\tau}\int_0^\tau \frac{d}{ds} L(Du + sDv, u + sv, x)\, ds =$
$\frac{1}{\tau}\int_0^\tau \sum_{i=1}^{n} L_{p_i}(Du + sDv, u + sv, x) v_{x_i} + L_z(Du + sDv, u + sv, x)\, v\, ds$. Using $ab \leq q^{-1}a^q + (q')^{-1}b^{q'}$ with $q^{-1} + (q')^{-1} = 1$ and the growth inequalities we have
$|L^\tau(x)| \leq C(|Du|^q + |u|^q + |Dv|^q + |v|^q + 1) \in L^1(U)$, for each $\tau \neq 0$.

So, L^{τ} is bounded independently of τ. By dominated convergence theorem, we conclude that $i'(0)$ exists and equals $\int_U \sum_{i=1}^{n} L_{p_i}(Du, u, x) v_{x_i} + L_z(Du, u, x) v \, dx$. But, since u is a minimizer, $i(\cdot)$ has a minimum for $\tau = 0$ so $i'(0) = 0$ and u is a weak solution.

Remark: The preceding result showed $I[u] = \min_{w \in \mathcal{A}} I[w] \Longrightarrow u$ is a weak solution of EL. The converse is not necessarily true. However, in the special case where the joint mapping $(p, z) \longmapsto L(p, z, x)$ is convex for each $x \in U$ then weak solution $\Longrightarrow$ minimizer. Why? Suppose $u \in \mathcal{A}$ is a weak solution of EL. Select any $w \in \mathcal{A}$. By convexity of $(p,z) \longmapsto L(p,z,x)$ we have $L(p,z,x) + D_p L(p,z,x) \cdot (q - p) + D_z L(p,z,x) \cdot (w - z) \leq L(q, w, x)$ since the supporting hyperplane lies below a convex surface. Let $p = Du(x)$, $q = Dw(x)$, $z = u(x)$, $w = w(x)$ and integrate over U to get

$$I[u] + \int_U D_p L(Du, u, x) \cdot (Dw - Du) + D_z L(Du, u, x)(w - u) \, dx \leq I[w].$$

The integral on the left side may be viewed as the defining condition for a weak solution wherein we have chosen $w - u$ as the test function. Therefore, this integral vanishes since u is a weak solution and $I[u] \leq I[w]$ $\forall w \in \mathcal{A}$.

Regularity of weak solutions of elliptic Euler-Lagrange equations

$U \subset \mathbb{R}^n$, bdd, open with smooth ∂U. $\mathcal{A} = \{ w : U \to \mathbb{R} ; w \in W^{1,q}(U), w = g \text{ on } \partial U \}$.

$I[w] = \int_U L(Dw(x), u(x), x) \, dx$. We have obtained the following so far:

- certain convexity conditions on $L \Longrightarrow \exists \, u \in \mathcal{A}$ s.t. $I[u] = \min_{w \in \mathcal{A}} I[w]$.
- The minimizer u is a weak solution of the boundary value problem

$$\text{EL} \begin{cases} -\sum_{i=1}^{n} \left(L_{p_i}(Du, u, x) \right)_{x_i} + L_z(Du, u, x) = 0 & \text{in } U \\ u = g & \text{in } \partial U. \end{cases}$$

The weak sense in which u is a solution is that $u \in \mathcal{A}$ and $\forall v \in W_0^{1,q}(U)$

$$\int_U \sum_{i=1}^{n} L_{p_i}(Du, u, x) v_{x_i} + L_z(Du, u, x) v \, dx = 0.$$

We will now specialize the discussion by assuming L has a specific form and, under an ellipticity hypothesis, prove that weak solutions of EL enjoy higher regularity in the interior of U.

We specialize the exponent q to $q=2$ and assume the action functional is of the more restricted form $I[w] = \int_U L(Dw) - wf \, dx$ for $f \in L^2(U)$. We suppose that $\boxed{|D_p L(p)| \leq C(|p|+1)}$, $\forall p \in \mathbb{R}^n$. Under these conditions, we don't satisfy the hypotheses of the theorem since we also need coercivity and convexity. However, the growth condition and the assumption that u is a minimizer ensures that u is a weak solution of EL. Any minimizer $u \in \mathcal{A}$ satisfies

$$\forall v \in H_0^1(U), \quad \int_U \sum_{i=1}^n L_{p_i}(Du) v_{x_i} \, dx = \int_U f v \, dx. \quad \Longleftarrow$$

We wish to show that any weak solution of $-\sum_{i=1}^n (L_{p_i}(Du))_{x_i} = f$ in U is actually better than H_g^1. In particular, we'd like to show $u \in H^2_{loc}(U)$. Without extra conditions on L, this is manifestly false since the family of problems considered at this stage also includes wave equations which do not regularize.

We impose the ellipticity hypothesis that $\exists \theta > 0$ s.t. $\boxed{\sum_{i,j=1}^n L_{p_i p_j}(p) \xi_i \xi_j \geq \theta |\xi|^2}$ $\forall p, \xi \in \mathbb{R}^n$. Let's also assume $\boxed{|D^2 L(p)| \leq C.}$

Theorem (H^2 regularity)

(i) Let $u \in H^1(U)$ be a weak solution of $-\sum_{i=1}^n (L_{p_i}(Du))_{x_i} = f$ in U where L satisfies the (boxed) conditions above. Then $u \in H^2_{loc}(U)$.

(ii) If in addition $u \in H_0^1(U)$ and ∂U is C^2 then $u \in H^2(U)$ with the estimate $\|u\|_{H^2(U)} \leq C \|f\|_{L^2(U)}$.

proof: The proof mimics the proof of H^2 regularity of weak solutions of 2nd order elliptic PDE in divergence form. The idea is to indirectly differentiate u by using difference quotients and eventually prove the difference quotient is upper bounded independent of the differencing perturbation to justify the desired limit.

① Fix open $V \subset\subset U$ and another open W s.t. $V \subset\subset W \subset\subset U$. Choose smooth cutoff $\zeta \in C^\infty(W)$, $0 \leq \zeta \leq 1$, $\zeta = 1$ on V. For small h, we define the difference quotient operator on the kth slot acting on functions $G: \mathbb{R}^n \to \mathbb{R}$ by $D_k^h G(x_1, \dots, x_n) = \frac{1}{h}\{G(x_1, \dots, x_k + h, x_{k+1}, \dots, x_n) - G(x_1, \dots, x_n)\}$. The smallness condition we impose on $|h|$ is related to dist$(\partial W, \partial U)$. Now, substitute the cleverly chosen test function $v = -D_k^{-h}(\zeta^2 D_k^h u)$ into the condition that u be a weak solution to obtain

$$* \quad \sum_{i=1}^n \int_U D_k^h (L_{p_i}(Du)) (\zeta^2 D_k^h u)_{x_i} \, dx = -\int_U f D_k^{-h} (\zeta^2 D_k^h u) \, dx$$

$$② \quad D_k^h L_{p_i}(Du(x)) = \frac{L_{p_i}(Du(x+he_k)) - L_{p_i}(Du(x))}{h}$$

$$= \frac{1}{h}\int_0^1 \frac{d}{ds} L_{p_i}\big(s\,Du(x+he_k) + (1-s)\,Du(x)\big)\,ds$$

$$= \frac{1}{h}\int_0^1 \sum_{j=1}^n L_{p_i p_j}\big(s\,Du(x+he_k) + (1-s)Du(x)\big)\,ds \cdot \big[u_{x_j}(x+he_k) - u_{x_j}(x)\big]$$

$$= \sum_{j=1}^n a_{ij}^h(x)\, D_k^h u_{x_j}(x).$$

Where $a_{ij}^h(x) = \int_0^1 L_{p_i p_j}\big(s\,Du(x+he_k) + (1-s)\,Du(x)\big)\,ds \quad i,j = 1,\dots,n.$

Insert this into $*$.

$$\sum_{i,j=1}^n \int_V a_{ij}^h(x)\, D_k^h u_{x_j}(x)\,\big(\underset{(A2)}{\zeta^2}\, \underset{(A1)}{D_k^h u}\big)_{x_i}\,dx = -\int_V f\, \underset{(B)}{D_k^{-h}(\zeta^2 D_k^h u)}\,dx$$

$$(A1) = \sum_{i,j=1}^n \int_V \zeta^2 a_{ij}^h(x)\, D_k^h u_{x_j}\, D_k^h u_{x_i}\,dx \geq \theta \int_V \zeta^2 |D_k^h Du|^2\,dx.$$

$$(A2) = \sum_{i,j=1}^n \int_V a_{ij}^h(x)\, D_k^h u_{x_j}\, 2\zeta\zeta_{x_i}\, D_k^h u\,dx.$$

Using $|D^2 L(p)| \leq C \Rightarrow a_{ij}^h \in L^\infty$ and standard bounds on ζ together with $ab = a\sqrt{\varepsilon}\,\frac{1}{\sqrt{\varepsilon}}b \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$ we bound $A2$:

$$|(A2)| \leq C\int_V \zeta\,\varepsilon\,|D_k^h u_{x_j}|^2\,dx + C\,\frac{1}{\varepsilon}\int_V \zeta\,|D_k^h u|^2\,dx.$$

Choosing ε small enough, the 1st term in the $|(A2)|$ upper bound may be absorbed into the lower bound on $(A1)$ at the expense of replacing θ by $\frac{\theta}{2}$. For the term (B) we have

$$|(B)| \leq C\varepsilon\int_V \zeta^2\,|D_k^h Du|^2\,dx + \frac{C}{\varepsilon}\int_V (f^2 + |Du|^2)\,dx$$

Again, the 1st term can be absorbed into the ellipticity lower bound for appropriate ε. Combining the estimates, we obtain

$$c_\theta \int_V \zeta^2\,|D_k^h Du|^2\,dx \leq C\int_V (f^2 + |Du|^2)\,dx \quad \text{for } k = 1,\dots,n.$$

Since RHS is independent of h we obtain $Du \in H^1(V)$ so $u \in H^2_{loc}(U)$ since V is an arbitrary open subset of U.

What about higher regularity? We distinguish the regularity theory for (nonlinear) EL
boundary value problem under the ellipticity hypothesis from the simpler case of
linear 2nd order elliptic equations in divergence form. For the equation

$$-\sum_{i,j=1}^{n} (a^{ij}(x) u_{x_i})_{x_j} + \sum_{i=1}^{n} b^{i}(x) u_{x_i} + c(x) u = f \text{ in } U \text{ with } a^{ij} \text{ elliptic:}$$

- ∂U is C^2; $a^{ij} \in C^1(\bar{U})$; $b^i, c \in L^\infty(U)$; $f \in L^2(U) \Longrightarrow u \in H^2(U)$.
- ∂U is C^{m+2}; $a^{ij}, b^i, c \in C^{m+1}(\bar{U})$; $f \in H^m(U) \Longrightarrow u \in H^{m+2}(U)$.

We are considering instead $-\sum_{i=1}^{n} (L_{p_i}(Du))_{x_i} = f$ in U with $|D^2L(p)| \le C$,
L uniformly convex in p. We do not have the luxury of hypothesizing extra
regularity of a^{ij} in this case since the regularity of a^{ij} w.r.t. x
is linked with the regularity of $u(x)$ through the smooth function L.
This linkage is exploited using a bootstrap argument: weak regularity of u
$\Longrightarrow$ weak regularity of $a^{ij}(u) \Longrightarrow$ less weak regularity of $u \Longrightarrow$ less weak
regularity of $a^{ij}(u) \Longrightarrow$ strong regularity of $u \Longrightarrow$ strong regularity of $a^{ij}(u) \Longrightarrow \cdots$

Sketch of higher regularity u is a weak solution of EL so $\int_U \sum_{i=1}^{n} L_{p_i}(Du) v_{x_i}\, dx =$
$\int_U f v\, dx \quad \forall v \in H_0^1(U)$. Let's assume $f = 0$ for simplicity and for some
$w \in C_c^\infty(U)$, $k \in \{1, \ldots, n\}$, choose $v = -w_{x_k}$. We have then that

$$0 = \int_U -\sum_{i=1}^{n} L_{p_i}(Du(x))\, w_{x_i x_k}\, dx = \int_U \sum_{i,j=1}^{n} L_{p_i p_j}(Du(x))\, \underbrace{u_{x_j x_k}}_{L^2 \text{ by previous theorem}}\, w_{x_i}\, dx.$$

The H^2 regularity result justifies this integration by parts. Set $\tilde{u} = u_{x_k}$ and
$a^{ij}(u) = L_{p_i p_j}(Du)$. $\forall$ open $V \subset\subset U$, an approximation argument gives $\forall w \in H_0^1(V)$

$$\int_V \sum_{ij} a^{ij}(u)(x)\, \tilde{u}_{x_j}\, w_{x_i}\, dx = 0$$

so $\tilde{u} \in H^1(V)$ is a weak solution of

$$\# \quad -\sum_{i,j=1}^{n} (a^{ij}(u)(x)\, \tilde{u}_{x_j})_{x_i} = 0 \text{ in } V.$$

But $a^{ij}(u) \in L^\infty(V)$ and we do not
have here that $a^{ij}(u) \in C^1(V)$. We want to infer regularity of the weak
solution $\tilde{u}$ in the extreme situation where a^{ij} is merely bounded and measurable.

Deep Theorem (De Giorgi–Nash) Any weak solution of # (with $a^{ij} \in L^\infty$) must be locally
Hölder cts. for some exponent $\gamma > 0$. Thus, if $W \subset\subset V$ then $\tilde{u} \in C^{0,\gamma}(W)$
Since $a^{ij} = L_{p_i p_j}(Du)$ we have $a^{ij} \in C^{0,\gamma}_{loc}(U)$.

Schauder's Theorem Any weak solution of # (with $a^{ij} \in C^{0,\gamma}_{loc}(U)) \Longrightarrow u \in C^{2,\gamma}_{loc}(U)$.
$\Longrightarrow u \in C^{1,\gamma}_{loc}(U)$.
But then $a^{ij} \in C^{1,\gamma}_{loc}(U)$ so $u \in C^{3,\gamma}_{loc}(U)$ and we bootstrap to $u \in C^\infty(U)$.

Hörmander - Mikhlin Multiplier Theorem Let $m(\xi)$ be a Fourier multiplier (symbol) such that $|\nabla^j m(\xi)| \lesssim |\xi|^j$ for all $j \geq 0$ (with implicit constant depending upon j). Let T_m be the associated Fourier multiplier operator: $(T_m f)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$. Then T_m is bounded on L^p for all $1 < p < \infty$.

proof: Let $\tilde{P}_k = P_k T_m$, $\tilde{\tilde{P}}_k = \sum_{k-2 \leq k' \leq k+2} P_{k'}$. Then $T_m = \sum_{k,k'} P_k T_m P_{k'} = \sum_k \tilde{P}_k \tilde{\tilde{P}}_k$. Observe that $\tilde{P}_k$ and $\tilde{\tilde{P}}_k$ are both smooth frequency localization operators to annulus $\{|\xi| \sim 2^m\}$.

We now estimate

$$\|T_m f\|_{L^p} = \Big\|\sum_k \tilde{P}_k(\tilde{\tilde{P}}_k f)\Big\|_{L^p} \overset{*'}{\lesssim} \Big\|\Big(\sum_k |\tilde{\tilde{P}}_k f|^2\Big)^{\frac{1}{2}}\Big\|_{L^p} \overset{*}{\lesssim} \|f\|_{L^p}.$$

Application: $(\Delta f)^\wedge(\xi) = -|\xi|^2 \hat{f}(\xi)$; $(\partial_{x_j}\partial_{x_k} f)^\wedge(\xi) = -\xi_j \xi_k \hat{f}(\xi)$. We can relate the off-diagonal second derivatives to the trace of second derivative matrix (the Laplacian) with the formula $(\partial_{x_j}\partial_{x_k} f)^\wedge(\xi) = \frac{\xi_j \xi_k}{|\xi|^2}(\Delta f)^\wedge(\xi)$. Consider the multiplier $m_{jk}(\xi) = \frac{\xi_j \xi_k}{|\xi|^2}$. This is homogeneous of degree zero and satisfies $|\nabla^\alpha m_{jk}(\xi)| \lesssim |\xi|^{|\alpha|}$. By Hörmander - Mikhlin multiplier theorem, we see that $\|\partial_{x_j}\partial_{x_k} f\|_{L^p(\mathbb{R}^d)} \lesssim \|\Delta f\|_{L^p(\mathbb{R}^d)}$. Thus, the control on the Laplacian implies control on all the second derivatives. This is an example of elliptic regularity theory.

Function Spaces

Since $\|\nabla P_k f\|_{L^p} \sim 2^k \|P_k f\|$ we can embellish our new way to represent the L^p norm via Littlewood-Paley inequality to define various scales of function space. For example,

$$\Big\|\Big(\sum_k |(1+2^k)^s |P_k f(x)||^q\Big)^{\frac{1}{q}}\Big\|_{L^p_x} = \|u\|_{F^{s,p}_q}$$

Triebel-Lizorkin

$$\Big\{\sum_k |(1+2^k)^s \|P_k f\|_{L^p_x}|^q\Big\}^{\frac{1}{q}} = \|u\|_{B^{s,p}_q}$$

Besov.

Littlewood-Paley inequality $\Longrightarrow$ $F^{0,p}_2 = L^p$ when $1 < p < \infty$.

$W^{s,p} = F^{s,p}_2$ when $1 < p < \infty$.