Navier Stokes Equations

1. Reformulations, structural properties, Hodge decomposition

The fluid velocity $\underline{u}(t,x)$ of a viscous, homogeneous, incompressible fluid that fills the entire space $\mathbb{R}^d$ evolves according to the Navier-Stokes equations

$$(1)\qquad \begin{cases} \rho\, \partial_t \underline{u} = \mu \Delta \underline{u} - \rho(\underline{u}\cdot\underline{\nabla})\underline{u} - \underline{\nabla} p + \underline{f} \\ \underline{\nabla}\cdot\underline{u} = 0. \end{cases}$$

The condition $\underline{\nabla}\cdot\underline{u} = 0$ express incompressibility of the fluid. ρ is the (constant) density of the fluid. μ is the viscosity coefficient. p is the (unknown) pressure. Without loss of generality, we may assume $\rho = 1$ and will redenote $\mu = \nu$. (We could also assume $\nu = 1$ but will retain ν for various reasons.) f is external force.

In coordinate notation, we denote $\underline{u} = (u^1, \dots, u^d)$ and can reexpress (1) as

$$(2)\qquad \begin{cases} \partial_t u^l + u^j \partial_j u^l = \nu \Delta u^l - \partial_l p + f^l \\ \partial_l u^l = 0. \end{cases}$$

The tensor product of two vectors is a matrix

$$(\underline{v}\otimes\underline{w})_{lk} = v^l w^k.$$

We calculate

$$\underline{\nabla}\cdot(\underline{v}\otimes\underline{w}) = \partial_l (v^l w^k) = (\partial_l v^l) w^k + v^l \partial_l w^k$$
$$= (\underline{\nabla}\cdot\underline{v})\underline{w} + (v\cdot\underline{\nabla})\underline{w}.$$

For divergence free $\underline{u}$ (so $\underline{\nabla}\cdot\underline{u} = \partial_k u^k = 0$)

$$\underline{\nabla}\cdot(\underline{u}\otimes\underline{u}) = (\underline{u}\cdot\underline{\nabla})\underline{u} = u^j \partial_j u^k.$$

Thus, we can reexpress the Navier - Stokes equation as

$$(3) \quad \begin{cases} \partial_t \underline{u} + \underline{\nabla} \cdot (\underline{u} \otimes \underline{u}) = \nu \Delta \underline{u} - \underline{\nabla} p + \underline{f} \\ \underline{\nabla} \cdot \underline{u} = 0 . \end{cases}$$

This course will mostly address well-posedness properties of the Navier - Stokes <u>initial value problem</u>

$$NS_\nu(\mathbb{R}^d) \begin{cases} \partial_t \underline{u} + (\underline{u} \cdot \underline{\nabla}) \underline{u} = \nu \Delta \underline{u} - \underline{\nabla} p + \underline{f} \\ \underline{\nabla} \cdot \underline{u} = 0 \\ \underline{u}(0, x) = \underline{u}_0(x). \qquad x \in \mathbb{R}^d \end{cases}$$

<u>Q</u>: What happens if we take the <u>divergence</u> of NS_ν equation?

Apply ∂_ℓ to the formulation in (2) and sum ...

$$\underbrace{\partial_t \partial_\ell u^\ell}_{0} + \partial_\ell (u^j \partial_j u^\ell) = \nu \Delta \underbrace{\partial_\ell u^\ell}_{0} - \underbrace{\partial_\ell \partial_\ell}_{\Delta} p + \partial_\ell f^\ell$$

$$(\partial_\ell u^j)(\partial_j u^\ell) + u^j \partial_j \underbrace{\partial_\ell u^\ell}_{0}$$

we find that the pressure p satisfies the <u>pressure equation</u>

$$(4) \quad \boxed{- \Delta p = (\partial_\ell u^j)(\partial_j u^\ell) - \partial_\ell f^\ell.}$$

a Poisson-type equation for p.

The <u>pressure equation</u> is ... Elliptic PDE theory may be used to solve for the pressure p in terms of the velocity $\underline{u}$ and external force $\underline{f}$.

<u>Q</u>: What happens if we take the <u>inner product</u> of NS_ν with $\underline{u}$?

Multiply by u^ℓ and sum ... take $\int_{\mathbb{R}^d_x} \cdots$.

$$\frac{1}{2} \partial_t u^\ell u^\ell + u^\ell (u^j \partial_j u^\ell) = u^\ell \nu \Delta u^\ell - u^\ell \partial_\ell p + u^\ell f^\ell.$$

$$\frac{1}{2} \partial_t \int_{\mathbb{R}^d_x} |\underline{u}|^2 + \underbrace{\int_{\mathbb{R}^d_x} u^\ell (u^j \partial_j u^\ell)}_{?} = - \nu \int_{\mathbb{R}^d_x} |\nabla u|^2 dx + \underbrace{\int_{\mathbb{R}^d_x} (\partial_\ell u^\ell) p \, dx}_{0} + \int_{\mathbb{R}^d_x} u^\ell f^\ell dx$$

$$(\partial_j u^\ell) u^j u^\ell = \frac{1}{2} \partial_j (u^\ell)^2 u^j$$

IBP

$$\sim - \frac{1}{2} (u^\ell)^2 \underbrace{\partial_j u^j}_{0}.$$

Thus, we have the <u>energy identity</u>

$$(5) \quad \frac{1}{2} \partial_t \int_{\mathbb{R}^d_x} |\underline{u}|^2 + \nu \int_{\mathbb{R}^d_x} |\nabla u|^2 dx = \int_{\mathbb{R}^d_x} \underline{u} \cdot \underline{f} \, dx.$$

Integrating in time yields

$$(6) \quad \boxed{\frac{1}{2} \int_{\mathbb{R}^d_x} |\underline{u}(T, x)|^2 dx + \nu \int_0^T \int_{\mathbb{R}^d_x} |\nabla u(t,x)|^2 dx \, dt = \frac{1}{2} \int_{\mathbb{R}^d_x} |\underline{u}_0(x)|^2 dx + \int_{\mathbb{R}^d_x} \underline{u} \cdot \underline{f} \, dx}$$

When $\underline{f} \equiv 0$, we see that the L^2_x - norm of the velocity $\underline{u}$ (formally) decreases under the NS_ν evolution for $\nu > 0$ and is (formally) conserved during the $NS_0 = $ Euler evolution.

Solving the pressure equation

(following [Bertozzi - Majda])

With the notation

$$\mathrm{tr}\,(\nabla \underline{u})^2 = \sum_{j,l=1}^{d} \partial_j u^l \partial_l u^j$$

we can express the pressure equation

$$-\Delta p = \mathrm{tr}\,(\nabla \underline{u})^2 - \nabla \cdot \underline{f} .$$

Lemma Let f be a smooth function on $\mathbb{R}^d$ vanishing fast enough as $|x| \longrightarrow \infty$. Then the solution u of the Poisson equation

(7) $$\Delta u = f$$

with ∇u vanishing as $|x| \nearrow \infty$ is given by

(8) $$u(x) = \int_{\mathbb{R}^d} N(x-y)\, f(y)\, dy$$

where N is the Newtonian Potential

$$N(x) = \begin{cases} \frac{1}{2\pi} \ln|x| & d = 2 \\ \frac{1}{(2-d)\omega_N} |x|^{2-d} & d \geq 3 \end{cases}$$

and $\omega_N = |S^{d-1}|$.

proof (Exercise)

Apply the lemma to the pressure equation (with external force $\underline{f} \equiv 0$). We obtain

$$p(x) = c_d \int_{\mathbb{R}^d} N(x-y) \left(-\mathrm{tr}\,(\nabla \underline{u})^2 (y) \right) dy ,$$

and

(9) $$\nabla p(x) = c_d \int_{\mathbb{R}^d} \frac{(x-y)}{|x-y|^d} \left(-\mathrm{tr}\,(\nabla \underline{u})^2 (y) \right) dy .$$

With this formula for ∇p, we can reexpress NS_ν as

(10) $$\begin{cases} \partial_t \underline{u} + (\underline{u} \cdot \nabla) \underline{u} = \nu \Delta \underline{u} + c_d \int \frac{(y-x)}{|x-y|^d} \left(\mathrm{tr}\,(\nabla u)^2 (y) \right) dy . \\ \nabla \cdot \underline{u} = 0 . \end{cases}$$

In this formulation, the divergence free condition is superfluous.

Why? Reexpress in coordinates, then apply ∂_l, ...

$$\partial_t u^l + u^j \partial_j u^l = \nu \Delta u^l + c_d \int \frac{(x-y)^l}{\cdots} dy .$$

$$\partial_t \underline{\underline{\partial_l u^l}} + \partial_l (u^j \partial_j u^l) = \nu \Delta \underline{\underline{\partial_l u^l}} + \underbrace{c_d \partial^l \int \frac{(x-y)^l}{\cdots} dy}_{\substack{\parallel \\ -\Delta p \\ \parallel \\ \mathrm{tr}\,(\nabla \underline{u})^2}} .$$

$$\mathrm{tr}\,(\nabla \underline{u})^2 + u^j \partial_j \underline{\underline{\partial_l u^l}}$$

Thus, the evolution equation in (10) implies

(11) $$\partial_t \underline{(\partial_l u^l)} + u^j \partial_j \underline{(\partial_l u^l)} = \nu \Delta \underline{(\partial_l u^l)}$$

Suppose V_1 and V_2 are scalars[@] which satisfy

$$\partial_t V_\alpha + u^j \partial_j V_\alpha = \nu \Delta V_\alpha \qquad \text{for } \alpha = 1, 2 .$$

[@] Here we imagine $\underline{u}$ is a fixed solution of the evolution equation in (10). Then zero is a solution of (11) and the Grönwall argument shows that 0 is unique.

Consider the expression $v_1 - v_2$. It satisfies a PDE.

$$\partial_t (v_1 - v_2) + u^j \partial_j (v_1 - v_2) = \nu \, \Delta (v_1 - v_2).$$

Multiply by $v_1 - v_2$ and $\int_{\mathbb{R}^d_x} \cdots$

$$\frac{1}{2} \partial_t \int_{\mathbb{R}^d_x} (v_1 - v_2)^2 \, dx = -\frac{1}{2} \int_{\mathbb{R}^d} \underline{u} \cdot \underline{\nabla} (v_1 - v_2)^2 + \nu \int_{\mathbb{R}^d} (v_1 - v_2) \Delta (v_1 - v_2) \, dx$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} (\underline{\nabla} \cdot \underline{u}) \, (v_1 - v_2)^2 - \nu \int |\nabla (v_1 - v_2)|^2 \, dx.$$

Provided(#) $\| \underline{\nabla} \cdot \underline{u} \|_{L^\infty_x} \le 2C$, we thus have

$$\frac{1}{2} \partial_t \int_{\mathbb{R}^d_x} (v_1 - v_2)^2 \, dx \le C \int (v_1 - v_2)^2 \, dx.$$

By Gronwall's Lemma, we have

$$\int_{\mathbb{R}^d_x} (v_1 - v_2)^2 \, dx \, (T) \le e^{2CT} \int_{\mathbb{R}^d_x} (v_1 - v_2)^2 \, dx \, (0).$$

Returning to (11): If we initially have $\partial_\ell u_0^\ell = \underline{\nabla} \cdot \underline{u} = 0$
then the dynamical unfolding of (11) guarantees that
$\partial_\ell u^\ell (T) = \underline{\nabla} \cdot \underline{u}(T) = 0$ at future times $T > 0$.
Thus, the divergence free condition is superfluous in (10).

We have proved ...

(#) To ensure this condition requires a lot of regularity on $\underline{u}$: e.g. $H^{\frac{d}{2}+1}$.

<u>Proposition (Leray's reformulation of NS_ν)</u>
Solving the NS_ν equation ((11), (12) or (3) with $\underline{f} \equiv 0$) with smooth initial data $\underline{u}_0$ satisfying $\underline{\nabla} \cdot \underline{u}_0 = \partial_\ell u_0^\ell = 0$
is equivalent to solving

$$\partial_t \underline{u} + (\underline{u} \cdot \underline{\nabla}) \underline{u} = \nu \Delta \underline{u} + c_d \int_{\mathbb{R}^d_x} \frac{(\underline{x} - \underline{y})}{|x - y|^d} \, \mathrm{tr} (\underline{\nabla} \underline{u}(t,y))^2 \, dy. \tag{12}$$

The pressure $p(t,x)$ can be recovered from the velocity $\underline{u}(t,x)$ by solving the Poisson equation

$$-\Delta p = \mathrm{tr} (\underline{\nabla} \underline{u})^2.$$

The derivation of (12) may be viewed as a <u>projection</u> of the Navier-Stokes equation <u>onto the divergence free vector fields</u>. We make this notion more precise with some further discussion.

<u>Lemma</u> Let w be a smooth, divergence-free vector field in $\mathbb{R}^d$ and let g be a smooth scalar such that

$$|w(x)| \, |g(x)| = \mathcal{O}(|x|^{1-d}) \quad \text{as} \quad |x| \nearrow \infty. \qquad o \text{ vs. } \mathcal{O}$$

Then, w and ∇g are orthogonal:

$$\int_{\mathbb{R}^d} w \cdot \nabla g \, dx = 0.$$

Proposition (Hodge's decomposition in $\mathbb{R}^d$): Every vector field $\underline{v} \in L^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ has a unique orthogonal decomposition

(13) $$v = w + \nabla g, \quad \operatorname{div} w = 0$$

with the following properties:

(i) $w, \nabla g \in L^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$

(ii) $w \perp \nabla g$ in L^2 s.t. $(w, \nabla g)_0 \equiv \int_{\mathbb{R}^d} w \cdot \nabla g \, dx = 0$

(iii) $\forall$ multi-index $\beta \geq 0$

$$\| D^\beta v \|_{L^2}^2 = \| D^\beta w \|_{L^2}^2 + \| \nabla D^\beta g \|_{L^2}^2.$$

proof. Given v and the formula (13), we apply the divergence operator and obtain $\Delta g = \operatorname{div} v$. Solving this Poisson equation produces g in terms of v. We then determine $w = v - \nabla g$. The vector field w and the scalar function g we thus defined and we now validate the properties claimed in the statement.

$\nabla g \in L^2(\mathbb{R}^d)$. First, assume that $v \in C_0^\infty(\mathbb{R}^d)$ and $\operatorname{spt} v \subset \{|x| \leq R\}$.

We know that

$$\nabla g(x) = c_d \int \frac{x-y}{|x-y|^d} \operatorname{div} v(y) \, dy.$$

For $|x| \geq 2R$, $|y| \leq R$ we have the identity

$$|x-y|^{-d} = |x|^{-d} \left| 1 - 2\frac{x \cdot y}{|x|^2} + \frac{y \cdot y}{|x|^2} \right|^{-\frac{d}{2}}$$

so by Taylor expansion for large $|x|$ we get

$$\nabla g(x) = c_d \frac{x}{|x|^d} \int_{|y| \leq R} \operatorname{div} v(y) \, dy + \mathcal{O}(|x|^{-d}).$$

Since $\operatorname{spt} v \subset \{|x| \leq R\}$, the first term is zero. So, $|\nabla g(x)| = \mathcal{O}(|x|^{-d})$ for large $|x|$, say for $|x| > 2R$. By polar coordinates,

$$\int_{|x| \geq 2R} |\nabla g|^2 \, dx \lesssim c_d \int r^{-2d} r^{d-1} \, dr < \infty.$$

Thus, $\nabla g \in L^2$. Since $v \in L^2$ and $w = v - \nabla g$, $w \in L^2$.

$w \perp \nabla g$ in L^2

By the lemma, all we have to verify is that $|w(x)||g(x)| = o(|x|^{1-d})$ as $|x| \nearrow \infty$. For large $|x|$, by the fundamental solution of the Laplace equation, we have that

$$g(x) \sim \begin{cases} \mathcal{O}(\ln |x|), & d = 2 \\ \mathcal{O}(|x|^{2-d}), & d \geq 3. \end{cases}$$

Since $w = v - \nabla g \sim \mathcal{O}(|x|^{-d})$ (since ∇g does this and $v \in C_0^\infty$. This reveals that $|w(x)||g(x)| \lesssim |x|^{2-2d}$ $(d \geq 3)$ + $\lesssim |x|^{-2} \ln |x|$ which are both $o(|x|^{1-d})$ so the orthogonality lemma implies (ii). Thus (iii) holds for $\beta = 0$.

$g \in C^\infty$?

$$\Delta D^\beta g = \operatorname{div} D^\beta v$$

$$\longrightarrow \| D^\beta v \|_{L^2}^2 = \| D^\beta w \|_{L^2}^2 + \| \nabla D^\beta g \|_{L^2}^2$$

$$\forall \beta \quad \| \nabla D^\beta g \|_{L^2} \leq \| D^\beta v \|_{L^2} \longrightarrow \nabla g \in C^\infty(\mathbb{R}^d).$$

Thus, $w \in C^\infty(\mathbb{R}^d)$.

For general $v \in L^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$, take $\rho \in C_0^\infty(\mathbb{R}^d)$ s.t. $\rho \sim \chi_{B_1}$ and set $v_n(x) = v(x)\, \rho\left(\frac{|x|}{n}\right)$. Then $v_n \in C_0^\infty(\mathbb{R}^n)$ and $v_n \longrightarrow v$ in L^2 so pass to the limit as $n \nearrow \infty$ to conclude the proof.

2. <u>Energy Methods for $NS_\nu(\mathbb{R}^d)$</u> (following [Bertozzi-Majda Ch.3]),

- remarks
- uniqueness of smooth solutions; stability estimate.
- local-in-time existence for $v_0 \in H^m(\mathbb{R}^d)$, $m \geq [\frac{d}{2}] + 2$.
  - Sobolev space Machinery
  - A regularized approximate equation $NS_{\nu,\varepsilon}(\mathbb{R}^d)$
  - Local existence for $NS_{\nu,\varepsilon}$ via Picard Contraction
  - Compactness of $\varepsilon \to 0$ limit.

<u>Remarks</u>

The Euler and Navier-Stokes equations $NS_\nu(\mathbb{R}^d)$ possess a natural physical energy

$$E[v](t) := \frac{1}{2} \int_{\mathbb{R}^d_x} |v(t,x)|^2 \, dx.$$

We have seen (see (5)) that, in the absence of an external force,

$$\frac{dE}{dt} = -\nu \int_{\mathbb{R}^d_x} |\nabla v(t,x)|^2 \, dx. \tag{14}$$

This identity was obtained by multiplying the $\partial_t v^\ell \cdots$ equation by v^ℓ, summing over ℓ, and integrating over $\mathbb{R}^d_x$. <u>Energy methods</u> involve a generalization of this idea: We multiply the differential equation by some functional of v and integrate by parts to obtain useful identities or estimates.

<u>A basic energy estimate for smooth solutions</u>

Let v_1, v_2 be solutions of $NS_\nu(\mathbb{R}^d)$ with external force F_1, F_2, respectively:

$$\begin{cases} \frac{d}{dt} v_\alpha^\ell + v_\alpha^j \cdot \partial_j v_\alpha^\ell = \nu \Delta v_\alpha^\ell - \partial_\ell P_\alpha + F_\alpha \\ \nabla \cdot v_\alpha = 0 \\ v_\alpha\big|_{t=0} = v_{0\alpha} \end{cases} \quad \text{for } \alpha = 1, 2. \tag{15}$$

Assume that the solutions v_1, v_2 exist on $[0,T]$ and they vanish fast enough as $|x| \to \infty$, in particular $v_i \in L^2(\mathbb{R}^d)$.

Write $\tilde{v} = v_1 - v_2$, $\tilde{p} = p_1 - p_2$, $\tilde{F} = F_1 - F_2$. We take the difference between the equations in (15):

$$\partial_t \tilde{v} + v_1 \cdot \nabla \tilde{v} + \tilde{v} \cdot \nabla v_2 = -\nabla \tilde{p} + \nu \Delta \tilde{v} + \tilde{F}.$$

Multiply by $\tilde{v}$ to get

$$\langle \tilde{v}_t, \tilde{v} \rangle + \overset{0}{\langle v_1 \cdot \nabla \tilde{v}, \tilde{v} \rangle} + \langle \tilde{v} \cdot \nabla v_2, \tilde{v} \rangle = \overset{0}{\langle -\nabla \tilde{p}, \tilde{v} \rangle} + \nu \langle \Delta \tilde{v}, \tilde{v} \rangle + \langle \tilde{F}, \tilde{v} \rangle$$

Now integrate on $\mathbb{R}^d_x$ and integrate by parts to get

$$\langle \tilde{v}_t, \tilde{v} \rangle + \nu \langle \nabla \tilde{v}, \nabla \tilde{v} \rangle = -\langle \tilde{v} \cdot \nabla v_2, \tilde{v} \rangle + \langle \tilde{F}, \tilde{v} \rangle,$$

the <u>basic energy identity</u>.

Recall,

$$\|f\|_{L^2} = \left(\int |f|^2 \, dx \right)^{\frac{1}{2}} \quad \text{is the } L^2 \text{ norm}$$

and we have Cauchy-Schwarz inequality

$$\langle f, g \rangle = \int f g \, dx \le \|f\|_{L^2} \|g\|_{L^2}.$$

We will also use

$$\|f\|_{L^\infty} = \text{ess sup} |f| \quad (= \sup |f| \text{ for cts. } f).$$

With the L^2 and L^∞ norms, we return to the basic energy identity to get

(16) $$\frac{1}{2} \frac{d}{dt} \|\tilde{v}(\cdot,t)\|_{L^2}^2 + \nu \|\nabla \tilde{v}(\cdot,t)\|_{L^2}^2 \le \|\nabla v_2(\cdot,t)\|_{L^\infty} \|\tilde{v}(\cdot,t)\|_{L^2}^2 + \|\tilde{F}\|_{L^2} \|\tilde{v}(\cdot,t)\|_{L^2}.$$

Since $\nu > 0$, the second term is manifestly nonnegative so it can be dropped. We then obtain

(17) $$\frac{d}{dt} \|\tilde{v}\|_{L^2} \le \|\nabla v_2\|_{L^\infty} \|\tilde{v}\|_{L^2} + \|\tilde{F}\|_{L^2}.$$

Apply the Gronwall Lemma yields . . .

<u>Proposition.</u> (Basic Energy Estimate) Let v_1, v_2 be two smooth solutions of $NS_\nu(\mathbb{R}^d)$ with external forces F_1, F_2 and the same viscosity $\nu \ge 0$. Suppose v_1, v_2 are defined on $[0,T]$ and, for fixed time, decay fast enough as $|x| \to \infty$ to belong to $L^2(\mathbb{R}^d)$. Then

(18) $$\sup_{0 \le t \le T} \|v_1 - v_2\|_{L^2} \le \left[\|(v_1 - v_2)|_{t=0}\|_{L^2} + \int_0^T \|F_1 - F_2\|_{L^2} \, dt \right] \exp\left(\int_0^T \|\nabla v_2\|_{L^\infty} \, dt \right)$$

We thus obtain uniqueness of smooth, fast enough decaying, solutions. Note that the right side does <u>not</u> depend upon viscosity ν.

<u>Corollary</u>: Let v_1, v_2 be two smooth L^2 solutions to (15) with the same forcing, with the same initial data, and both defined on $[0,T]$. Then $v_1 = v_2$.

If we integrate (16) in time and use (15), we can obtain

<u>Gradient Control</u>

(19) $$\nu \int_0^T \|\nabla (v_1 - v_2)(\cdot,t)\|_{L^2}^2 \, dt \le C(v_2, T) \left\{ \left(\|v_1 - v_2(\cdot,0)\|_{L^2}^2 \right) + \left(\int_0^T \|F_1 - F_2(\cdot,t)\|_{L^2} \, dt \right)^2 \right\}$$

where $C(v_2, T)$ depends upon $\int_0^T \|\nabla v_2\|_{L^\infty_x} \, dt$. $\|F_1 - F_2\|_{L^1_T L^2_x}^2$

Small Viscosity Navier-Stokes approximates Euler

Suppose v_1 solves $NS_0(\mathbb{R}^d)$, smooth and $L^2(\mathbb{R}^d)$, with $F_1 = 0$.

Suppose v_2 solves $NS_0(\mathbb{R}^d)$, smooth, $L^2(\mathbb{R}^d)$, with $F_1 = -\nu \Delta v_2$

Then v_2 also solves $NS_\nu(\mathbb{R}^d)$. Let's rename $v_1 = v_E$ and $v_\nu = v_2$. The preceding estimates imply "$v_\nu \to v_0$" in the sense that

$$\sup_{t \in [0,T]} \| v_\nu - v_E \|_{L^2} \leq \int_0^T \| \nu \Delta$$

Small Viscosity $NS_\nu(\mathbb{R}^d)$ solutions approximate Euler solutions

Let v_ν solve $NS_\nu(\mathbb{R}^d)$ with zero forcing and initial data $v_\nu(0)$

Let v_0 solve Euler ($NS_{\nu=0}(\mathbb{R}^d)$) with forcing $F = \nu\Delta v_0$ and the same initial data $v_0(0) = v_\nu(0)$. With this choice of forcing we may also view v_0

We can reexpress the L^2-control appearing in (18) as

(20) $$\|\tilde v\|_{L^\infty_T L^2_x} \leq \left[\|\tilde v(0)\|_{L^2_x} + \|\tilde F\|_{L^1_T L^2_x} \right] \exp\left(\|\nabla v_2\|_{L^1_T L^\infty_x} \right)$$

and the gradient control in (19) as

(21) $$\nu \|\nabla \tilde v\|^2_{L^2_T L^2_x} \lesssim \left[\|\tilde v(0)\|^2_{L^2_x} + \|\tilde F\|^2_{L^1_T L^2_x} \right] C\left(\|\nabla v_2\|_{L^1_T L^\infty_x} \right).$$

Small Viscosity Navier Stokes solutions approximate Euler solutions

Let v_ν denote the solution of $NS_\nu(\mathbb{R}^d)$ with zero forcing emerging from initial data $v_\nu(0)$.

Let v_0 denote the solution of $NS_\nu(\mathbb{R}^d)$ with forcing $F = -\nu\Delta v_0$ emerging from initial data $v_0(0)$. With this forcing we are in fact considering v_0 a solution of $NS_0(\mathbb{R}^d)$ with zero forcing, the Euler equation.

The basic energy estimate implies $v_\nu - v_0$ stays small if it is initially small, at least for a short enough time interval.

Let's assume $v_0(0) = v_\nu(0)$. Then (18) implies

$$\sup_{0 \leq t \leq T} \| v_0(t) - v_\nu(t) \|_{L^2_x} \leq \int_0^T \| \nu \Delta v^0 \|_{L^2} \, dt \; \exp\left(\int_0^T \| \nabla v_0 \|_{L^\infty_x}(t) \, dt \right)$$

and (19) implies

$$\nu \int_0^T \| \nabla (v_\nu - v_0) \|^2_{L^2} \, dt \leq C(v_0, T) \left(\int_0^T \nu \| -\Delta v_0 \|_{L^2} \, dt \right)^2$$

$$\leq T^{1/2} \frac{1}{\nu^{1/2}} \left(\int_0^T \| \nu \Delta v_0 \|_{L^2} \, dt \right)^{1/2}$$

Small Viscosity NS_ν approximates Euler (assuming smooth solutions).

Suppose $\quad v_\nu(\cdot) \longmapsto v_\nu(t)$ solves NS_ν with zero forcing, $\forall t \in [0,T]$

Suppose $\quad v_0(\cdot) \longmapsto v_0(t)$ solves NS_ν with $-\nu\Delta v_0$ forcing, $\forall t \in [0,T]$.

Then v_0 may also be viewed as an $NS_0^0 = $ Euler evolution with zero forcing.

Let $\tilde{v} = v_\nu - v_0$ and apply (20), (21). From (20), we have

$$\|\tilde{v}\|_{L^\infty_T L^2_x} \lesssim \|\nu\Delta v_0\|_{L^1_T L^2_x} \exp\left(\|\nabla v_0\|_{L^1_T L^\infty_x}\right)$$

$$\lesssim \nu T^{1/2} \|\Delta v_0\|_{L^2_T L^2_x} \underbrace{\exp\left(\|\nabla v_0\|_{L^1_T L^\infty_x}\right)}.$$

From (21), we have

$$\cancel{\nu} \|\nabla\tilde{v}\|^2_{L^2_T L^2_x} \lesssim \|\nu\Delta v_0\|^2_{L^2_T L^2_x} \, c\left(\|\nabla v_0\|_{L^1_T L^\infty_x}\right)$$

$$\lesssim \nu^{\cancel{2}} T \|\Delta v_0\|_{L^2_T L^2_x} \underline{c\left(\|\nabla v_0\|_{L^1_T L^\infty_x}\right)}.$$

$$\|\nabla\tilde{v}\|^2_{L^2_T L^2_x} \lesssim \nu T \|\Delta v_0\|_{L^2_T L^2_x} \, c\left(\|\nabla v_0\|_{L^1_T L^\infty_x}\right).$$

If we assume that v_0 is a smooth solution on $[0,T]$ (in fact we need$^{(*)}$ ν-independent control on $\|\Delta v_0\|_{L^2_T L^2_x} + \|\nabla v_0\|_{L^1_T L^\infty_x}$) the underlined terms are bounded. Thus, we obtain then the underlined terms are bounded. Thus, we obtain control on $\tilde{v}$ in $L^\infty_T L^2_x \cap L^2_T H^1_x$ which improves to zero in the limit $\nu \to 0$.

(*) We could tolerate small explosions of these norms in the limit $\nu \to 0$ while retaining a good approximation of v_ν by v_0. The requirement is that the explosions are overwhelmed by the vanishing prefactor.

Strategy for proving local existence for NS_ν.

[8M]

We seek $\underline{v} : [0,T] \times \mathbb{R}^d_x \longrightarrow \mathbb{R}^d_x$ satisfying

$$NS_\nu(\mathbb{R}^d) \quad \begin{cases} \partial_t \underline{v} + \underline{v}\cdot\underline{\nabla} v = \nu\Delta\underline{v} - \nabla p \\ \underline{\nabla}\cdot\underline{v} = 0 \\ \underline{v}\big|_{t=0} = \underline{v}_0 . \end{cases}$$

To construct $\underline{v}$, we will first construct solutions $\underline{v}^\varepsilon$ of the regularized equation

$$NS_{\nu,\varepsilon}(\mathbb{R}^d) \quad \begin{cases} \partial_t \underline{v}^\varepsilon + J_\varepsilon\left[J_\varepsilon \underline{v}^\varepsilon \cdot \underline{\nabla}(J_\varepsilon v^\varepsilon)\right] = \nu J_\varepsilon^2 \Delta v^\varepsilon - \nabla p^\varepsilon \\ \underline{\nabla}\cdot\underline{v}^\varepsilon = 0 \\ \underline{v}\big|_{t=0} = \underline{v}_0 . \end{cases}$$

The operator J_ε is a mollifier defined: Given radial $\rho(|x|) \in C_0^\infty(\mathbb{R}^d)$, $\rho \geq 0$, $\int_{\mathbb{R}^d} \rho \, dx = 1$ we set

(22) $$(J_\varepsilon v)(x) = \varepsilon^{-d} \int_{\mathbb{R}^d} \rho\left(\frac{x-y}{\varepsilon}\right) v(y)\, dy, \quad \varepsilon > 0.$$

The solutions $\underline{v}^\varepsilon$ of $NS_{\nu,\varepsilon}(\mathbb{R}^d)$ will be obtained by a fixed-point/contraction mapping argument in an appropriate Banach space.

We will then extract a subsequence $\{\underline{v}^{\varepsilon_n}\}$ with $\varepsilon_n \to 0$ using compactness which converges to $\underline{v}$, the solution of $NS_\nu(\mathbb{R}^d)$.

We pause the development of fluid theory to record some useful functional analysis. In particular, we will describe the theory of Sobolev spaces.

[CDGG]

Sobolev Spaces on $\mathbb{R}^d$

The Fourier transform of v, denoted $\mathcal{F}v$ or $\hat{v}$, is defined by

$$\mathcal{F}v(\xi) = \hat{v}(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} v(x)\,dx.$$

The inverse Fourier transform allows us to recover v from $\hat{v}$.

$$v(x) = \mathcal{F}^{-1}\hat{v}(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{+ix\cdot\xi}\hat{v}(\xi)\,d\xi.$$

$\forall s \in \mathbb{R}$, define the inhomogeneous Sobolev space $H^s(\mathbb{R}^d)$

$$H^s(\mathbb{R}^d) := \left\{ v \in \mathcal{S}' : \|v\|_{H^s} := \left\{\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{v}(\xi)|^2\,d\xi\right\}^{\frac{1}{2}} < \infty \right\},$$

and similarly we define the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$

$$\dot{H}^s(\mathbb{R}^d) := \left\{ v \in \mathcal{S}' \text{ s.t. } \hat{v} \in L^1_{loc} \text{ and } \|v\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^d} |\xi|^{2s}|\hat{v}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} < \infty \right\}$$

Sobolev Embeddings

"The" Sobolev embedding estimate.

$$f(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{+ix\cdot\xi}\hat{f}(\xi)\,d\xi.$$

$$|f(x)| \le \int_{\mathbb{R}^d} |\hat{f}(\xi)|\,d\xi$$

$$|f(x)| \lesssim \int_{\mathbb{R}^d} \underbrace{(1+|\xi|^2)^{-\frac{s}{2}}}\,\underbrace{(1+|\xi|^2)^{\frac{s}{2}}|\hat{f}(\xi)|}\,d\xi$$

$$\lesssim \left(\int_{\mathbb{R}^d}(1+|\xi|^2)^{-2s}\,d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\hat{f}(\xi)|^2\,d\xi\right)^{\frac{1}{2}}$$

$$\le c_d \|f\|_{H^s(\mathbb{R}^d)} \quad \text{provided } s > \tfrac{d}{2}.$$

Thus,

$$\|f\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{H^s(\mathbb{R}^d)} \quad \text{if } s > \tfrac{d}{2}.$$

Meanwhile, we have the obvious estimate

$$\|f\|_{L^2} \le \|f\|_{H^s}.$$

So for $p \in (2,\infty)$ perhaps there is some $s \in (0, \frac{d}{2})$ s.t.

$$(\circledast) \qquad \|f\|_{L^p(\mathbb{R}^d)} \overset{?}{\le} C\|f\|_{\dot{H}^s(\mathbb{R}^d)} \qquad ??$$

The relationship between p and s is dictated by scaling. Suppose the estimate held, then it must also hold for functions of the form $f_\sigma(x) = f(\frac{x}{\sigma})$; $\forall\, \sigma > 0$.

We can calculate

$$\|f_\sigma\|_{L^p(\mathbb{R}^d_x)} = \left(\int |f_\sigma(x)|^p\,dx\right)^{\frac{1}{p}} = \left(\int |f(\tfrac{x}{\sigma})|^p\,dx\right)^{\frac{1}{p}}$$

$$y = \tfrac{x}{\sigma}\ ;\ dy = \left(\tfrac{1}{\sigma}\right)^d dx$$

$$= (\sigma)^{\frac{d}{p}}\left(\int |f(y)|^p\,dy\right) = (\sigma)^{\frac{d}{p}}\|f\|_{L^p(\mathbb{R}^d)}.$$

Similarly,

$$\|f_\sigma\|_{\dot{H}^s(\mathbb{R}^d)} = \left(\tfrac{1}{\sigma}\right)^s \sigma^{\frac{d}{2}}\|f\|_{\dot{H}^s(\mathbb{R}^d)} = \sigma^{\frac{d}{2}-s}\|f\|_{\dot{H}^s(\mathbb{R}^d)}.$$

Thus, combining estimates we have

$$\|f_\sigma\|_{L^p(\mathbb{R}^d)} = \sigma^{\frac{d}{p}}\|f\|_{L^p(\mathbb{R}^d)} \overset{\circledast}{\le} \sigma^{\frac{d}{p}}C\|f\|_{\dot{H}^s(\mathbb{R}^d)}$$

$$= \sigma^{\frac{d}{p}}C\,\sigma^{s-\frac{d}{2}}\|f_\sigma\|_{\dot{H}^s(\mathbb{R}^d)}.$$

From these observations, we can conclude that the only way $(\circledast)$ can hold on $\mathbb{R}^d$ is if s, p and d satisfy the homogeneity requirement

$$\frac{d}{p} + s - \frac{d}{2} = 0.$$

Theorem (Sobolev Embedding) If s is positive and smaller than $\frac{d}{2}$ (s, $0 < s < \frac{d}{2}$) then the space $\dot H^s(\mathbb{R}^d)$ is continuously embedded in $L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$. (Sometimes written $\dot H^s(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$)
This also means there is the estimate

(25) $$\|f\|_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \le c\|f\|_{\dot H^s(\mathbb{R}^d)}$$

proof: By Fubini's theorem,

$$\|f\|_{L^p}^p = \int_{\mathbb{R}^d} |f|^p\,dx = \int_{\mathbb{R}^d}\int_0^{|f(x)|} p\lambda^{p-1}\,d\lambda\,dx$$

(26) $$= p\int_0^\infty \lambda^{p-1}\,\text{meas}\,\{x\in\mathbb{R}^d : |f(x)|>\lambda\}\,d\lambda.$$

Next, we make a decomposition into low and high frequencies. We write

$$f = f_{1,A} + f_{2,A} = f_{|\xi|<A} + f_{|\xi|>A}$$

with $$f_{1,A} = \mathcal{F}^{-1}(\chi_{B(0,A)}\hat f) \quad + \quad f_{2,A} = \mathcal{F}^{-1}(\chi_{B(0,A)^c}\hat f)$$

where $A>0$ will be determined later.

The low frequency piece $f_{1,A}$ is bounded.

Lemma Let $s\in(-\infty, \frac{d}{2})$ and K be a compact subset of $\mathbb{R}^d$. If $f\in\dot H^s(\mathbb{R}^d)$ and $\operatorname{spt}\hat f \subset K$ then

(27) $$\|f\|_{L^\infty} \le (2\pi)^{-d}\left(\int_K \frac{d\xi}{|\xi|^{2s}}\right)^{\frac12}\|f\|_{\dot H^s}.$$

(proof: follows direct imitation of "the" Sobolev embedding estimate.)

Thus, $$\|f_{1,A}\|_{L^\infty} \le C_{s,d}\,A^{\frac{d}{2}-s}.$$

By the triangle inequality, $\forall\ A>0$ we have

$$\{x\in\mathbb{R}^d : |f(x)|>\lambda\} \subset \{x\in\mathbb{R}^d : 2|f_{1,A}(x)|>\lambda\} \cup \{x\in\mathbb{R}^d : 2|f_{2,A}(x)|>\lambda\}.$$

For fixed λ we will choose $A = A_\lambda$ by the condition that

$$\lambda = C_{s,d}\,A^{\frac{d}{2}-s}. \qquad \frac{d}{p} = \frac{d}{2}-s.$$

Thus $$A_\lambda = c\,\lambda^{\frac{2}{d-2s}} = \lambda^{\frac{p}{d}} \quad \text{if } p = \frac{2d}{d-2s}.$$

with this choice of A, we know that

$$\{x\in\mathbb{R}^d : 2|f_{1,A}(x)|>\lambda\} = \emptyset;$$

so all the action in (26) takes place in the high frequency part of f:

$$\|f\|_{L^p(\mathbb{R}^d)}^p \le \int_{\mathbb{R}^d}|f|^p\,dx \le p\int_0^\infty \lambda^{p-1}\,\text{meas}\,\{x\in\mathbb{R}^d : |f_{2,A_\lambda}(x)| > \tfrac{\lambda}{2}\}\,d\lambda.$$

By Tchebychev's inequality,

$$\text{meas}\,\{x\in\mathbb{R}^d : |f_{2,A_\lambda}(x)| > \tfrac{\lambda}{2}\} = \int_{\{x\in\mathbb{R}^d : |f_{2,A_\lambda}(x)|>\frac{\lambda}{2}\}} dx$$

$$\le \int_{\{\cdots\}} \frac{4|f_{2,A_\lambda}(x)|^2}{\lambda^2}\,dx$$

$$\le \frac{4}{\lambda^2}\|f_{2,A_\lambda}\|^2_{L^2(\mathbb{R}^d)}.$$

Thus, we have

$$\|f\|^p_{L^p(\mathbb{R}^d)} \le 4p\int_0^\infty \lambda^{p-3}\|f_{2,A_\lambda}\|^2_{L^2(\mathbb{R}^d)}\,d\lambda.$$

By Plancherel, we thus have

$$\|f\|^p_{L^p(\mathbb{R}^d)} \le 4p \int_0^\infty \Lambda^{p-3} \left(\int_{|\xi| > A_\Lambda} |f(\xi)|^2 \, d\xi \right) d\Lambda$$

By the definition A_Λ, we have

$$|\xi| > A_\Lambda \iff |\xi| > c\Lambda^{\frac{p}{d}} \iff \Lambda < C|\xi|^{\frac{d}{p}}.$$

By Fubini, we thus have

$$\|f\|^p_{L^p(\mathbb{R}^d)} \le 4p(2\pi)^{-d} \int_{\mathbb{R}^d} \left(\int_0^{C|\xi|^{\frac{d}{p}}} \Lambda^{p-3} \, d\Lambda \right) |\hat{f}(\xi)|^2 \, d\xi$$

$$\le \left(\frac{4p}{p-2}\right)(2\pi)^{-d} \left(C, \quad\right)^{p-2} \int_{\mathbb{R}^d} |\xi|^{\frac{d(p-2)}{p}} |\hat{f}(\xi)|^2 \, d\xi$$

$$\le C_{p,d,s} \|f\|^2_{\dot{H}^s} \quad \text{since} \quad 2s = \frac{d(p-2)}{p}.$$

Remark. This establishes $\dot{H}^s \hookrightarrow L^p$. Further arguments prove the inequality (25).

Corollary: If $p \in (1,2]$ then

$$L^p(\mathbb{R}^d) \subset \dot{H}^s(\mathbb{R}^d) \quad \text{with} \quad s = -d\left(\frac{1}{p} - \frac{1}{2}\right).$$

proof: Duality: $\|g\|_{\dot{H}^s} = \sup_{\|\varphi\|_{\dot{H}^{-s}} \le 1} \langle g, \varphi \rangle$.

Since $-s = d\left(\frac{1}{p} - \frac{1}{2}\right) = d\left(\frac{1}{2} - \left(1 - \frac{1}{p}\right)\right)$ we have

$\|\varphi\|_{\dot{H}^{-s}} \ge c\|\varphi\|_{L^{p'}}$, so

$$\|g\|_{\dot{H}^s} \le C \sup_{\|\varphi\|_{L^{p'}} \le 1} \langle g, \varphi \rangle \le \|g\|_{L^p}.$$

Corollary (Gagliardo-Nirenberg Estimates)

If $p \in [2, \infty)$ is such that $\frac{1}{p} > \frac{1}{2} - \frac{1}{d}$ then $\exists\, C$ s.t. for any domain $\Omega \subset \mathbb{R}^d$, we have for any $v \in H^1_0(\Omega) = \overline{\{f \in C_0^\infty(\Omega) : \|f\|_{H^1} < \infty\}}$ that

(28) $$\|v\|_{L^p(\Omega)} \le C \|v\|^{1-\sigma}_{L^2(\Omega)} \|\nabla v\|^{\sigma}_{L^2(\Omega)} \quad \text{where} \quad \sigma = \frac{d(p-2)}{2p}.$$

proof. We may assume, by density, that $v \in C_0^\infty(\mathbb{R}^d)$.

Then, we have by Sobolev embedding that

$$\|v\|_{L^p(\Omega)} \le C \|v\|_{\dot{H}^{\frac{d(p-2)}{2p}}(\mathbb{R}^d)}.$$

Finally, we use the convexity of the Sobolev norms

(29) $$\|v\|_{\dot{H}^\sigma(\mathbb{R}^d)} \le \|v\|^{1-\sigma}_{L^2(\mathbb{R}^d)} \|v\|^{\sigma}_{\dot{H}^1(\mathbb{R}^d)}, \quad \forall \sigma \in [0,1].$$

Exercise: Prove (29). Hint: Use Hölder's inequality on the Fourier transform side.

Bernstein's inequality

Let B_λ denote a ball of radius λ located somewhere in $\mathbb{R}^d$.
Then

(30) $$\operatorname{spt} \hat{v} \subset B_\lambda \implies \|v\|_{L^q} \le C \lambda^{d\left(\frac{1}{p} - \frac{1}{q}\right)} \|v\|_{L^p} \quad \forall\, 1 \le p \le q \le \infty.$$

proof: By rescaling, we may restrict attention to $\lambda = 1$. Let B_2 denote the ball of radius 2 which is concentric with $B_{\lambda=1}$. Introduce $\hat{\psi} \in C_0^\infty(B_2)$ with $0 \le \hat{\psi} \le 1$ and $\hat{\psi} = 1$ on B_1.

Then

$$\hat{v} = \hat{\psi}\hat{v}$$

by the support property of $\hat{v}$. Thus, upon taking inverse Fourier transf.

$v = \ \ + * \ v.$

By Young's inequality for convolutions,

$$\|v\|_{L^{\delta}} \leq \|\psi * v\|_{L^{q}} \leq \|\psi\|_{L^{r}} \|v\|_{L^{p}} \ ; \quad \frac{1}{\delta} + 1 = \frac{1}{r} + \frac{1}{q}.$$

Thus, we will complete the proof if we show $\|\psi\|_{L^r} \leq C$.

By Hölder's inequality, we have

$$\left(\int |\psi|^{r} \, dx\right)^{\frac{1}{r}} = \left(\int |\psi|^{r-1} |\psi| \, dx\right)^{\frac{1}{r}} \leq \|\psi\|_{L^\infty}^{1-\frac{1}{r}} \|\psi\|_{L^1}^{\frac{1}{r}}.$$

By the convexity inequality, $\forall\ a,b \in \mathbb{R}^+$, $0 \leq \theta < 1$,

$$ab \leq \theta a^{\frac{1}{\theta}} + (1-\theta) b^{\frac{1}{1-\theta}}$$

we get

$$\|\psi\|_{L^r} \leq \|\psi\|_{L^\infty} + \|\psi\|_{L^1}.$$

Then, we write

$$\|\psi\|_{L^1} \leq \|(1+|x|)^{-2d}(1+|x|)^{2d}\psi\|_{L^1}$$

$$\leq C \|(1+|x|)^{2d}\psi\|_{L^\infty}$$

so

$$\|\psi\|_{L^r} \leq C \|(1+|x|)^{2d}\psi\|_{L^\infty}$$

$$\leq C \|(Id - \Delta)^{d} \hat{\psi}\|_{L^1}$$

$$\leq C.$$

Tracing through the proof, we can establish the more general statement: $\forall\ k \geq 0$, $\forall\ p,q$ s.t. $1 \leq p \leq q \leq +\infty$, $\forall\ v \in L^p$:

$$\operatorname{supp} \hat{v} \subset B_\lambda \implies \sup_{|\alpha|=k} \|D^\alpha v\|_{L^q} \leq C^{k+1} \lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)} \|v\|_{L^p}$$

Lemma (Properties of Mollifiers) [BM]

Let $\rho \in C_0^\infty(\mathbb{R}^d)$, $\rho \geq 0$, $\int_{\mathbb{R}^d} \rho \, dx = 1$, ρ radial.

Let $(J_\varepsilon v)(x) = \varepsilon^{-d} \int_{\mathbb{R}^d} \rho\left(\frac{x-y}{\varepsilon}\right) v(y)\, dy$, $\varepsilon > 0$.

Then $J_\varepsilon v \in C^\infty(\mathbb{R}^d)$ and

(i) $\forall\ v \in C^0(\mathbb{R}^d)$, $J_\varepsilon v \longrightarrow v$ uniformly on compact sets $\Omega \subset\subset \mathbb{R}^d$ and

(31) $$\|J_\varepsilon v\|_{L^\infty} \leq \|v\|_{L^\infty}.$$

(ii) (Mollifiers commute with Distribution Derivatives.)

(32) $$D^\alpha J_\varepsilon v = J_\varepsilon D^\alpha v \qquad \forall\ |\alpha| \leq m, \ \forall\ v \in H^m.$$

(iii) $\forall\ v \in L^p(\mathbb{R}^d)$, $V \in L^q(\mathbb{R}^d)$, $\frac{1}{p} + \frac{1}{q} = 1$

(33) $$\int_{\mathbb{R}^d} (J_\varepsilon v)\, V \, dx = \int_{\mathbb{R}^d} v \, (J_\varepsilon V)\, dx$$

(iv) $\forall\ v \in H^s(\mathbb{R}^d)$, $J_\varepsilon v \longrightarrow v$ in H^s and the rate of convergence in the H^{s-1} norm is linear in ε.

$$\lim_{\varepsilon \downarrow 0} \|J_\varepsilon v - v\|_{H^s} = 0$$

$$\|J_\varepsilon v - v\|_{H^{s-1}} \leq C\varepsilon \|v\|_{H^s}.$$

(v) $\forall\ v \in H^m(\mathbb{R}^d)$, $k \in \mathbb{Z}^+ \cup \{0\}$, and $\varepsilon > 0$

(34) $$\|J_\varepsilon v\|_{H^{m+k}} \leq \frac{C_{mk}}{\varepsilon^k} \|v\|_{H^m}$$

(35) $$\|J_\varepsilon D^k v\|_{L^\infty} \leq \frac{C_k}{\varepsilon^{\frac{d}{2}+k}} \|v\|_{L^2}.$$

parts of proof

(i) $$\left| J_\varepsilon v(x) - v(x) \right| = \left| \varepsilon^{-d} \int \rho\left(\frac{x-y}{\varepsilon}\right)\left[v(y) - v(x)\right] dy \right|$$

If $v \in C^0$ then $\forall \varepsilon' \; \exists \; \delta > 0$ s.t. $|x-y| < \delta$, $x, y \in \Omega \subset\subset \mathbb{R}^d$
$\Longrightarrow |v(x) - v(y)| < \varepsilon'$. Thus, provided $\varepsilon < \varepsilon'$ (and we assume without loss of generality that $\operatorname{supp} \rho \subset \bar{B}_1(0)$)
we have

$$\left| J_\varepsilon v(x) - v(x) \right| \leq \varepsilon'.$$

This proves $J_\varepsilon v \longrightarrow v$ uniformly on Ω.

(ii) $$\begin{aligned} D^\alpha J_\varepsilon v(x) &= D_x^\alpha \varepsilon^{-d} \int_{\mathbb{R}^d} \rho\left(\frac{x-y}{\varepsilon}\right) v(y)\, dy \\ &= \varepsilon^{-d} \int_{\mathbb{R}^d} D_x^\alpha \rho\left(\frac{x-y}{\varepsilon}\right) v(y)\, dy \\ &= \varepsilon^{-d} (-1)^{|\alpha|} \int_{\mathbb{R}^d} D_y^\alpha \rho\left(\frac{x-y}{\varepsilon}\right) v(y)\, dy \\ &= \varepsilon^{-d} (-1)^{|\alpha|} (-1)^{|\alpha|} \int_{\mathbb{R}^d} \rho\left(\frac{x-y}{\varepsilon}\right) D_y^\alpha v(y)\, dy \\ &= J_\varepsilon D^\alpha v. \end{aligned}$$

part of

(v) $$\begin{aligned} |J_\varepsilon D^\alpha v(x)| &= \varepsilon^{-d} \left| \int \rho\left(\frac{x-y}{\varepsilon}\right) D^\alpha v(y)\, dy \right| \\ &= \varepsilon^{-d-\kappa} \left| \int (D^\alpha \rho)\left(\frac{x-y}{\varepsilon}\right) v(y)\, dy \right| \\ &\leq \varepsilon^{-\kappa-d} \left(\int |(D^\alpha \rho)\left(\frac{x-y}{\varepsilon}\right)|^2 dx \right)^{\frac{1}{2}} \left(\int |v(y)|^2 dy \right)^{\frac{1}{2}} \\ &\leq \varepsilon^{-\kappa-\frac{d}{2}} \left(\varepsilon^{-d} \int \left|(D^\alpha \rho)\left(\frac{x-y}{\varepsilon}\right)\right|^2 dx \right)^{\frac{1}{2}} \|v\|_{L^2}. \end{aligned}$$

The Hodge decomposition (13) of L^2 vector fields has a Sobolev space generalization. [BM]

Lemma (Hodge Decomposition in H^m) Every vector field $v \in H^m(\mathbb{R}^d)$, $m \in \mathbb{Z}^+ \cup \{0\}$ has the unique orthogonal decomposition

(36) $$\underline{v} = \underline{w} + \underline{\nabla} \varphi$$

such that Leray's projection operator $P\underline{v} = \underline{w}$ on the divergence-free vector fields satisfies

(i) $P\underline{v}, \underline{\nabla}\varphi \in H^m$, $\int_{\mathbb{R}^d} P\underline{v} \cdot \underline{\nabla}\varphi \, dx = 0$, $\operatorname{div} P\underline{v} = 0$ and

$$\|P\underline{v}\|^2_{H^m} + \|\underline{\nabla}\varphi\|^2_{H^m} = \|\underline{v}\|^2_{H^m},$$

(ii) P commutes with distribution derivatives

$$P D^\alpha \underline{v} = D^\alpha P \underline{v} \qquad \forall \; \underline{v} \in H^m, \quad |\alpha| \leq m,$$

(iii) P commutes with mollifiers J_ε

$$P(J_\varepsilon \underline{v}) = J_\varepsilon (P\underline{v}) \qquad \forall \; \underline{v} \in H^m, \quad \varepsilon > 0$$

(iv) P is symmetric

$$(P\underline{v}, \underline{v})_{H^m} = (\underline{v}, P\underline{v})_{H^m}$$

The proof is an adaptation of the L^2-based decomposition and a consideration of mollifier properties and the Sobolev space definition.

$$V^s(\mathbb{R}^d) := \left\{ \underline{v} \in H^s(\mathbb{R}^d) : \operatorname{div} \underline{v} = 0 \right\}.$$

Theorem (Picard Theorem on a Banach space) Let $\mathcal{O} \subset B$ (a Banach space) be an open set and let $F : \mathcal{O} \longrightarrow B$ be a mapping that satisfies:

F is locally Lipschitz continuous, i.e. $\forall X \in \mathcal{O}$ $\exists L > 0$ and an open neighborhood $U_X \subset \mathcal{O}$ with $X \in U_X$ such that $\forall X', X'' \in U_X$

$$\| F(X') - F(X'') \|_B \leq L \| X' - X'' \|_B .$$

Then $\forall X_0 \in \mathcal{O}$ $\exists$ time $T > 0$ such that the ODE

$$\begin{cases} \dfrac{dX}{dt} = F(X) \\ X\big|_{t=0} = X_0 \in \mathcal{O} \end{cases}$$

has a unique (local) solution $X \in C^1([-T, T]; \mathcal{O})$.

ε-Regularized Navier-Stokes initial value problem on $\mathbb{R}^d$.

$$NS_{\nu,\varepsilon}(\mathbb{R}^d) \begin{cases} \partial_t \underline{v}^\varepsilon + J_\varepsilon \left[J_\varepsilon v^\varepsilon \cdot \underline{\nabla} J_\varepsilon \underline{v}^\varepsilon \right] = -\underline{\nabla} p^\varepsilon + \nu J_\varepsilon (J_\varepsilon \Delta v^\varepsilon) \\ \operatorname{div} \underline{v}^\varepsilon = 0 \\ v^\varepsilon\big|_{t=0} = v_0 . \end{cases}$$

Because the Leray projection operator commutes with mollifiers and derivatives, we can recast this problem as

$$P\,NS_{\nu,\varepsilon}(\mathbb{R}^d) \begin{cases} \partial_t \underline{v}^\varepsilon + P J_\varepsilon \left[J_\varepsilon \underline{v}^\varepsilon \cdot \nabla J_\varepsilon \underline{v}^\varepsilon \right] = \nu J_\varepsilon^2 \Delta \underline{v}^\varepsilon \\ \underline{v}^\varepsilon\big|_{t=0} = v_0 , \quad \operatorname{div} v_0 = 0 . \end{cases}$$

The initial value problem $P\,NS_{\nu,\varepsilon}(\mathbb{R}^d)$ may be viewed as an ODE problem on the Banach space V^s:

$$\ast_\varepsilon \begin{cases} \dfrac{d}{dt} \underline{v}^\varepsilon = F_\varepsilon(\underline{v}^\varepsilon) \\ \underline{v}^\varepsilon\big|_{t=0} = \underline{v}_0 \end{cases}$$

where

$$F_\varepsilon(\underline{v}^\varepsilon) = \nu J_\varepsilon^2 \Delta \underline{v}^\varepsilon - P J_\varepsilon \left[(J_\varepsilon \underline{v}^\varepsilon) \cdot \underline{\nabla} J_\varepsilon v^\varepsilon \right]$$
$$:= F_\varepsilon^1(\underline{v}) - F_\varepsilon^2(\underline{v}^\varepsilon).$$

Theorem (Global well-posedness of $PNS_{\nu,\varepsilon}(\mathbb{R}^d)$) Given an initial condition $v_0 \in V^m$, $m \in \mathbb{Z}^+ \cup \{0\}$, $\forall \varepsilon > 0$ $\exists$ a unique solution $v^\varepsilon \in C^1([0,\infty); V^m)$ defined for all time of the ε-regularized Navier-Stokes equation $PNS_{\nu,\varepsilon}(\mathbb{R}^d)$.

proof (3 steps: ① local-in-time
② continuation property
③ a priori bound $\Rightarrow$ continuation property hypothesis.)

①
Local-in-time Proposition If $v_0 \in V^m$, $m \in \mathbb{Z}^+ \cup \{0\}$ then

(i) $\forall \varepsilon > 0$ $\exists !$ solution $\underline{v}^\varepsilon \in C^1([0, T_\varepsilon); V^m)$ to the ODE $\ast_\varepsilon$ where $T_\varepsilon = T(\|v_0\|_{H^m}, \varepsilon)$.

(ii) On any time interval $[0, T]$ where this solution belongs to $C^1([0,T]; V^0)$, an energy estimate holds

$$(37) \qquad \sup_{0 \leq t \leq T} \| \underline{v}^\varepsilon \|_{L^2} \leq \| v_0 \|_{L^2} \qquad \text{(independent of } \varepsilon, \nu).$$

proof of local-in-time proposition

First, note that $F_\varepsilon : V^m \to V^m$. Next, we establish the required Lipschitz estimate. Note that

$$\| F_\varepsilon^1(v^1) - F_\varepsilon^1(v^2) \|_{H^m} = \nu \| J_\varepsilon^2 \Delta(v^1 - v^2) \|_{H^m}$$

$$\le \nu \| J_\varepsilon^2 (v^1 - v^2) \|_{H^{m+2}}$$

$$\le \frac{C\nu}{\varepsilon^2} \| v^1 - v^2 \|_{H^m}$$

by the smoothing estimate (34) for mollifiers.

$$\| F_\varepsilon^2(v^1) - F_\varepsilon^2(v^2) \|_{H^m} \le \| P J_\varepsilon [J_\varepsilon v^1 \cdot \nabla (J_\varepsilon [v^1 - v^2])] \|_{H^m}$$

$$+ \| P J_\varepsilon [J_\varepsilon (v^1 - v^2) \cdot \nabla J_\varepsilon v^2] \|_{H^m}$$

$$\le C \| J_\varepsilon v^1 \|_{L^\infty} \| D^m \nabla J_\varepsilon (v^1 - v^2) \|_{L^2}$$

$$+ C \| D^m J_\varepsilon v^1 \|_{L^2} \| \nabla J_\varepsilon (v^1 - v^2) \|_{L^\infty}$$

$$+ C \| J_\varepsilon (v^1 - v^2) \|_{L^\infty} \| D^m J_\varepsilon \nabla v^2 \|_{L^2}$$

$$+ C \| D^m J_\varepsilon (v^1 - v^2) \|_{L^2} \| \nabla J_\varepsilon v^2 \|_{L^\infty} .$$

mollifier properties

$$\le \frac{C}{\varepsilon^{\frac{d}{2}+1+m}} \left(\| v^1 \|_{L^2} + \| v^2 \|_{L^2} \right) \| v^1 - v^2 \|_{H^m} .$$

Combining the two estimates, we obtain

$$\| F_\varepsilon(v^1) - F_\varepsilon(v^2) \|_{H^m} \le C (\| v^i \|_{L^2}, \varepsilon, d) \| v^1 - v^2 \|_{H^m} .$$

Thus, on any open set $\mathcal{O}^M = \{ v \in V^m : \|v\|_{H^m} \le M \}$ we see that F_ε is locally Lipschitz continuous. By the Picard theorem, we learn that, given any initial condition $v_0 \in H^m$ $\exists !$ $v^\varepsilon \in C^1([0, T_\varepsilon); V^m \cap \mathcal{O}^M)$, $m \in \mathbb{Z}^+ \cup \{0\}$ for some $T_\varepsilon > 0$.

Remark: In proving the Lipschitz bound on F_ε, we could be a bit cavalier with the mollifiers J_ε. But the specific form of F_ε will play a role in the proof of the energy estimate (37).

Finally, we prove the energy estimate. Take the L^2 inner product of $PNS_{\nu,\varepsilon}(\mathbb{R}^d)$ with v^ε we get

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |v^\varepsilon|^2 \, dx = \nu \int_{\mathbb{R}^d} v^\varepsilon J_\varepsilon^2 \Delta v^\varepsilon \, dx - \int_{\mathbb{R}^d} v^\varepsilon P J_\varepsilon [(J_\varepsilon v^\varepsilon) \cdot \nabla (J_\varepsilon v^\varepsilon)] dx$$

$$= I + II .$$

Since mollifiers are symmetric and commute with derivatives,

$$I = -\nu \int_{\mathbb{R}^d} |\nabla J_\varepsilon v^\varepsilon|^2 \, dx .$$

For II, let's reexpress in components (suppressing the ε):

$$II = - \int_{\mathbb{R}^d} v^k P [J_\varepsilon (J_\varepsilon v^\ell) \partial_\ell (J_\varepsilon v)^k] \, dx$$

$$= - \int P(J_\varepsilon v^k) (J_\varepsilon v^\ell) \partial_\ell (J_\varepsilon v^k) \, dx$$

$$= - \int (J_\varepsilon v^\ell) \frac{1}{2} \partial_\ell (J_\varepsilon v^k)^2 \, dx = 0 .$$

Thus,

(38) $$\frac{d}{dt}\|v^\varepsilon\|_{L^2}^2 + 2\nu\|\nabla J_\varepsilon v^\varepsilon\|_{L^2}^2 = 0.$$

Since $\nu \geq 0$, we establish the energy identity (37)

(2) Continuation property.

If $v^\varepsilon(T_\varepsilon) \in \mathcal{O}^{\underline{M}}$ then we may consider $v^\varepsilon(T_\varepsilon)$ as new initial data. The resulting initial value problem then has a solution valid on a time interval which extends our solution past time T_ε. we may always extend provided $v^\varepsilon(\cdot) \in \mathcal{O}^{\underline{M}}$. Thus, we need to show that $v^\varepsilon(t) \in \mathcal{O}^{\underline{M}}$ $\forall t$ we need to show

$$\|v^\varepsilon(t)\|_{H^m} \leq \underline{M}.$$

(3) A priori bounds.

For $m=0$, this follows from the energy identity. But the claim also concerns $m \in \mathbb{Z}^+$. What about $m>0$?

Reconsidering the continuation property, note that don't strictly need to show $\|v^\varepsilon(t)\|_{H^m}$ is bounded uniformly. It would suffice to show that $\|v^\varepsilon(t)\|_{H^m} \leq \underline{M}(t)$ for some (possibly growing unboundedly) function $M(t)$.

Revisiting the bound on $\|F^\varepsilon(v^1) - F^\varepsilon(v^2)\|_{H^m}$ and setting $v^2 = 0$ we observe with $v^1 = v^\varepsilon$ that

(39) $$\frac{d}{dt}\|v^\varepsilon(t)\|_{H^m} \leq C(\|v^\varepsilon\|_{L^2}, \varepsilon, d)\,\|v^\varepsilon(t)\|_{H^m}.$$

By Gronwall's inequality, we obtain

$$\|v^\varepsilon(t)\|_{H^m} \leq \|v^\varepsilon(0)\|_{H^m}\, e^{C(\|v^\varepsilon(0)\|_{L^2}, \varepsilon, d)\, t}.$$

Thus $v_0^\varepsilon \longmapsto v^\varepsilon(t)$ solving $PNS_{\nu,\varepsilon}(\mathbb{R}^d)$ exists for all time $t>0$.

[BM]

We have established that for any viscosity $\nu \geq 0$ and any regularization parameter $\varepsilon > 0$, there exist solutions $v^\varepsilon \in C^1([0,\infty); V^m)$, $m \in \mathbb{Z}^+ \cup \{0\}$, to the regularized Navier-Stokes system

$NS_{\nu,\varepsilon}(\mathbb{R}^d)$ $$\begin{cases} \partial_t v^\varepsilon = \nu J_\varepsilon^2 \Delta v^\varepsilon - P J_\varepsilon [J_\varepsilon v^\varepsilon \cdot \nabla J_\varepsilon v^\varepsilon] \\ v^\varepsilon|_{t=0} = v_0. \end{cases}$$

Next, we show, provided that $m > \frac{d}{2} + 2$, $\exists$ a time interval $[0,T]$ and a subsequence $\{v^\varepsilon\}$ which converges to a limit function $v \in C([0,T]; C^2) \cap C^1([0,T]; C)$ that solves the Euler or Navier-Stokes equation.

We begin with a proposition which refines the estimate (39) by carrying the energy type calculation through after applying D^m to $NS_{\nu,\varepsilon}(\mathbb{R}^d)$.

<u>Proposition</u> (H^m energy estimate) Let $v_0 \in V^m$. Then the unique solution $v^\varepsilon \in C^1([0,\infty); V^m)$ of $NS_{\nu,\varepsilon}(\mathbb{R}^d)$ satisfies

(40) $$\frac{d}{dt}\frac{1}{2}\|v^\varepsilon\|_{H^m}^2 + \nu\|J_\varepsilon \nabla v^\varepsilon\|_{H^m}^2 \leq C_m \|\nabla J_\varepsilon v^\varepsilon\|_{L^\infty}\|v^\varepsilon\|_{H^m}^2.$$

proof Apply D^α with $|\alpha| \le m$ to $NS_{\nu,\varepsilon}(\mathbb{R}^d)$. Take L^2 inner product of the result with $D^\alpha v^\varepsilon$

$$(D^\alpha v_t, D^\alpha v) = (D^\alpha J_\varepsilon^2 \Delta v, D^\alpha v) - (D^\alpha P J_\varepsilon[J_\varepsilon v \cdot \nabla J_\varepsilon v], D^\alpha v)$$

$$= -\nu \|J_\varepsilon \nabla D^\alpha v\|_{L^2}^2 - (P J_\varepsilon[J_\varepsilon v \cdot \nabla D^\alpha J_\varepsilon v], D^\alpha v)$$

$$- (D^\alpha P J_\varepsilon[J_\varepsilon v \cdot \nabla J_\varepsilon v] - P J_\varepsilon[J_\varepsilon v \cdot D^\alpha \nabla J_\varepsilon v], D^\alpha v)$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

$$\mathrm{II} = \int P J_\varepsilon [J_\varepsilon v^\ell \partial_\ell D^\alpha J_\varepsilon v^k] D^\alpha v^k \, dx$$

$$= \int P [J_\varepsilon v^\ell \tfrac{1}{2} \partial_\ell (D^\alpha J_\varepsilon v^k)^2] \, dx = 0.$$

IBP

$\Longrightarrow$

$$\tfrac{1}{2} \tfrac{d}{dt} \|D^\alpha v\|_{L^2}^2 + \nu \|J_\varepsilon \nabla v\|_{L^2} \le |\mathrm{III}|.$$

Summing over $|\alpha| \le m$ gives

$$\tfrac{1}{2} \tfrac{d}{dt} \|v\|_{H^m}^2 + \nu \|J_\varepsilon \nabla v\|_{H^m}^2 \le \|J_\varepsilon v\|_{H^m} \sum_{|\alpha|\le m} \| D^\alpha P J_\varepsilon [J_\varepsilon v \cdot \nabla J_\varepsilon v] - P J_\varepsilon [J_\varepsilon v \cdot D^\alpha \nabla v] \|_{L^2}$$

$$\le \|J_\varepsilon v\|_{H^m} \left\{ \|D^{m-1} \nabla J_\varepsilon v\|_{L^2} \|\nabla J_\varepsilon v\|_{L^\infty} \right\}$$

$$\le \|J_\varepsilon v\|_{H^m}^2 \|\nabla J_\varepsilon v\|_{L^\infty}.$$

$$\le \|\nabla J_\varepsilon v\|_{L^\infty} \|v\|_{H^m}^2. \quad \blacksquare$$

If $m > \frac{d}{2} + 1$ then by Sobolev's inequality we have

(41) $$\|\nabla J_\varepsilon v\|_{L^\infty} \le c_m \|v\|_{H^m} \qquad (\text{independent of } \varepsilon > 0)$$

so we can infer from (40) that

$$\frac{d}{dt} \|v^\varepsilon\|_{H^m} \le c_m \|v^\varepsilon\|_{H^m}^2.$$

Hence, for all $\varepsilon > 0$, we have

$$\sup_{0 \le t \le T} \|v^\varepsilon\|_{H^m} \le \frac{\|v_0\|_{H^m}}{1 - c_m T \|v_0\|_{H^m}} = \|v_0\|_{H^m} + \frac{\|v_0\|_{H^m}^2 c_m T}{1 - c_m T \|v_0\|_{H^m}}.$$

This implies that the family $\{v^\varepsilon\}$ is <u>uniformly bounded</u> in $C([0,T]; H^m)$ provided $m > \frac{d}{2} + 1$ and $T < (c_m \|v_0\|_{H^m})^{-1}$. Furthermore, the family of time derivatives $\left\{\frac{dv^\varepsilon}{dt}\right\}$ is <u>uniformly bounded</u> in

$$C([0,T]; H^{m-2}).$$

<u>Lemma</u> The family $\{v^\varepsilon\}$ forms a Cauchy sequence in $C([0,T]; L^2(\mathbb{R}^d))$. In particular, $\exists\, c > 0$ s.t. $c = c(\|v_0\|_{H^m}, T)$ so that for all $\varepsilon, \varepsilon'$

(42) $$\sup_{0 \le t \le T} \|v^\varepsilon - v^{\varepsilon'}\|_{L^2} \le C \max(\varepsilon, \varepsilon').$$

proof:

$$\frac{d}{dt}\frac{1}{2}\|v^\varepsilon - v^{\varepsilon'}\|_{L^2}^2 = \nu\left(J_\varepsilon^2\Delta v^\varepsilon - J_{\varepsilon'}^2\Delta v^{\varepsilon'}, v^\varepsilon - v^{\varepsilon'}\right)$$

$$- \left(P J_\varepsilon\left[J_\varepsilon v^\varepsilon \cdot \nabla J_\varepsilon v^\varepsilon\right] - \text{same}^{\varepsilon'}, v^\varepsilon - v^{\varepsilon'}\right)$$

$$= T1 + T2.$$

$$T1 \le \nu\left[\left((J_\varepsilon^2 - J_{\varepsilon'}^2)\Delta v^\varepsilon, v^\varepsilon - v^{\varepsilon'}\right)\right] + \nu\left(J_{\varepsilon'}^2\Delta(v^\varepsilon - v^{\varepsilon'}), (v^\varepsilon - v^{\varepsilon'})\right)$$

$$\le \|(J_\varepsilon^2 - J_{\varepsilon'}^2)\Delta v^\varepsilon\|_{L^2}\|v^\varepsilon - v^{\varepsilon'}\|_{L^2} - \nu\|J_{\varepsilon'}\nabla(v^\varepsilon - v^{\varepsilon'})\|_{L^2}^2.$$

$$\le C\max(\varepsilon,\varepsilon')\|v^\varepsilon\|_{H^3}\|v^\varepsilon - v^{\varepsilon'}\|_{L^2}$$

(using the mollifier property $\|J_\varepsilon v - v\|_{H^{s-1}} \le \varepsilon\|v\|_{H^s}$.)

For $T2$, using Sobolev's inequality, some mollifier properties and the zero divergence condition we complete ~~the~~ the estimate to prove

$$\frac{1}{2}\frac{d}{dt}\|v^\varepsilon - v^{\varepsilon'}\|_{L^2} \le C(M)\left[\max(\varepsilon,\varepsilon') + \|v^\varepsilon - v^{\varepsilon'}\|_{L^2}\right]$$

where $M = \sup_{0\le t\le T}\|v^\varepsilon\|_{H^m}$ which we control. Gronwall's estimate then gives us that

$$\sup_{0\le t\le T}\|v^\varepsilon - v^{\varepsilon'}\|_{L^2} \le e^{C(M)T}\left[\max(\varepsilon,\varepsilon') + \|v_0^\varepsilon - v_0^{\varepsilon'}\|_{L^2}\right] - \max(\varepsilon,\varepsilon')$$

$$\le e^{C(M)T}\max(\varepsilon,\varepsilon') \quad \text{since } v_0^\varepsilon = v_0^{\varepsilon'}.$$

Thus, $\{v^\varepsilon\}$ is a Cauchy sequence in $C([0,T]; L^2(\mathbb{R}^d))$ provided $v_0 \in V^m$, $m > \frac{d}{2} + 2$. Therefore v^ε converges strongly in $C([0,T]; L^2(\mathbb{R}^d))$ to a value $v \in C([0,T]; L^2(\mathbb{R}^d))$.

We have just proved that there exists $v \in C([0,T]; L^2)$ such that

$$\sup_{0\le t\le T}\|v^\varepsilon - v\|_{L^2} \le C\varepsilon.$$

We also know that $\sup_{0\le t\le T}\|v^\varepsilon\|_{H^m} \le M$.

Since $\{v^\varepsilon\}$ is uniformly bounded in $C([0,T]; H^m)$ we know that v is also bounded in this space.

This allows us to upgrade the convergence of v^ε to v in L^2 to convergence in $H^{m'}$ for any $0 < m' < m$. How? Recall the convexity property

$$\|f\|_{H^{s'}} \le C_s\|f\|_{L^2}^{1-\frac{s'}{s}}\|f\|_{H^s}^{\frac{s'}{s}}.$$

Applying this to

$$\sup_{0\le t\le T}\|v^\varepsilon - v\|_{H^{m'}} \le C_m \sup_{0\le t\le T}\left[\|v^\varepsilon - v\|_{L^2}^{1-\frac{m'}{m}}\|v^\varepsilon - v\|_{H^m}^{\frac{m'}{m}}\right]$$

$$\le C_m \sup(\quad_{L^2})\sup(\quad_{H^m})$$

$$\le C_m C\varepsilon \underline{M}.$$

We have just proved:

Theorem (Local-in-time existence for Euler and Navier-Stokes equations)

Given $u_0 \in V^m$, $m \geq \left[\frac{d}{2}\right] + 2$, then we have

(i) $\exists$ $T > 0$ with the rough upper bound

(43) $$T \leq \frac{1}{c_m \|u_0\|_{H^m}}$$

such that for any viscosity $0 \leq \nu < \infty$ $\exists!$ solution

$u^\nu \in C([0,T]; C^2(\mathbb{R}^d)) \cap C^1([0,T]; C(\mathbb{R}^d))$ to $NS_\nu(\mathbb{R}^d)$.

(ii) The approximate solutions u^ε and the limit u^ν satisfy higher order energy estimates

(44) $$\sup_{0 \leq t \leq T} \|u^\nu\|_{H^m} + \sup_{0 \leq t \leq T} \|u^\varepsilon\|_{H^m} \leq \frac{\|u_0\|_{H^m}}{1 - c_m T \|u_0\|_{H^m}}$$

(iii) The approximate solutions and the limit u^ν are uniformly bounded in the spaces

$$L^\infty([0,T]; H^m(\mathbb{R}^d)), \quad Lip([0,T]; H^{m-2}(\mathbb{R}^d)).$$

We want to upgrade this to $C([0,T]; H^m(\mathbb{R}^d)) \cap C^1([0,T]; H^{m-2})$

To upgrade the L^∞_T flow property to C_T flow, we first show continuity in the weak topology. Then, we show that the norm $\|v(t)\|_{H^m}$ is continuous in time.

<u>Weak Convergence in Hilbert space</u>

Given a Hilbert space H with inner product $(u,v)_H$, the sequence $\{v_\varepsilon\}$ is said to <u>converge weakly to v</u> in H, written $v_\varepsilon \rightharpoonup v$, if $\forall\ v \in H$ $(v_\varepsilon, v) \longrightarrow (v, v)$ as $\varepsilon \longrightarrow 0$. This should be distinguished from <u>strong convergence.</u> The sequence $\{v_\varepsilon\}$ is said to converge strongly in a Banach space B if $\exists\ f \in B$ such that $\|f_\varepsilon - f\|_B \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

<u>Examples</u> Let $\psi \in C_0^\infty(\mathbb{R})$. Define $\psi_n(x) = \psi(x-n)$. Then $\psi_n \rightharpoonup 0$ in L^2 as $n \longrightarrow \infty$. Define $\rho_n(x) = \psi(x) \sin nx$. Then $\rho_n \rightharpoonup 0$ in L^2 as $n \longrightarrow \infty$. Define $\delta_n(x) = n^{\frac{1}{2}} \psi(nx)$. Then $\delta_n \rightharpoonup 0$ in L^2 as $n \longrightarrow \infty$. However $\psi_n \not\longrightarrow 0$, $\rho_n \not\longrightarrow 0$, $\delta_n \not\longrightarrow 0$ in L^2 as $n \longrightarrow \infty$.

A consequence of the Banach-Alaoglu Theorem is that a bounded sequence $\|v_\varepsilon\|_{H^m} \leq C$ in $H^m(\mathbb{R}^d)$ has a subsequence that converges weakly to some limit $v \in H^m$, $v_\varepsilon \rightharpoonup v$ as $\varepsilon \to 0$.

Since $v_\varepsilon \in L^\infty([0,T]; H^m(\mathbb{R}^d))$ we have $v_\varepsilon \in L^2([0,T]; H^m(\mathbb{R}^d))$ (and $L^2([0,T]; H^m(\mathbb{R}^d))$ is a Hilbert space) so $v_\varepsilon \rightharpoonup v$ in $L^2([0,T]; H^m(\mathbb{R}^d))$. Also for fixed $t \in [0,T]$, we have $\|v^\varepsilon(t)\|_{H^m} \leq C$ so $v^\varepsilon(t) \rightharpoonup v(t)$ in H^m. So, for each fixed t, $\|v(t)\|_{H^m}$ is bounded and $\|v(t)\|_{H^m} \in L^2([0,T])$. This implies that $v \in L^\infty([0,T]; H^m(\mathbb{R}^d))$. Similar arguments apply to show $\frac{\partial v}{\partial t} \in L^\infty([0,T]; H^{m-2}(\mathbb{R}^d))$.

Next, we show that $v \in C_w([0,T], H^m(\mathbb{R}^d))$. What is this?

Definition: $C_w([0,T]; H^s(\mathbb{R}^d))$ denotes continuity on the time interval $[0,T]$ with values in the weak topology of $H^s(\mathbb{R}^d)$. That is, $\forall$ fixed $\varphi \in (H^s(\mathbb{R}^d))^* = H^{-s}(\mathbb{R}^d)$, $\langle \varphi, v(t) \rangle_{L^2}$ is a continuous scalar function on $[0,T]$.

Since $v^\varepsilon \to v$ in $C([0,T]; H^{m'}(\mathbb{R}^d))$ it follows that $\langle \varphi, v^\varepsilon(\cdot,t)\rangle_{L^2} \longrightarrow \langle \varphi, v^\varepsilon(\cdot,t)\rangle$ uniformly on $[0,T]$ $\forall \varphi \in H^{-m'}$ where $0 \le m' < m$. Since $H^{-m'}(\mathbb{R}^d)$ is dense in $H^{-m}(\mathbb{R}^d)$ for $m' < m$ and $\|v^\varepsilon(t)\|_{H^m} \le C$ it follows by an $\frac{\varepsilon}{2}$ argument that $\langle \varphi, v^\varepsilon(t)\rangle_{L^2} \longrightarrow \langle \varphi, v(t)\rangle_{L^2}$ uniformly on $[0,T]$ $\forall \varphi \in H^{-m}$. This implies that $\langle \varphi, v^\varepsilon(\cdot,t)\rangle$ is a cts. scalar function on $[0,T]$.

If $f_n \rightharpoonup f$ and $\|f_n\| = \|f\|$ then $f_n \to f$. So, to upgrade weak to strong convergence it suffices to show that $\|v(t)\|_{H^m}$ is a continuous function of time. The point here is that we are considering $t_n \to t$, we know by the $C_w([0,T]; H^m(\mathbb{R}^d))$ property that $v(t_n) \rightharpoonup v(t)$ in $H^m(\mathbb{R}^d)$ and, if we prove $\|v(t_n)\|_{H^m} \longrightarrow \|v(t)\|_{H^m}$ then $v(t_n) \longrightarrow v(t)$ in $H^m(\mathbb{R}^d)$.

To show the continuity we have 2 different arguments in case $\nu = 0$ or $\nu > 0$.

$\boxed{\text{Case 1: } \nu = 0}$

In the proof of higher order energy estimates, we recorded the estimate valid $\forall \varepsilon > 0$

$$\sup_{0 \le t \le T} \|v^\varepsilon(t)\|_{H^m} \le \|v^\varepsilon(0)\|_{H^m} + \frac{\|v^\varepsilon(0)\|_{H^m}^2 \, C_m T}{1 - C_m T \|v_0^\varepsilon\|_{H^m}^2}.$$

Passing to the limit as $\varepsilon \to 0$ using $\liminf_{\varepsilon \to 0} \|v^\varepsilon(t)\|_{H^m} \ge \|v(t)\|_{H^m}$ we obtain

$$\sup_{0 \le t \le T} \|v(t)\|_{H^m} - \|v(0)\|_{H^m} \le \frac{\|v(0)\|_{H^m}^2 \, C_m T}{1 - C_m T \|v(0)\|_{H^m}^2}.$$

Since $v \in C_w([0,T]; H^m(\mathbb{R}^d))$ we know that $\liminf_{t \to 0^+} \|v(t)\|_{H^m} \ge \|v(0)\|_{H^m}$. This gives us strong continuity on the right as $t \to 0^+$. For $\nu = 0$, the Euler equation is reversible so we can also obtain strong continuity on the left as $t \to 0^-$.

We revisit the argument at time $t_0 \in (0,T)$ to prove continuity at time t_0. We use the uniqueness theorem to patch solutions together ...

$\boxed{\text{Case 2: } \nu > 0}$

As above, we have strong continuity on the right as $t \to 0^+$. But, $\nu > 0$ introduces But $NS_\nu(\mathbb{R}^d)$ is not time reversible. diffusive smoothing for $t > 0$ which allows us to prove continuity at times $t > 0$.

From energy bounds, we know that $\nu \int_0^T \| J_\varepsilon \nabla v^\varepsilon \|^2_{H^m}$ is bounded independent of ε. This guarantees that the limit $v \in L^2([0,T]; H^{m+1})$ (This bound depends very badly upon ν and is not true for the Euler equation.) Thus, $\forall\ \delta > 0\ \exists\ T_0 \in (0,\delta)$ such that $v(\cdot, T_0) := v_0^{T_0} \in H^{m+1}$. Take $v_0^{T_0}$ as new initial data. We can replace m by $m+1$ and we know by earlier arguments that $v \in C([T_0, T']; H^{\tilde m})$ $\forall\ \tilde m < m+1$. This observation combined with earlier stuff unpacks to complete the proof that

Theorem
If $v_0 \in H^m(\mathbb{R}^d)$, $m \geq \frac{d}{2} + 2$ then $v_0 \longmapsto v(t)$ solving $NS_\nu(\mathbb{R}^d)$ exists and is unique $\forall\ \nu \geq 0$ and

$$v \in C([0,T]; Y^m) \cap C^1([0,T]; Y^{m-2})$$

with

$$T \gtrsim \frac{1}{C_m \|v_0\|_{H^m}}.$$

3. Properties of Leray Solutions

[Beale-Kato-Majda] 1984 "Remarks on the Breakdown of Smooth solutions for the 3D Euler Equations"

Take $x \in \mathbb{R}^3$, $m = 5 \geq \frac{3}{2} + 2$. $H^m \ni v_0 \longmapsto v$ solves $NS_{\nu=0}(\mathbb{R}^3) = (E)$ on $[0,T]$ with $T \sim \|v_0\|^{-1}_{H^m}$. Suppose the maximal forward existence time T^* is finite &

$$v \in C([0,T^*); H^m) \cap C^1([0,T^*); H^{m-2}).$$

Necessarily,

$$\limsup_{t \uparrow T^*} \|v(t)\|_{H^m} = +\infty.$$

If not, $\|v(t)\|_{H^m} \leq C_0$ $\forall\ t < T^*$ and we use LWP at t_1 near T^*.

The [BKM] result is that

$$[\text{BKM}] \qquad \int_0^{T^*} \|\nabla \times v(t)\|_{L^\infty_{x\in\mathbb{R}^3}}\, dt = +\infty.$$

Thus,
$$\limsup_{t\uparrow T^*} \|\nabla \times v(t)\|_{L^\infty} = +\infty.$$

This result is also valid for $NS_\nu(\mathbb{R}^3)$.

Remark If $(v,p) : [0,T^*)_t \times \mathbb{R}^3_x \longrightarrow \mathbb{R}^3 \times \mathbb{R}$ solves $NS_\nu(\mathbb{R}^3)$ then if we form for $\lambda > 0$

$$v_\lambda(\tau, y) = \frac{1}{\lambda} v\left(\frac{\tau}{\lambda^2}, \frac{y}{\lambda}\right)$$

$$p_\lambda(\tau, y) = \frac{1}{\lambda^2} p\left(\frac{\tau}{\lambda^2}, \frac{x}{\lambda}\right)$$

we observe that $(v_\lambda, p_\lambda) : [0, \lambda^2 T^*)_\tau \times \mathbb{R}^3_y \longrightarrow \mathbb{R}^3 \times \mathbb{R}$ solves $NS_\nu(\mathbb{R}^3)$.
Furthermore, we can calculate the λ dependence of spacetime norms:

$$(45) \qquad \|\nabla^s v_\lambda\|_{L^\delta_{\tau\in[0,\lambda^2 T)} L^p_{y\in\mathbb{R}^3}} = \left(\frac{1}{\lambda}\right)^{1+s-\frac{2}{\delta}-\frac{3}{p}} \|D^s v\|_{L^\delta_{t\in[0,T)} L^p_{x\in\mathbb{R}^3}}.$$

Observe that $1 + s - \frac{2}{\delta} - \frac{3}{p} = 0$ if $s = 1$, $\delta = 1$, $p = \infty$.
This reveals that [BKM] is a scaling invariant estimate.

To prove [BKM], we assume otherwise: Assume $T^* < \infty$ and

$$(46) \qquad \int_0^{T^*} \|\nabla \times v(t)\|_{L^\infty} dt = M_1 < \infty.$$

We then show $\exists\ C_0 < \infty$ such that

$$(47) \qquad \|v(t)\|_{H^m} \leq C_0 \quad \forall\ t < T^* \quad (\Longrightarrow T^* \text{ not maximal}).$$

C!

The task is to show that a $L^1_{T^*} L^\infty_x$ bound on $\omega = \nabla \times v$ implies H^m bounds.

Energy arguments have shown us that $\forall\, m \geq 1$

(48) $$\|u(t)\|_{H^m} \leq \|u(0)\|_{H^m} \exp\left(c \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right)$$

Thus, H^m bounds follow directly from $L^1_T L^\infty_x$ bounds on ∇u.

[BKM] follows from a delicate control on ∇u using $\nabla \times u = \omega$.

for Euler solutions which also satisfy $\nabla \cdot u$.

A similar argument establishes (which we will provide soon)

(49) $$\|\omega(t)\|_{L^2} \leq \|\omega_0\|_{L^2} \exp\left(c \int_0^T \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right).$$

Potential Theory Estimate

Let u be a smooth, divergence free vector field, $u \in L^2 \cap L^\infty$ and let $\omega = \nabla \times u$. Then

(50) $$\|\nabla u\|_{L^\infty} \leq C\left(1 + \ln^+ \|u\|_{H^3} + \ln^+ \|\omega\|_{L^2} \right)\left(1 + \|\omega\|_{L^\infty} \right).$$

$$\left(\ln^+(x) = \max(0, \ln x) \right).$$

Let's postpone the proof of (49), (50) for the moment. Insert (48) with $m = 3$ and (49) into (50).

$$\ln^+ \|u(t)\|_{H^m} \leq \ln^+ \|u(0)\|_{H^m} + c \int^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau$$

$$\leq C\left(1 + \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right).$$

$$\ln^+ \|\omega(t)\|_{L^2} \leq C\left(1 + \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right)$$

$\Longrightarrow$

$$\|\nabla u_{(t)}\|_{L^\infty} \leq C\left(1 + \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right)\left(1 + \|\omega(t)\|_{L^\infty} \right).$$

By Gronwall's estimate, we conclude

(51) $$\|\nabla u(t)\|_{L^\infty} \lesssim \|\nabla u_0\|_{L^\infty} \, e^{\int_0^t \|\omega(\tau)\|_{L^\infty} d\tau}.$$

Under the contradictory assumption (46) we get uniform control on $\|\nabla u(t)\|_{L^\infty_x}$ $\forall\, t \in [0, T^*)$. Thus, we obtain (through (48)) uniform control on $\|u(t)\|_{H^m}$ whenever $\|u(0)\|_{H^m} < \infty$. This contradicts the maximality of T^*.

Using (51) in (48) shows that

(52) $$\|u(t)\|_{H^m} \leq \|u(0)\|_{H^m} \, e^{c \int_0^t \|\nabla u_0\|_{L^\infty} e^{\int_0^\tau \|\omega(\tau')\|_{L^\infty} d\tau'} d\tau}.$$

Remark For $NS_\nu(\mathbb{R}^2)$, it can be show that $\|\omega(t)\|_{L^\infty} = \|\omega_0\|_{L^\infty}$ for $\nu = 0$ and $\|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty}$ for $\nu > 0$. Thus, (52) in 2D reveals that

$$\|u(t)\|_{H^m} \leq \|u(0)\|_{H^m} \, e^{c \frac{\|\nabla u_0\|_{L^\infty}}{\|\omega_0\|_{L^\infty}} e^{\|\omega_0\|_{L^\infty} t}}$$

$$\leq C e^{e^t}.$$

Can this be realized? Best examples are e^t?

There exist global-in-time solutions of $NS_\nu(\mathbb{R}^3)$ for which the vorticity can grow exponentially without bound.

Reading week

Let's take the curl of the Navier-Stokes equation:

$$\partial_t u + u\cdot\nabla u = -\nabla p + \nu\Delta u.$$

Apply $\nabla\times u$. On the linear terms, we just commute. On ∇p we get zero:

$$\det\begin{bmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ \partial_1 p & \partial_2 p & \partial_3 p\end{bmatrix} = e_1\underbrace{(\partial_2\partial_3 p - \partial_3\partial_2 p)}_{0} + e_2\cdots = 0.$$

~~Let's calculate then the curl of the nonlinear term. First component is~~

$$\cancel{e_1\left[\partial_2(u^j\partial_j u^3) - \partial_3(u^j\partial_j u^2)\right] = e_1\left[(\partial_2 u^j\right.}$$

Take curl of $u\cdot\nabla u$ term.

$$\nabla\times(\underline{u}\cdot\nabla\underline{u}) = (\nabla\times\underline{u})\cdot\nabla\underline{u} + \underline{u}\cdot(\nabla\times\nabla\underline{u}).$$

But $\nabla\times\partial_j = \partial_j\nabla\times$ so

The second term may be express $u^j\nabla\times\partial_j\underline{u}$.

$\nabla\times\nabla\underline{u} = \nabla(\nabla\times\underline{u})$. Thus, $\underline{u}\cdot(\nabla\times\nabla\underline{u}) = (u\cdot\nabla)(\nabla\times u)$.

If we define $\omega = \nabla\times u$ we then obtain the vorticity equation

(53) $$\partial_t\omega + \omega\cdot\nabla u + u\cdot\nabla\omega = \nu\Delta\omega.$$

Take L^2 inner product with ω.

$$\tfrac{1}{2}\partial_t\|\omega\|_{L^2}^2 + \underbrace{\int\omega^l\partial_l u^j\omega_j\,dx}_{\mathrm{I}} + \underbrace{\int(u^j\partial_j\omega^l)\,\omega^l\,dx}_{\mathrm{II}} = -\nu\|\nabla\omega\|_{L^2}^2.$$

$$|\mathrm{I}| \le \|\nabla u\|_{L^\infty}\|\omega\|_{L^2}^2.$$

$$\mathrm{II} = \int\omega^l u^j\partial_j\omega^l\,dx = -\int\partial_j(\omega^l u^j)\,\omega^l\,dx = 0.$$

Thus,

$$\tfrac{1}{2}\partial_t\|\omega\|_{L^2}^2 + \nu\|\nabla\omega\|_{L^2}^2 \le \|\nabla u\|_{L^\infty}\|\omega\|_{L^2}^2.$$

For $\nu\ge 0$ we obtain from Gronwall's estimate that

$$\|\omega(t)\|_{L^2} \le \|\omega(0)\|_{L^2}\exp\left(c\int_0^T\|\nabla u(s)\|_{L^\infty}\,ds\right).$$

This establishes (49).

Note that the vorticity equation (53) has the velocity u appearing in two places. If we could express u in terms of the vorticity ω we would obtain an evolution equation for ω.

Finding the velocity in terms of the vorticity

Suppose we are given ω. Can we determine u from the conditions

(54) $$\begin{cases}\nabla\times u = \omega \\ \nabla\cdot u = 0.\end{cases} \quad ?$$

This system is overdetermined since we have 4 equations in the 3 unknowns (u^1,u^2,u^3). Nevertheless, this problem has a solution.

Proposition (Hodge Decomposition) Let $\omega\in L^2$ be a smooth vector field on $\mathbb{R}^3$ vanishing sufficiently rapidly as $|x|\to\infty$. Then

(i) Equations (54) have a smooth solution that vanishes as $|x|\to\infty$ if and only if

$$\operatorname{div}\omega = 0$$

(ii) If $\operatorname{div}\omega = 0$, then u is determined constructively by

$$u = -\operatorname{curl}\psi$$

where ψ is the vector-stream function satisfying

$$\Delta\psi = \omega.$$

There is an explicit formula for u in terms of ω.

(55) $$u(x) = \int_{\mathbb{R}^3} K_3(x-y)\,\omega(y)\,dy$$

where the 3×3 matrix kernel K_3 is

$$K_3(x)\,h = \frac{1}{4\pi}\,\frac{x \times h}{|x|^3}.$$

Formally, we then have that

$$u = \nabla \times \Delta^{-1}\omega$$

so

$$\nabla u = \nabla \times \Delta^{-1}\nabla\omega, \qquad \nabla u \sim \omega \text{ in function spaces.}$$

Proposition (∇u from ω)

Let u be given by (55). Then

(56) $$[\nabla u(x)]\,h = -PV\int_{\mathbb{R}^3}\left\{\frac{1}{4\pi}\,\frac{\omega(y)\times h}{|x-y|^3} + \frac{3}{4\pi}\,\frac{\left(\{[x-y]\times\omega(y)\}\otimes(x-y)\right)h}{|x-y|^5}\right\}dy$$

$$+\ \frac{1}{3}\,\omega(x)\times h.$$

$$= P_3\,\omega(x) + c\,\omega(x).$$ (Thus (50) boils down to an estimate on P_3.)

Recall $z \otimes w = z_i w_j$.

proof of Hodge Decomposition Proposition

For smooth vector fields $\psi : \mathbb{R}^3 \to \mathbb{R}^3$, we have the identities

(57) $$\operatorname{div}\operatorname{curl}\psi = 0$$

(58) $$-\operatorname{curl}\operatorname{curl}\psi + \nabla\operatorname{div}\psi = \Delta\psi.$$

e.g. $\partial_{x_1}(\partial_{x_2}\psi^3 - \partial_{x_3}\psi^2) + \partial_{x_2}(\partial_{x_3}\psi^1 - \partial_{x_1}\psi^3) + \partial_{x_3}(\partial_{x_1}\psi^2 - \partial_{x_2}\psi^1) = 0.$

Suppose u solves the div-curl system (54). Then $\operatorname{curl} v = \omega$ and the vector identity implies that $\operatorname{div}\omega = 0$. Now, suppose $\operatorname{div}\omega = 0$. We prove that there exists a vector field u that solves the div-curl system.

First, we consider the Poisson Equation

$$\Delta\psi = \omega.$$

We can obtain a solution ψ by convolving ω with the fundamental solution, a.k.a. the Newtonian potential. In $\mathbb{R}^3$, we thus have that

$$\psi(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\omega(y)}{|x-y|}\,dy$$

is a solution. Now that we have ψ, we can define

$$h = -\operatorname{curl}\operatorname{curl}\psi \quad\text{and}\quad K = \nabla\operatorname{div}\psi.$$

The vector identity (58) and the Poisson equation reveal that

$$h + K = \omega.$$

We take the L^2 inner product of this equation with K to get

$$\langle K, K\rangle = \langle\omega, K\rangle - \langle h, K\rangle.$$

Since $\operatorname{div}\omega = 0$ and K is the gradient of $\operatorname{div}\psi$ we have

$$\langle\omega, K\rangle = \langle\omega, \nabla(\operatorname{div}\psi)\rangle = \langle\operatorname{div}\omega, \operatorname{div}\psi\rangle = 0.$$

Similarly, since $\operatorname{div} h = 0$ (using (57)), we have

$$\langle h, K\rangle = 0.$$

Therefore, $\langle K, K\rangle = 0$ and $K \equiv 0$. Returning to (58), we see that

$$\omega = \operatorname{curl}(-\operatorname{curl}\psi).$$

Now, define $u = -\operatorname{curl}\psi$. By (57), $\nabla\cdot u = \operatorname{div} u = 0$. Since ψ is given by the convolution with the Newtonian potential, direct calculations validate (55). ∎

Remark

To eliminate u from the vorticity equation requires that we calculate ∇u. It is tempting to take ∇ of formula (55). But we encounter a convolution operator on $\mathbb{R}^3$ with kernel ∇K_3 which is homogeneous of degree -3. Thus, differentiation under the integral sign is very subtle. We get a SIO.

Distribution Derivatives of functions of homogeneity degree $1-d$ and singular integrals

We make some general observations in the $\mathbb{R}^d$ setting which when specialized to $d=3$ and $K = K_3$ explain how (56) follows from (55).

Consider a function K defined on $\mathbb{R}^d$ which is homogeneous of degree $1-d$:

$$K(\lambda x) = \lambda^{1-d} K(x) \qquad \forall \ \lambda > 0, \ x \in \mathbb{R}^d.$$

Definition The distribution derivative ∂_{x_j} of f is the linear functional $\partial_{x_j} f$ defined by the formula

$$\langle \partial_{x_j} f, \varphi \rangle = - \langle f, \partial_{x_j} \varphi \rangle \qquad \forall \ \varphi \in C_0^\infty(\mathbb{R}^d).$$

We compute the distribution derivative of K. Since $K \in L^1_{loc}(\mathbb{R}^d)$, $\forall \varphi \in C_0^\infty(\mathbb{R}^d)$ we have (by dominated convergence and Green's theorem):

$$(K, \partial_{x_j}\varphi) = \lim_{\varepsilon \downarrow 0} \int_{|x| \geq \varepsilon} K \, \partial_{x_j} \varphi \, dx$$

$$= \lim_{\varepsilon \downarrow 0} \left\{ \int_{|x| \geq \varepsilon} (-\partial_{x_j} K) \varphi \, dx + \int_{|x| = \varepsilon} K \varphi \frac{x_j}{|x|} \, ds \right\}.$$

The first term is a Cauchy Principle Value integral,

(57) $$PV \int_{\mathbb{R}^n} f \, dx = \lim_{\varepsilon \downarrow 0} \int_{|x| \geq \varepsilon} f \, dx.$$

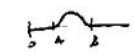
The integral in the second term may be reexpressed as an integral over a sphere of radius 1 by rescaling $x \longrightarrow \varepsilon x$ and using the homogeneity of K as

$$\lim_{\varepsilon \downarrow 0} \int_{|x|=1} \varepsilon^{1-d} K(x) \varphi(\varepsilon x) \, x_j \, \varepsilon^{d-1} \, ds = \varphi(0) \int K(x) \, x_j \, ds.$$

With these remarks, (56) can be derived from (55).

Proposition Let K be a function smooth outside of $x=0$ and homogeneous of degree $1-d$. Then $\partial_{x_j} K$ is homogeneous of degree $-d$ and has mean zero on the sphere:

(58) $$\int_{|x|=1} \partial_{x_j} K \, ds = 0.$$

proof Pick a cutoff function $\rho \in C_0^\infty(\mathbb{R})$, $\rho(r) \geq 0$ and 
$\rho(r) = 0$ for $r \leq A$ and $\rho(r) = 0$ for $r \geq B > A > 0$.
Then, $$\int_0^\infty \rho'(r) \, dr = 0 \quad \text{and} \quad \int_0^\infty \rho(r) \frac{dr}{r} = c > 0.$$

We have

$$0 = \int_{\mathbb{R}^d} \partial_{x_j} \left[\rho(|x|) K(x) \right] dx = \underbrace{\int_{\mathbb{R}^d} \rho'(|x|) \frac{x_j}{|x|} K(x) \, dx}_{①} + \underbrace{\int_{\mathbb{R}^d} \rho(|x|) \, \partial_{x_j} K(x) \, dx}_{②}.$$

Reexpress ① using polar coordinates and homogeneity:

$$\int_0^\infty \rho'(r) \int_{|x|=r} \frac{x_j}{|x|} K(x) \, ds \; r^{d-1} \, dr = \int_0^\infty \rho'(r) \int_{|x|=1} x_j K(x) \, ds \, dr = 0.$$

Reexpress ② using polar coordinates and homogeneity:

$$\int_0^\infty \rho(r) \int_{|x|=r} (\partial_{x_j} K)(x) \, ds \; r^{d-1} \, dr = \int_0^\infty \frac{\rho(r)}{r} \int_{|x|=1} (\partial_{x_j} K)(x) \, ds \, dr.$$

Thus, $$\int_{|x|=1} (\partial_{x_j} K)(x) \, ds = 0.$$ ∎

Definition Any function P that is homogeneous of degree $-d$ on $\mathbb{R}^d$ and has mean value zero on the sphere defines a singular integral operator (SIO) through the convolution

(59) $$Pf(x) = PV \int P(x-y) f(y) \, dy = \lim_{\varepsilon \downarrow 0} \int_{|x-y| \geq \varepsilon} P(x-y) f(y) \, dy.$$

The operator P_3 in (56) is a SIO.

Potential Theory Estimates for P_3

Lemma Let $f \in C^\gamma(\mathbb{R}^d;\mathbb{R}^d)$, $0<\gamma<1$ with compact support, where $m(\mathrm{supp}\, f) = m_f < \infty$. Define a length scale R by $R^d = m_f$. Let P_d be a SIO satisfying, P_d smooth away from $x=0$

(60) $$P_d(\lambda x) = \lambda^{-d} P_d(x), \quad \forall \lambda>0,\ x\neq 0$$

(61) $$\int_{|x|=1} P_d \, ds = 0$$

(62) $$P_d f(x) = \mathrm{PV}\int_{\mathbb{R}^d} P_d(x-y)\, f(y)\, dy$$

Then $\exists$ constant c independent of R and ε such that

(63) $$\|P_d f\|_{L^\infty} \le c \left\{ \|f\|_{C^\gamma}\, \varepsilon^\gamma + \max\left(1, \ln \frac{R}{\varepsilon}\right)\|f\|_{L^\infty}\right\} \quad \forall \varepsilon>0$$

(64) $$\|P_d f\|_{C^\gamma} \le c\, \|f\|_{C^\gamma}.$$

proof: We split the integration in (62) into two regions. $|x-y| \le \varepsilon$ and $|x-y|\ge\varepsilon$. Denote the associated integrals I_1, I_2. We rewrite

$$I_1(x) = \mathrm{PV}\int_{|x-y|\le\varepsilon} P_d(x-y)\{f(y)-f(x)\}\, dy \qquad (\text{using } (61))$$

$$= \mathrm{PV}\int_{|y|\le\varepsilon} P_d(y)\{f(x-y)-f(x)\}\, dy$$

So

$$|I_1(x)| \le \mathrm{PV}\int_{|y|\le\varepsilon} |P_d(y)|\, \frac{|f(x-y)-f(x)|}{|y|^\gamma}\, |y|^\gamma\, dy$$

$$\le C\|f\|_{C^\gamma}\int_{|y|\le 2\varepsilon} |y|^{-d+\gamma}\, dy \le C\|f\|_{C^\gamma}\,\varepsilon^\gamma, \quad 0<\gamma<1.$$

The Hölder continuity of was was used to compensate for the singularity $|y|^{-d}$ of the kernel.

For I_2, we have

$$|I_2(x)| \le \int_{R\ge|x-y|\ge\varepsilon} |P_d(x-y)\, f(y)|\, dy + \int_{R\le|x-y|} |P_d(x-y)|\,|f(y)|\, dy$$

$$\le C\|f\|_{L^\infty} \ln\left(\frac{R}{\varepsilon}\right) + C R^{-d}\|f\|_{L^\infty}(m_f).$$

$$\le C\|f\|_{L^\infty}\left(\ln\left(\frac{R}{\varepsilon}\right)+1\right)$$

Combining the estimates on I_1 and I_2 proves (63).

We omit the proof of (64) since it is not required to complete the proof of the potential theory estimate which underpins [BKM]. ∎

We now prove the potential theory estimate (50). Since $\nabla u = P_\varepsilon w + cw$ and we wish to control $\|\nabla u\|_{L^\infty}$ in terms of $\|w\|_{L^\infty}$, matters boil down to proving (50) with ∇u on the left side replaced by $P_3 w$. Recall that $P_3 w(x)$ is a SIO defined by the Cauchy Principal Value integral

$$P_3 w(x) = \mathrm{PV}\int_{\mathbb{R}^3} \nabla K_3(x-y)\, w(y)\, dy.$$

Introduce a radial bump function ρ s.t. $\rho(r) = 1$ for $r<R_0$ and $\rho(r)=0$ for $r>2R_0$ with $\rho\ge 0$. We decompose the integral

$$P_3 w(x) = \mathrm{PV}\int_{\mathbb{R}^3} \nabla K_3(x-y)\,\rho(|x-y|)\,w(y)\, dy + \mathrm{PV}\int_{\mathbb{R}^3} \nabla K_3(x-y)\left[1-\rho(|x-y|)\right] w(y)\, dy.$$

$$= I_1 + I_2.$$

For I_1, we use (63) with $R = R_0$ and ε to be selected to observe

$$\| I_1 \|_{L^\infty} \le C \left\{ \|w\|_{C^\gamma} \varepsilon^\gamma + \max\left(1, \ln \frac{R}{\varepsilon}\right) \|w\|_{L^\infty} \right\}, \ \forall \varepsilon > 0.$$

Since $H^2 \hookrightarrow C^\gamma$ on $\mathbb{R}^3$ we have

$$\|w\|_{C^\gamma} \le c \|w\|_{H^2}.$$

Since $w = \nabla \times u$ we have

$$\|w\|_{H^2} \le c \|u\|_{H^3}.$$

Therefore,

$$\| I_1 \|_{L^\infty} \le C \left\{ \|u\|_{H^3} \varepsilon^\gamma + \max\left(1, \ln \frac{R_0}{\varepsilon}\right) \|w\|_{L^\infty} \right\}, \ \forall \varepsilon > 0.$$

For I_2 we use Cauchy-Schwarz to obtain

$$\| I_2 \|_{L^\infty} \le \left(\int_{|x-y| > R_0} |\nabla K_3(x-y)|^2 \, dy \right)^{\frac{1}{2}} \|w\|_{L^2}$$

$$\le R_0^{-\frac{3}{2}} \|w\|_{L^2}.$$

We choose

$$\varepsilon = \begin{cases} 1 & \text{if } \|u\|_{H^3} \le 1 \\ \|u\|_{H^3}^{-\frac{1}{\gamma}} & \text{otherwise} \end{cases}$$

and R_0 so that $R_0^{\frac{3}{2}} = \|w\|_{L^2}$ to obtain

$$\| \nabla u \|_{L^\infty} \le C \left\{ 1 + \ln^+ \|u\|_{H^3} + \ln^+ \|w\|_{L^2} \right\} (1 + \|w\|_{L^\infty})$$

which is the potential theory estimate (50).

This completes the proof of the Beale-Kato-Majda theorem.

If $u_0 \longmapsto u$ solves NS_ν with $\nu \ge 0$ on $[0, T^*)$, T^* finite and $[0, T^*)$ maximal then

$$\int_0^{T^*} \| \nabla \times u(t) \|_{L^\infty} \, dt = +\infty.$$

Remark: The potential theory estimate is similar to an estimate of [Brezis-Gallouet]: $\forall u \in H^2(\mathbb{R}^2)$ satisfying $\|u\|_{H^1(\mathbb{R}^2)} \le 1$ we have

(65) $$\|u\|_{L^\infty} \le C \left(1 + \sqrt{\log(1 + \|u\|_{H^2})} \right).$$

proof of (65): Since $u(x) = \int e^{ix\cdot\xi} \hat{u}(\xi) \, d\xi$ we have $\|u\|_{L^\infty} \le \|\hat{u}\|_{L^1}$. Let $R > 0$ be a number to be determined.

$$\|\hat{u}\|_{L^1} = \int |\hat{u}(\xi)| \, d\xi = \int_{|\xi| < R} |\hat{u}(\xi)| \, d\xi + \int_{|\xi| \ge R} |\hat{u}(\xi)| \, d\xi.$$

$$= I_1 + I_2.$$

$$I_1 = \int_{|\xi| < R} (1 + |\xi|) |\hat{u}(\xi)| \frac{1}{(1 + |\xi|)} \, d\xi$$

$$\le \left\{ \int_{|\xi| < R} (1 + |\xi|)^2 |\hat{u}(\xi)|^2 \, d\xi \right\}^{\frac{1}{2}} \left\{ \int_{|\xi| < R} \frac{1}{(1 + |\xi|)^2} \, d\xi \right\}^{\frac{1}{2}}$$

$$\le c \|u\|_{H^1} \sqrt{\log(1 + R)} \le C \sqrt{\log(1 + R)}$$

$$I_2 = \int_{|\xi| > R} (1 + |\xi|^2) |\hat{u}(\xi)| \frac{1}{(1 + |\xi|^2)} \, d\xi \le \|u\|_{H^2} \left\{ \int_{|\xi| > R} \frac{1}{(1 + |\xi|^2)^2} \, d\xi \right\}^{\frac{1}{2}}$$

$$\le c \|u\|_{H^2} \frac{1}{1 + R}.$$

Combining the estimates, we have for every $R > 0$

$$\|u\|_{L^\infty} \leq C\sqrt{1+\log R} + C\|u\|_{H^2}(1+R)^{-1}.$$

Now, choose $R = \|u\|_{H^2}$ to establish (65).

Uniqueness of Leray Solutions (following [Lemarié])

We have seen earlier in this course that solutions $u_0 \mapsto u$ of $NS_\nu(\mathbb{R}^d)$ may be defined by taking limits as $\varepsilon \to 0$ of solutions $u_0 \mapsto u^\varepsilon$ of the regularized $NS_{\nu,\varepsilon}(\mathbb{R}^d)$ equations provided that $u_0 \in H^m(\mathbb{R}^d)$ with $m > \frac{d}{2}+2$. Furthermore, for data of that regularity, we have seen that uniqueness holds. Using a different regularization procedure, it can be established that there are weak solutions of $NS_\nu(\mathbb{R}^d)$ with merely L^2 initial data. Two observations give a hint as to why this is the case. First, the nonlinear term $(u\cdot\nabla)u$ may be reexpressed using $\nabla\cdot u = 0$ as $\nabla\cdot(u\otimes u)$. So, to make sense of this term in the weak sense requires that $u \in L^2_{t,x}$ at least locally. Second, the regularized solutions are constructed so that the energy equality holds uniformly w.r.t. ε and thus implies $u \in L^\infty_T L^2_x \cap L^2_T \dot H^1_x$ which in turn implies that $u \in L^2_{tx}$ locally. Here, a statement...

Theorem (Leray) $\forall\, u_0 \in L^2(\mathbb{R}^d)$ s.t. $\nabla\cdot u_0 = 0$ $\exists$ a weak solution of $NS_\nu(\mathbb{R}^d)$ on $(0,\infty)\times\mathbb{R}^d$ satisfying
s.t. $u \in L^\infty((0,\infty);L^2_x) \cap L^2((0,\infty);\dot H^1)$ s.t. it satisfies $\lim_{t\downarrow 0^+}\|u(t)-u_0\|_{L^2} = 0$. Moreover, we may choose u s.t. it satisfies the Leray Energy inequality: $\forall\, t > 0$

(66)
$$\|u(t,\cdot)\|^2_{L^2} + 2\nu\int_0^t\int_{\mathbb{R}^d}|\nabla u|^2\,dx\,ds \leq \|u_0\|^2_{L^2}.$$

Remarks. The theorem does not include uniqueness. The theorem does not guarantee that initially regular solutions stay regular for all time.

Definition (Leray Solution)

A. Let $u_0 \in L^2(\mathbb{R}^d)$ with $\nabla\cdot u_0 = 0$. A Leray solution on $(0,T)$ for $NS(\mathbb{R}^d)$ with initial value u_0 is a weak solution u for

$$\begin{cases} \exists\, p \in \mathcal{D}'((0,T)\times\mathbb{R}^d) & \partial_t u + \nabla\cdot(u\otimes u) = \Delta u - \nabla p \\ & \nabla\cdot u = 0 \end{cases}$$

such that $u \in L^\infty((0,T);L^2_x) \cap L^2((0,T);\dot H^1_x)$, $\lim_{t\downarrow 0^+}\|u(t)-u_0\|_{L^2} = 0$ and
$\forall\, t \in [0,T)$, $\|u(t,\cdot)\|^2_{L^2} + 2\int_0^t\int_{\mathbb{R}^d_x}|\nabla u|^2\,dx\,ds \leq \|u_0\|^2_{L^2}$.

B. A restricted Leray solution on $(0,T)$ is a solution obtained by the limiting process from solutions of a regularized equation. ← This is [illegible]

Proposition Let $T \in (0,+\infty]$, u_1, u_2 [illegible]
for $i = 1,2$
$$\begin{cases} \exists\, p_i \in \mathcal{D}'((0,T)\times\mathbb{R}^d) \text{ s.t. } & \partial_t \vec u_i + \nabla\cdot(\vec u_i\otimes\vec u_i) = \Delta\vec u_i - \nabla p_i \\ & \nabla\cdot\vec u_i = 0. \end{cases}$$

Assume that
(i) $u_1, u_2 \in L^\infty((0,T);L^2_x) \cap L^2((0,T);\dot H^1_x)$ (ii)
(iii) For some $q \in [d,\infty)$, $u_1 \in L^p((0,T),L^q_x)$ with $\frac{2}{p}+\frac{d}{q}-1 = 0$.

Then $t \mapsto \int u_1(t,x)\cdot u_2(t,x)\,dx$ is cts on $[0,T)$ and $\forall\, 0 \leq \tau < t < T$

(67)
$$\int_{\mathbb{R}^d} u_1(t,x)\cdot u_2(t,x)\,dx + 2\int_\tau^t\int_{\mathbb{R}^d}(\nabla\otimes u_1)\cdot(\nabla\otimes u_2)\,dx\,ds =$$
$$\int_\tau^t\int_{\mathbb{R}^d} u_1\cdot(u_1\cdot\nabla)u_2\,dx\,ds - \int_\tau^t\int_{\mathbb{R}^d} u_1\cdot(u_2\cdot\nabla)u_2\,dx\,ds + \int u_1(\tau,x)\cdot u_2(\tau,x)\,dx.$$

Corollary: Let $T \in (0,+\infty]$, let u solve $NS(\mathbb{R}^d)$. Assume (i) $u \in L^\infty_T L^2_x$, (ii) $u \in L^2_T \dot H^1_x$ and (iii) $\exists\, q \in [d,\infty)$ s.t. $u \in L^p_T L^q_x$ with $\frac{2}{p}+\frac{d}{q}-1 = 0$. Then $u \in C_T L^2_x$ and the energy equality holds $\forall\, 0 \leq \tau < t < T$

$$\|u(t,\cdot)\|^2_{L^2} + 2\int_\tau^t\int_{\mathbb{R}^d}|\nabla u|^2\,dx\,ds \underset{\text{energy equality}}{=} \|u(\tau,\cdot)\|^2_{L^2}.$$

Serrin's Uniqueness Theorem

Let $u_0 \in L^2_x$ with $\nabla\cdot u_0 = 0$. Assume $\exists$ a solution u of $NS(\mathbb{R}^d)$ on $(0,T)\times\mathbb{R}^d$ with initial value u_0 which satisfies: (i) $u \in L^\infty_T L^2_x$, (ii) $u \in L^2_T \dot H^1_x$ and (iii) $\exists\, q \in (d,\infty)$ with $u \in L^p_T L^q_x$ for $\frac{2}{p}+\frac{d}{q}-1 = 0$. Then u is the unique Leray solution associated to u_0 on $(0,T)$.

Proof of Serrin's uniqueness theorem assuming the Proposition: If $u_0 \mapsto v$ is another Leray solution then
$$\|u(t)-v(t)\|^2_{L^2} = \|u(t)\|^2_{L^2} + \|v(t)\|^2_{L^2} - 2\int_{\mathbb{R}} v(t,x)\cdot v(t,x)\,dx$$
$$\leq \cancel{\|u_0\|^2_{L^2}} - 2\int_0^t\int_{\mathbb{R}^d}|\nabla\otimes u|^2\,dx\,ds + \cancel{\|u_0\|^2_{L^2}} - 2\int_0^t\int_{\mathbb{R}^d}|\nabla\otimes v|^2\,dx\,ds$$
$$+ 4\int_0^t\int_{\mathbb{R}^d_x}(\nabla\otimes u)\cdot(\nabla\otimes v)\,dx\,ds - 2\int_0^t\int_{\mathbb{R}^d} u\cdot(u\cdot\nabla)v\,dx\,dx$$
$$+ 2\int_0^t\int_{\mathbb{R}^d_x} u\cdot(v\cdot\nabla)v\,dx\,ds - 2\cancel{\|u_0\|^2_{L^2}}.$$
$$= -2\int_0^t\int_{\mathbb{R}^d}|\nabla\otimes(u-v)|^2\,dx\,ds - 2\int_0^t\int_{\mathbb{R}^d} u\cdot((u-v)\cdot\nabla)v\,dx\,ds.$$

We also have that $\int_{\mathbb{R}^d} v\cdot(w\cdot\nabla)v\,dx = \int_{\mathbb{R}^d} v^k w^j\partial_j v^k\,dx = -\int_{\mathbb{R}^d}\partial_j(v^k w^j)v^k\,dx = 0$

So, in the last term we can replace the last v by $v-u$.

For $d \le q < \infty$, define $r = \frac{d}{q}$. Since u, v are Leray solutions we have
$u - v \in L^\infty_T L^2_x \cap L^2_T \dot H^1_x \subset L_t^{\frac{2}{r}} \dot H^r_x$ (since $\|f\|_{\dot H^r} \le \|f\|_{L^2}^{1-r} \|f\|_{\dot H^1}^r$).
By Sobolev embedding $\dot H^r_x \hookrightarrow L^\sigma_x$ where $\frac{d}{\sigma} = \frac{d}{2} - r$ so $u - v \in L_t^{2/r} L^\sigma_x$.
We also know $\nabla \otimes (u-v) \in L^2_T L^2_x$ and $v \in L^p_T L^q_x$ for some p, q. (Hypothesis (iii)).
Thus, $\forall\ 0 \le \tau < t < T$, and since $\frac{1}{q} + \frac{1}{\sigma} + \frac{1}{2} = 1$ and $\frac{1}{p} + \frac{r}{2} + \frac{1}{2} = 1$,

$$\left| \int_\tau^t \int_{\mathbb{R}^d} v \cdot \big((u - v) \cdot \nabla \big) (v - u)\, dx\, ds \right| \le \|v\|_{L^p_{[\tau,t]} L^q_x} \|u - v\|_{L^{\frac{2}{r}}_{[\tau,t]} L^\sigma_x} \|\nabla (v - u)\|_{L^2_{[\tau,t]} L^2_x}$$

$$\lesssim \|v\|_{L^p_{[\tau,t]} L^q_x} \|u - v\|_{L^{\frac{2}{r}}_{[\tau,t]} \dot H^r_x} \|\nabla (v - u)\|_{L^2_{[\tau,t]} L^2_x}$$

$$\le \|v\|_{L^p_{[\tau,t]} L^q_x} \Big\| \|u - v\|_{L^2}^{1-r} \|u - v\|_{\dot H^1}^{r} \Big\|_{L^{2/r}_{[\tau,t]}} \|\nabla (u - v)\|_{L^2_t L^2_x}$$

$$\lesssim \|v\|_{L^p_{[\tau,t]} L^q_x} \|u - v\|^{1-r}_{L^\infty_{[\tau,t]} L^2_x} \|\nabla (u - v)\|^{1+r}_{L^2_t L^2_x}.$$

(68)
$$\lesssim \|v\|_{L^p_{[\tau,t]} L^q_x} \left\{ \frac{1+r}{2} \int_\tau^t \int_{\mathbb{R}^d_x} |\nabla \otimes (u - v)|^2\, dx\, ds + \frac{1-r}{2} \|u - v\|^2_{L^\infty_t L^2_x} \right\}.$$

If $u \equiv v$ on $[0, \tau]$ and $t > \tau$ is such that $\frac{1+r}{2} \|v\|_{L^p_t L^q_x} < 1$ then we can absorb this contribution into (0) or (0) to obtain

(69)
$$\|u - v\|^2_{L^\infty_{[\tau,t]} L^2_x} \le \underbrace{\frac{1-r}{2} \|v\|_{L^p_{[\tau,t]} L^q_x}}_{< 1} \|u - v\|^2_{L^\infty_{[\tau,t]} L^2_x}.$$

This implies $u \equiv v$ up to time t. When $p < \infty$ this gives uniqueness. We obtain the same on all of $[0, T)$. When $p = \infty$ ($\Rightarrow q = d$) we obtain the theorem is proved for $q < \infty$. ∎
Conclusion if $\|v\|_{L^\infty_T L^d} << 1$. The

<u>Remark</u>: Sohr and Von Wahl obtained uniqueness with (iii) replaced by the condition $v \in C([0,T); L^d)$. The case when $\|v\|_{L^\infty_T L^d}$ is bounded but large remains open as far as I know for $d = 3$. For $d \ge 4$, uniqueness in $L^\infty_T L^d_x$ has recently been established by [Lions - Masmoudi:].

We pause to summarize the main ideas in the proof of Serrin's uniqueness theorem.

(70)
$$\|(u - v)(t)\|^2_{L^2_x} = -2 \int_\tau^t \int_{\mathbb{R}^d_x} |\nabla \otimes (u - v)|^2\, dx\, ds + 2 \iint_{\tau\ \mathbb{R}^d_x}^{t} v \cdot \big[(u - v) \cdot \nabla \big] (u - v)\, dx\, ds$$

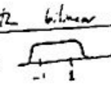
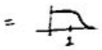
energy equality
zero divergence antisymmetry property.

(71)
$$\left| \int_\tau^t \int_{\mathbb{R}^d_x} v \cdot \big[(u - v) \cdot \nabla \big] (u - v)\, dx\, ds \right| \lesssim \|v\|_{L^p_{[\tau,t]} L^q_x} \left\{ C_r \int_\tau^t \int_{\mathbb{R}^d_x} |\nabla \otimes (u - v)|^2\, dx\, ds + C_r' \|u - v\|^2_{L^\infty_{[\tau,t]} L^2_x} \right\}.$$

Thus, the first term on right side of (71) may be absorbed by the first negative term on right side of (70) to obtain

$$\|(u - v)(t)\|^2_{L^2_x} \le \|v\|_{L^p_{[\tau,t]} L^q_x}\ C_r' \|u - v\|^2_{L^\infty_{[\tau,t]} L^2_x}$$

and this implies $u \equiv v$ on $[\tau, t]$.

"<u>proof" of the bilinear energy equality proposition</u>. Let $\theta(t,x) = \alpha(t) \beta(x) \in \mathcal{D}(\mathbb{R}^{d+1})$
where $\alpha =$  and $\beta(x) = \rho(|x|) =$  ... normalized so that $\iint \theta\, dx\, dt = 1$.
$\forall\ \varepsilon > 0$ define $\theta_\varepsilon(t, x) = \frac{1}{\varepsilon^{d+1}} \theta\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right)$. Then $\theta_\varepsilon * u_i$ is smooth on $(\varepsilon, T - \varepsilon) \times \mathbb{R}^d_x$
and we write

$$\partial_t \big((\theta_\varepsilon * u_1)(\theta_\varepsilon * u_2) \big) = \partial_t (\theta_\varepsilon * u_1)(\theta_\varepsilon * u_2) + (\theta_\varepsilon * u_1)\, \partial_t (\theta_\varepsilon * u_2)$$
$$= (\theta_\varepsilon * \partial_t u_1)(\theta_\varepsilon * u_2) + (\theta_\varepsilon * u_1)(\theta_\varepsilon * \partial_t u_2).$$

Then, we can use the equation. The resulting expression may be written, e.g.

$$(\theta_\varepsilon * \partial_t u_1)(\theta_\varepsilon * u_2) = \underset{①}{\nabla \cdot \big[(\theta_\varepsilon * [\nabla \otimes u_1]) \cdot (\theta * u_2) \big]} - \underset{②}{(\theta_\varepsilon * [\nabla \otimes u_1] \cdot (\theta_\varepsilon * [\nabla \otimes u_2])}$$
$$- \underset{③}{\nabla \cdot \big[(\theta_\varepsilon * [u_1 \otimes u_1] \big] \cdot (\theta_\varepsilon * u_2)} - \underset{④}{\big(\theta_\varepsilon * [u_1 \otimes u_1] \big) \cdot \big(\theta_\varepsilon * [\nabla \otimes u_2] \big)}$$
$$- \underset{⑤}{\nabla \cdot (\theta_\varepsilon * P_1)(\theta_\varepsilon * u_2)}.$$

Multiply this identity by $\psi(t)\, \varphi(x/R)$ with $\varphi \equiv 1$ on a neighborhood of the origin and let $R \to \infty$. Terms ①, ③, ⑤ go to zero upon passing the exposed divergence onto φ. This follows since the remnants of these terms (after moving the derivative) are $L^1((0,T) \times \mathbb{R}^d_x)$. We thus obtain a new identity valid in $\mathcal{D}'((\varepsilon, T-\varepsilon) \times \mathbb{R}^d)$

(72) $$\partial_t \int (\theta_\varepsilon * v_1)(\theta_\varepsilon * v_2)\,dx = -2\int \theta_\varepsilon * [\nabla\otimes v_1]\cdot \theta_\varepsilon * [\nabla\otimes v_2]\,dx$$
$$+ \int \theta_\varepsilon * [v_1\otimes v_2]\cdot \theta_\varepsilon * [\nabla\otimes v_2]\,dx$$
$$+ \underbrace{\int \theta_\varepsilon *_{\nabla\cdot} [v_2\otimes v_2]\cdot \theta_\varepsilon * [\nabla\otimes v_1]\,dx.}_{\text{may be suppressed}}$$

Restrict attention to $\mathbb{R}^d$; $d\geq 3$. We wish to prove that $v_1\otimes v_2 \in L^1((0,T)\times\mathbb{R}^d)$.
Since $L^2_t\dot H^1_x \cap L^\infty_t L^2_x \subset L^r_t L^r_x$ for $2\leq r\leq \frac{2d}{d-2}$ and $\frac{2}{r} = \frac{d}{2} - \frac{d}{r}$. We know that (iii) $v_1 \in L^p((0,T);L^q_x)$ for some (p,q) with $d\leq q<\infty$ and $\frac{2}{p} = 1 - \frac{d}{q}$.
We choose p and r so that $\frac{1}{r} = \frac{1}{2} - \frac{1}{q}$ so that $\frac{1}{r}+\frac{1}{q} = \frac{1}{2}$ and $\frac{1}{r} + \frac{1}{p} = \frac{d}{4} - \frac{d}{2}\left(\frac{1}{2}-\frac{1}{q}\right) + \frac{1}{2} - \frac{d}{2q} = \frac{1}{2}$.
By Hölder, we then have that

$$\|v_1\otimes v_2\|_{L^2_{(0,T)}L^2_x} \leq \|v_1\|_{L^r_t L^r_x}\|v_2\|_{L^p_t L^q_x} < \infty.$$

When $\varepsilon\to 0$ and $f\in L^2((0,T)\times\mathbb{R}^d)$, $\theta_\varepsilon * f \longrightarrow f$ in $L^2((0,T)\times\mathbb{R}^d)$.
Based on this convergence property, we have that, in $\mathcal{D}'$ sense

$$\partial_t\int(\theta_\varepsilon * v_1)(\theta_\varepsilon * v_2)\,dx \longrightarrow \partial_t\int v_1\cdot v_2\,dx$$

$$\int \theta_\varepsilon * [\nabla\otimes v_1]\cdot \theta_\varepsilon * [\nabla\otimes v_2]\,dx \longrightarrow \int \nabla\otimes v_1\cdot\nabla\otimes v_2\,dx$$

$$\int \theta_\varepsilon * [v_1\otimes v_1]\cdot \theta_\varepsilon * [\nabla\otimes v_2]\,dx \longrightarrow \int [v_1\otimes v_1]\cdot[\nabla\otimes v_2]\,dx = \int v_1\cdot(v_1\cdot\nabla)v_2\,dx$$

Thus, the only remaining issue is the convergence as $\varepsilon\to 0$ of the last integral in (72). Using the energy-based regularity bounds it can be justified that $\theta * \nabla\cdot[v_2\otimes v_2] = \theta * [(v_2\cdot\nabla)v_2]$. We have, by (iii), that $v_1\in L^p((0,T);L^q(\mathbb{R}^d))$ with $\frac{2}{p}+\frac{d}{q} = 1$. and $q<\infty$. We write $v_2\in L^r((0,T);L^r(\mathbb{R}^d))$ where we choose r so that $\frac{1}{r}+\frac{1}{q} = \frac{1}{2}$ and $\frac{1}{r}+\frac{1}{p} = \frac{1}{2}$. Since $\nabla\otimes v_2 \in L^2((0,T)\times\mathbb{R}^d)$ we learn that $(v_2\cdot\nabla)v_2 \in L^{p/(p-1)}((0,T);L^{q/(q-1)})$.
When $f\in L^{p/(p-1)}_T L^{q/(q-1)}_x$ we know that $\theta_\varepsilon * f \longrightarrow f$ strongly in $L^{p/(p-1)}_T L^{q/(q-1)}_x$ as $\varepsilon\to 0$. When $g\in L^p_T L^q_x$, $\theta_\varepsilon * g\longrightarrow g$ strongly in $L^p_T L^q_x$ as $\varepsilon\to 0$. This justifies the convergence

$$\int \theta_\varepsilon * [\nabla\cdot(v_2\otimes v_2)]\cdot(\theta_\varepsilon * v_1)\,dx \longrightarrow \int v_1\cdot(v_2\cdot\nabla)v_2\,dx$$

in $\mathcal{D}'((0,T))$ as $\varepsilon\to 0$.

We have thus justified the identity in $\mathcal{D}'((0,T))$:

$$\partial_t\int v_1\cdot v_2\,dx = -2\int \nabla\otimes v_1\cdot\nabla\otimes v_2\,dx + \int v_1\cdot(v_1\cdot\nabla)v_2\,dx - \int v_1\cdot(v_2\cdot\nabla)v_2\,dx.$$

It can be shown that $t\longmapsto \int v_1\cdot v_2\,dx$ is cts. So this identity may be integrated in time to prove the bilinear energy equality (67).

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4 Kato's Theory of Mild Solutions of NS($\mathbb{R}^d$) (following [Lemarié; Ch. 15])

We have seen (see (10)) that NS($\mathbb{R}^d$) may be reformulated using the Leray projector $\mathbb{P}$ onto divergence free vector fields as

$$\text{NS}(\mathbb{R}^d)\quad \begin{cases} (\partial_t - \Delta)v = -\mathbb{P}\nabla(v\otimes v) \\ \nabla\cdot v = 0 \\ v(0,x) = v_0(x). \end{cases}$$

A scheme to construct solutions of NS($\mathbb{R}^d$) is to find solutions to the (formally) equivalent integral equation

(73) $$v = e^{t\Delta}v_0 - \int_0^t e^{(t-s)\Delta}\,\mathbb{P}\nabla\cdot(v\otimes v)(s)\,ds.$$

Here $e^{t\Delta}$ is the heat semigroup defined by: $e^{t\Delta}v_0$ solves $\begin{cases}(\partial_t - \Delta)v = 0 \\ v(0,x) = v_0(x).\end{cases}$

Let us denote the bilinear Duhamel integral in (73) as $B(v,v)$.

One method to find solutions of (73) is to search for a fixed point of the transformation:

$$\underline{v} \longmapsto e^{t\Delta}\underline{v}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\nabla\cdot(\underline{v}\otimes\underline{v})(s)\,ds = e^{t\Delta}\underline{v}_0 - B(\underline{v},\underline{v}).$$

The Picard method to prove the existence of a fixed point suggests trying to prove a contraction estimate in an appropriate Banach space.

Structural Aspects of the Picard approach to solving $NS(\mathbb{R}^d)$

We wish to define a Banach space $\mathcal{E}_T$ consisting of solution candidates $\underline{w}:(0,T)\times\mathbb{R}^d\to\mathbb{R}^d$ s.t. $\forall\, \underline{w},\underline{v}\in\mathcal{E}_T$

$$B(\underline{v},\underline{w}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\nabla\cdot(\underline{w}\otimes v)(s)\,ds$$

is bounded

$$B:\mathcal{E}_T\times\mathcal{E}_T\longrightarrow\mathcal{E}_T.$$

Associated to the solution space $\mathcal{E}_T$ we consider the initial data space $E_T\subset\mathcal{S}'$ defined by

$$f\in E_T \iff e^{t\Delta}f\in\mathcal{E}_T.$$

$L^2_{uloc,x}$: $\forall\,\varphi\in\mathcal{D}((0,T)\times\mathbb{R}^d)$ $\sup_{x_0\in\mathbb{R}^d}\iint|\varphi(x-x_0,t)\,\underline{v}|^2\,dx\,dt<\infty$.

Theorem (Picard Contraction Principle)

Let $\mathcal{E}_T\subset L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d)$ be such that $B:\mathcal{E}_T\times\mathcal{E}_T\to\mathcal{E}_T$ is bounded. Then:

(a) If $v\in\mathcal{E}_T$ is a weak solution for $NS(\mathbb{R}^d)$

$$\partial_t v = \Delta v - \mathbb{P}\nabla\cdot(v\otimes v),\quad \nabla\cdot v = 0.$$

then the associated initial value $v_0\in E_T$.

(b) Conversely, $\exists\, C>0$ s.t. $\forall\, v_0\in E_T$ (with $\nabla\cdot v_0=0$) and $\|e^{t\Delta}v_0\|_{\mathcal{E}_T}<C$ $\exists$ weak solution $v\in\mathcal{E}_T$ of (73).

proof: (a) If $v\in\mathcal{E}_T$ and $B(v,v)\in\mathcal{E}_T$ then $e^{t\Delta}v_0\in\mathcal{E}_T\iff v_0\in E_T$.

(b) If $\|B(w,v)\|_{\mathcal{E}_T}\le C_0\|v\|_{\mathcal{E}_T}\|w\|_{\mathcal{E}_T}$ then if $\|e^{t\Delta}v_0\|_{\mathcal{E}_T}\le\delta<\frac{1}{4C_0}$ we find the map $\underline{v}\mapsto e^{t\Delta}v_0 - B(\underline{v},\underline{v})$ is contractive on the ball $B_{2\delta}=\{v:\|v\|_{\mathcal{E}_T}\le 2\delta\}$. Thus, the sequence defined by recursion $e^{t\Delta}v_0 = v^0$, $v^{(n+1)} = e^{t\Delta}v_0 - B(v^{(n)},v^{(n)})$ converges to a fixed point $\underline{v}$.

Theorem (Relaxed Picard [illegible])

Let $\mathcal{E}_T\subset\mathcal{F}_T\subset L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d)$, (with estimates) be such that

$$B:\mathcal{F}_T\times\mathcal{F}_T\longrightarrow\mathcal{E}_T.$$

(a) Let $v\in\mathcal{F}_T$ be a weak solution of $NS(\mathbb{R}^d)$. If $v_0\in E_T$ ($\Rightarrow e^{t\Delta}v_0\in\mathcal{E}_T$) then $v\in\mathcal{E}_T$.

(b) Conversely, $\exists\, C>0$ s.t. $\forall\, v_0\in E_T$ (satisfying $\nabla\cdot v_0=0$) and $\|e^{t\Delta}v_0\|_{\mathcal{F}_T}<C$ $\exists$ weak solution to (73) associated to v_0.

proof: Just check it...

Theorem (Regularity Criterion)

Let $v\in L^\infty((0,T)\times\mathbb{R}^d)$ solve $NS(\mathbb{R}^d)$. Then $v\in C^\infty((0,T)\times\mathbb{R}^d)$.

① Heat kernel: Consider the initial value problem for the heat equation: $\partial_t v - \Delta v = 0$; $v(0,x)=v_0(x)$. We pause to develop this regularity criterion. We take Fourier transforms and encounter $\partial_t\hat{v}(t,\xi)+|\xi|^2\hat{v}(t,\xi)=0$; $\hat{v}_0(0,\xi)=\hat{v}_0(\xi)$. This is an ODE family parametrized by ξ. We can solve it to find that $\hat{v}(t,\xi) = e^{-|\xi|^2 t}\hat{v}_0(\xi)$. If we define $K_t(x)$ to be the function whose Fourier transform is $e^{-t|\xi|^2}$ then $v(t,x) = (K_t * v_0)(x)$ by Fourier transform properties. Using properties of Gaussians and dilations under Fourier transform, we have that $K_t(x) = C_{\pi,n}\, t^{-\frac{d}{2}}e^{-\frac{|x|^2}{4t}}$, $t>0$. This function K_t is called the heat kernel or the Gauss kernel.

Note that $K_t(x) = t^{-\frac{d}{2}}K_1(x/\sqrt{t})$ and $\int K_t(x)\,dx = \hat{K}_t(0) = \text{const.}$ With this notation, with K_t in hand, we can define $e^{t\Delta}v_0 := K_t * v_0$. We can also define solutions of the inhomogeneous problem $\partial_t v - \Delta v = f$; $v(0,x)=v_0(x)$, with the Duhamel formula $v(t,x) = e^{t\Delta}v_0 + \int_0^t e^{(t-t')\Delta}f(t')\,dt'$. With the heat equation we have, by the flow property and uniqueness of solutions of the semigroup property: $e^{t_1\Delta}e^{t_2\Delta}v_0 = e^{(t_1+t_2)\Delta}v_0$.

② Young's Inequality

Next, recall Young's inequality for convolutions: If $f\in L^1$ and $g\in L^p$ ($1\le p\le\infty$) then $f*g\in L^p$, and $\|f*g\|_{L^p}\le\|f\|_{L^1}\|g\|_{L^p}$. Note that for $f\in L^p$, $g\in L^{p'}$ with $\frac{1}{p}+\frac{1}{p'}=1$, it is obvious by Hölder's inequality that $\|f*g\|_{L^\infty}\le\|f\|_{L^p}\|g\|_{L^{p'}}$. Interpolating between these bounds produces the general form of Young's inequality: $\|f*g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$ provided $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$.

Thus, we see that convolution with an L^1 kernel preserves L^p regularity.

③ Heat Kernel Derivatives

Recall $K_t(x) = t^{-d/2}K_1(x/\sqrt{t})$ where $K_1 = C_d e^{-|x|^2}$. We therefore see that $D_x^\alpha K_t(x) = t^{-d/2}(D_x^\alpha K_1)(\frac{x}{\sqrt{t}})(\frac{1}{\sqrt{t}})^{|\alpha|}$ by the chain rule for any multiindex α. A compact way to rewrite this property is that $(\sqrt{t\Delta})^\alpha K_t(x) = t^{-d/2}\left[(\sqrt{t\Delta})^\alpha(K_1)\right](\frac{x}{\sqrt{t}})$. Note also that $D_x^\alpha K_1 \in L^1$ for all multiindices α.

④ Parabolic Regularity

Suppose $u_0 \in L^\infty$. Then $u(t,x) = (K_t * u_0)(x)$ satisfies regularity estimates.

$\|u\|_{L^\infty_x} \le \|K_t\|_{L^1}\|u_0\|_{L^\infty} \Longrightarrow \|u(t)\|_{L^\infty} \le \|u_0\|_{L^\infty}$.

$\|D_x^\alpha u(t)\|_{L^\infty_x} \le \|(D_x^\alpha K_t)*u_0\|_{L^\infty_x} \le \left(\frac{1}{\sqrt{t}}\right)^{|\alpha|}\|t^{-d/2}D_x^\alpha K_1(\frac{\cdot}{\sqrt{t}})*u_0\|_{L^\infty}$

$\le \left(\frac{1}{\sqrt{t}}\right)^{|\alpha|}\|D_x^\alpha K_1\|_{L^1}\|u_0\|_{L^\infty} \le C_{t,u_0,\alpha} < \infty.$

$\Longrightarrow u \in C^\infty((0,\infty)\times\mathbb{R}^d_x)$

Suppose $f \in L^\infty((0,T), L^\infty(\mathbb{R}^d_x))$. Consider the Duhamel integral $\int_0^t K_{(t-t')}*f(t')\,dt'$ $:= w(t,x)$. $\|w(t)\|_{L^\infty_x} \le \int_0^t \|K_{(t-t')}*f(t')\|_{L^\infty_x}dt' \le \int_0^t \|f(t')\|_{L^\infty_x}dt'$ $\le t\|f\|_{L^\infty_{t,x}} \Longrightarrow w \in L^\infty_{t,x}$. Applying D_x^α and repeating the argument reveals that $\|D_x^\alpha w\|_{L^\infty} < \infty$ so $w \in C^\infty$ if $f \in L^\infty$.

Suppose $f \in L^\infty((0,T); L^\infty(\mathbb{R}^d_x))$ but we consider $\partial_t u - \Delta u = \nabla f$. Now, the forcing is potentially very bad. The associated Duhamel term is

$\int_0^t\int K_{(t-t')}(x-y)\,\nabla f(y)\,dy\,dt' = \int_0^t\int \nabla K_{(t-t')}(x-y)f(y)\,dy\,dt'$

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and, since ∇K_t is L_y^1 with size $\frac{1}{\sqrt{t}}$ we get

$\|\int_0^t K_{(t-t')}*\nabla f(t')dt'\|_{L^\infty} \le \int_0^t \frac{1}{\sqrt{t-t'}}\|f(t')\|_{L^\infty_y}dt' \le \|f\|_{L^\infty_{t,x}}$.

To avoid issues with an accumulation of $\sqrt{t}$ expressions in the denominator, it is natural to replace D_x^α by $(\sqrt{-t\Delta})^\alpha$

⑤ Navier-Stokes

The operator $e^{(t-t')\Delta}\mathbb{P}\nabla$ may be expressed as a convolution operator with a kernel whose L^1 norm is of size $(t-t')^{-1/2}$.

Combining these ideas produces a proof of the regularity criterion.

Kato-Fujita Theorem (Mild solutions in $H^s(\mathbb{R}^3)$, $s \ge \frac{1}{2}$): (following Lemarié).

(A) Let $s > \frac{1}{2}$. Then $\forall u_0 \in H^s(\mathbb{R}^3)$ with $\nabla\cdot u_0 = 0$ $\exists T^* > 0$ and a unique weak solution $u \in C([0,T^*); H^s(\mathbb{R}^d))$ for $NS(\mathbb{R}^d)$ on $(0,T^*)\times\mathbb{R}^3$ such that $u(0,\cdot) = u_0(\cdot)$. The solution is smooth on $(0,T^*)\times\mathbb{R}^3$.

(B) Let $s = \frac{1}{2}$. $\forall u_0 \in H^{1/2}(\mathbb{R}^3)$ with $\nabla\cdot u_0 = 0$ $\exists T^* > 0$ and a ~~unique~~ weak solution $u \in C([0,T^*); H^{1/2}(\mathbb{R}^3))$ for $NS(\mathbb{R}^3)$ on $(0,T^*)\times\mathbb{R}^3$ such that $u(0,\cdot) = u_0(\cdot)$. Moreover, this solution may be chosen to satisfy the extra condition that $\forall T \in (0,T^*)$, $u \in L^2((0,T); H^{3/2}(\mathbb{R}^3))$. With this extra condition on the solution, we have uniqueness and that the solution is smooth on $(0,T^*)\times\mathbb{R}^3$.

(C) $\exists \varepsilon_0 > 0$ s.t. if $\left(\int |\xi|\,|\hat{u}_0(\xi)|^2 d\xi\right)^{\frac{1}{2}} < \varepsilon_0$ then the existence time T^* in (A) or (B) for the smooth solution u is equal to ∞.

(D) The smooth solutions u in (A) or (B) satisfy $u(t,\cdot) \in C_0(\mathbb{R}^3)$ $\forall t < T^*$. The maximal existence time T^* is finite if and only if $\lim_{t\uparrow T^*}\|u(t)\|_{L^\infty} = \infty$.

Let's prove (B). We want to show that $H^{1/2}(\mathbb{R}^3)$ is a good initial value space for $NS(\mathbb{R}^3)$. We take $\mathcal{E}_T$, the spacetime function space for Picard contraction, to be $L^\infty([0,T]; H^{1/2}(\mathbb{R}^3)) \cap L^2([0,T]; H^{3/2}(\mathbb{R}^3))$. If $f \in \mathcal{E}_T$ then $\circledast$ $f \in L^4([0,T]; H^{1/2}(\mathbb{R}^3))$. Thus, if $f, g \in \mathcal{E}_T$ then $f \otimes g \in L^2([0,T], H^{1/2}(\mathbb{R}^3))$.

Next, consider the mapping $A : h(t,x) \longmapsto Ah(t,x) := \int_0^t e^{(t-t')\Delta} \Delta h(t',x)\, dt'$.

<u>Fact</u> (Maximal $L^p_t L^q_x$ regularity for the heat kernel):

$$A : L^p((0,T); L^q(\mathbb{R}^3)) \longrightarrow L^p((0,T); L^q(\mathbb{R}^3)).$$

We will prove this shortly for $p = q = 2$ [which will suffice for (B)].

Since $B(f,g) = A(\Delta^{-1}\mathbb{P}\nabla(f\otimes g))$, we have that $B(f,g) \in L^2([0,T]; H^{3/2}(\mathbb{R}^3))$.

<u>Lemma</u> Let $T \in (0, +\infty]$, $j \in \{1,2,3\}$. If $f \in L^2([0,T] \times \mathbb{R}^3)$ then $\int_0^t e^{(t-t')\Delta} \partial_j f(t')\, dt' \in L^\infty([0,T]; L^2(\mathbb{R}^3))$.

Assume the lemma for the moment. Then, since $f \otimes g \in L^2([0,T]; H^{1/2}(\mathbb{R}^3))$, ~~Thus, we find that~~ we find that $B(f,g) \in L^\infty([0,T]; H^{1/2}(\mathbb{R}^3))$. Thus, $B(f,g) \in L^\infty([0,T]; H^{1/2}(\mathbb{R}^3)) \cap L^2([0,T]; H^{3/2}(\mathbb{R}^3)) \subset L^4([0,T]; H^{1/2}(\mathbb{R}^3))$.

$B(f,g) \in L^\infty([0,T]; H^{1/2}(\mathbb{R}^3)) \cap L^4([0,T]; H^{1/2}(\mathbb{R}^3))$ that

In fact, we have shown for $\mathcal{F}_T = L^4([0,T]; H^{1/2}(\mathbb{R}^3))$ ~~In fact, we have that~~

$B : \mathcal{F}_T \times \mathcal{F}_T \longmapsto \mathcal{E}_T$. In fact, we have the estimate

~~$B : \mathcal{F}_T \times \mathcal{F}_T \longmapsto \mathcal{F}_T \subseteq \mathcal{E}$~~

$\|B(f,g)\|_{\mathcal{E}_T} \leq C_T \|f\|_{\mathcal{F}_T} \|g\|_{\mathcal{F}_T}$. (where C_T is a nondecreasing

not needed. → function of T that vanishes as $T \downarrow 0$) To validate the Picard function space framework contraction argument, we invoke the relaxed Picard Theorem. All that remains to verify is that $\forall\, u_0 \in H^{1/2}(\mathbb{R}^3)$,

$$\lim_{T\downarrow 0} \int_0^T \|e^{t\Delta} u_0\|^4_{H^{1/2}}\, dt = 0,$$ which is clear.

This proves (B) pending proof of the maximal $L^2_t L^2_x$ regularity fact and the lemma.

<u>proof of lemma</u> Write $g(t) = \int_0^t e^{(t-t')\Delta} \partial_j f(t')\, dt'$. Let $\varphi \in L^2(\mathbb{R}^3)$. We calculate

$$\langle g(t), \varphi\rangle_{L^2} = -\int_0^t \langle f(t'), e^{(t-t')\Delta} \partial_j \varphi \rangle_{L^2}\, dt'$$

So, by Cauchy-Schwarz, we have

$$|\langle g(t), \varphi\rangle_{L^2}| \leq \left(\int_0^t \int |f|^2\, dx\, dt'\right)^{\frac12} \left(\int_0^t \int |e^{(t-t')\Delta} \partial_j \varphi|^2\, dx\, dt'\right)^{\frac12}$$
$$\leq \left(\int_0^T \int |f|^2\, dx\, dt'\right)^{\frac12} \left(\int_0^{+\infty} \int |e^{(0+t')\Delta} \partial_j \varphi|^2\, dx\, dt'\right)^{\frac12}.$$

By Plancherel in space,

$$\int_0^{+\infty} \int |e^{t'\Delta} \partial_j \varphi|^2\, dx\, dt' = \frac{1}{(2\pi)^3} \int_0^{+\infty} \int |e^{-s|\xi|^2} \xi_j \hat{\varphi}(\xi)|^2\, d\xi\, ds$$
$$\leq C \|\varphi\|^2_{L^2}.$$

Thus, $g \in L^\infty([0,T]; L^2(\mathbb{R}^3))$ as claimed.

$\boxed{\text{skip}}$ <u>proof of Maximal Regularity</u>: We may suppose $T = +\infty$. (otherwise, replace h by an extension such that $h \equiv 0$ on (T, ∞).) We may extend h and Ah to $t < 0$ by zero extension: $f = Af = 0$ for $t \in (-\infty, 0)$.

Let W be the kernel of e^Δ: $W(x) = (4\pi)^{-\frac32} e^{-\frac{|x|^2}{4}}$, and define $\Omega(t,x) = \frac{1}{t^{d/2}}(\Delta W)\left(\frac{x}{\sqrt{t}}\right)$ for $t > 0$ and $\Omega(t,x) \equiv 0$ for $t < 0$.

Then, $Af(t,x) = \int_{t' \in \mathbb{R}} \int_{y \in \mathbb{R}^d} \frac{1}{\quad}$

<u>Heuristic</u>: $(\partial_t - \Delta)(Af) = \Delta f$ so $Af = (\partial_t - \Delta)^{-1} \Delta f$.

Thus, A appears to be a ~~ps~~ Fourier multiplier operator with symbol $\frac{-|\xi|^2}{i\tau + |\xi|^2}$. This multiplier satisfies

$$\left|\frac{-|\xi|^2}{i\tau + |\xi|^2}\right| \leq 1 \quad \text{so} \quad A : L^2_t L^2_x \longrightarrow L^2_t L^2_x.$$

$\circledast$ In fact we have $f \in L^\infty_T H^{1/2} \cap L^2_T H^{3/2} \Rightarrow f \in L^4_T H^1$. By Sobolev, we thus have that $f \in L^4_T L^6_x$ and that $D^{1/2} f \in L^4_T L^3_x$. Consider then $f \otimes g$. Apply $D^{1/2}$ and use the Leibniz rule to replace the considerations with $(D^{1/2} f) g$. But we want to estimate this expression in $L^2_T L^2_x$. $\|D^{1/2} f\, g\|_{L^2_x} \leq \|D^{1/2} f\|_{L^3_x} \|g\|_{L^6_x}$. Then we take the L^2_T

term and apply Cauchy-Schwarz to control

$$\| D^{1/2}(f \otimes g) \|_{L^2_T L^2_x} \leq \| D^{1/2} f \|_{L^4_T L^3_x} \, \| g \|_{L^4_T L^6_x}$$

Since $f, g \in L^\infty_T H^{1/2} \cap L^2_T H^{3/2}$, we know the right side is bounded.