ON BLOWUP SOLUTIONS OF NLS WITH LOW REGULARITY INITIAL DATA

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Abstract. The initial value problem for the focusing cubic nonlinear Schrödinger equation (NLS) on $\mathbb{R}^2$ is locally well-posed for initial data in $L^2$. The $L^2$ norm is invariant under the dilation symmetry of solutions, so this problem is called $L^2$-critical. Finite time blowup solutions of this problem are known to exist. Qualitative properties, such as mass concentration, of blowup solutions evolving from initial data much more regular than $L^2$ have been established. This talk will describe recent work toward a more descriptive theory of blowup solutions in the setting of $L^2$ initial data. Also, some comparisons with the (much less understood) blowup of $L^2$ supercritical NLS will be made.

1. Nonlinear Schrödinger Initial Value Problem

This talk concerns the initial value problem for the cubic nonlinear Schrödinger equation

$$\begin{cases}
i \partial_t u + \Delta u = -|u|^2 u, & u : (-T^*, T^*) \times \mathbb{R}^d \to \mathbb{C} \\
u(0, x) = u_0(x).
\end{cases}$$

(1.1)

We will mostly be interested in solutions with a finite (forward) maximal existence time interval $[0, T^*)$ and the issue to understand is the nature of the blowup as $t \uparrow T^*$.

Time Invariant Quantities

Sufficiently regular solutions of (1.1) evolve leaving the following quantities invariant. For all $t \in (-T^*, T^*)$,

$$\text{Mass} = \|u(t)\|_{L^2_x} = \|u_0\|_{L^2_x},$$

(1.2)

$$\text{Hamiltonian} = \int_{\mathbb{R}^d} |\nabla u(t)|^2 dx - \frac{1}{2} |u(t)|^4 dx = H[u_0].$$

(1.3)

Dilation Invariance

If $u : (-T^*, T^*) \times \mathbb{R}^2 \to \mathbb{C}$ solves (1.1) then $u_\lambda : (-T^* \lambda^2, T^* \lambda^2) \times \mathbb{R}^2 \to \mathbb{C}$ defined by

$$u_\lambda(\tau, y) = \lambda^{-1} u(\tau \lambda^{-2}, y \lambda^{-1})$$

(1.4)

also solves (1.1). Dilation invariant norms play a decisive role in the theory of (1.1). We specialize the discussion of (1.1) to the setting of two and three
spatial dimensions. For $x \in \mathbb{R}^2$, the dilation symmetry $u \mapsto u_\lambda$ leaves the $L^2_x$ norm invariant:

$$(1.5) \quad \| u_\lambda(\tau, \cdot) \|_{L^2_y} = \| u_0 \|_{L^2_y}.$$ 

For $x \in \mathbb{R}^3$, the dilation symmetry leaves the homogeneous Sobolev $\dot{H}^{\frac{1}{2}}_y$ and Lebesgue $L^3_y$ norms invariant:

$$(1.6) \quad \| D_x^{\frac{1}{2}} u_\lambda(\tau, \cdot) \|_{L^2_y} = \| D_x^{\frac{1}{2}} u(\lambda^{-2} \tau, \cdot) \|_{L^2_y},$$

$$(1.7) \quad \| u_\lambda(\tau, \cdot) \|_{L^3_y} = \| u(\lambda^{-2} \tau, \cdot) \|_{L^3_y}.$$ 

Based on these norm invariance properties of solutions, (1.1) with $x \in \mathbb{R}^2$ is called $L^2$-critical or mass critical. For $x \in \mathbb{R}^3$, (1.1) is called $L^3$ or $\dot{H}^{\frac{1}{2}}$ critical. Both of these problems are energy or $\dot{H}^1$ subcritical.

At this time, there is a more complete understanding of blowup solutions of mass critical than for mass supercritical (such as (1.1) with $x \in \mathbb{R}^3$) nonlinear Schrödinger equations. The goals of this talk are to

- survey some celebrated results,
- describe basic conjectures and open questions,
- discuss partial results & heuristic ideas towards the conjectures.

The talk will emphasize low regularity issues and some important recent advances in the theory of finite energy blowup solutions of the $L^2$-critical problem will not be discussed.

2. $L^2$-critical Focusing Cubic NLS: Well-posedness

An essentially complete local-in-time theory (see the textbook [2] by T. Cazenave for a survey) of the initial value problem (1.1) is now in place. This theory is based on spacetime dispersive decay estimates for the associated linear flow $e^{it\Delta}u_0$ and a fixed point argument. We recall some features of that theory.

- For all $u_0 \in L^2$ there exists $T^*(u_0), T_*(u_0) > 0$ and there exists a unique maximal solution $u$ of (1.1) for $x \in \mathbb{R}^2$ satisfying

$$(2.1) \quad u \in C((-T_*, T^*); L^2(\mathbb{R}^2)) \cap L^4((-T_*, T^*); L^4(\mathbb{R}^2)).$$

The forward-in-time maximality property of solutions is expressed by the property that $\| u \|_{L^4_{[0,T^*-\delta]} L^4_x} < \infty$ for all $\delta > 0$ but this norm diverges to infinity for $\delta = 0$. A similar property holds at the lower endpoint of the existence time interval.

- For all $s > 0$, if $u_0 \in H^s$ then $u(t) \in H^s$ for all $t \in [0, T^*)$. When $T^* < \infty$ there is the scaling lower bound

$$(2.2) \quad \| u(t) \|_{H^s} \gtrsim (T^* - t)^{-s/2}.$$
These is a scattering threshold mass $\mu_0$ (determined by sharp constants in the spacetime dispersive decay estimates) such that
\[(2.3) \quad \|u_0\|_{L^2} < \mu_0 \implies \|u\|_{L^4_t L^4_x} < 2\mu_0.\]
Thus, for small enough data $u_0$ the associated nonlinear evolution $u_0 \mapsto u$ is global-in-time and bounded in $L^4$ on spacetime. The finiteness of $L^4_t L^4_x$ allows for a precise description of the long-time asymptotic behavior in terms of the linear evolution of some related initial data $u_{\pm}$ as $t \to \pm \infty$.

There exists a sharp $H^1$-global well-posedness mass threshold $\|Q\|_{L^2}$ such that
\[(2.4) \quad \|u_0\|_{L^2} < \|Q\|_{L^2} \implies H^1 \ni u_0 \mapsto u\]
is global-in-time. The proof of this result, based on a sharp Gagliardo-Nirenberg estimate obtained by M. Weinstein, does not imply that the solution is bounded in $L^4_t L^4_x$. Here $Q$ is the ground state profile which arises as the unique (up to translations) positive solution of $-Q + \Delta Q = -Q^4$. There also exist excited state solutions of the $Q$ equation. This equation emerges when the time-periodic ansatz $u(t, x) = e^{it}Q(x)$ is inserted into (1.1). The result (2.4) can be relaxed \cite{4} \cite{5} to the setting of $H^s$ initial data in the range $s > 4/7$. A refinement of the Hamiltonian-based techniques in \cite{4} appears likely to extend this result to the range $s > 1/2$.

3. Blowup Solution Properties

Explicit finite time blowup solutions of (1.1) with $x \in \mathbb{R}^2$ are known to exist. Initial data with negative Hamiltonian and initially finite variance explode in finite time. Blowup solutions are known to concentrate mass into a point.

Time-periodic solutions $e^{it}Q(x)$ (with $Q$ a ground or excited state solution of the $Q$ equation) map under the pseudoconformal transformation to explicit finite time blowup solutions:
\[(3.1) \quad S(t, x) = \frac{1}{|t|}Q\left(\frac{x}{|t|}\right)e^{-i\frac{|x|^2}{4|t|} + \frac{i}{4|t|}}.\]
Since $\|S(-1, \cdot)\|_{L^2} = \|Q\|_{L^2}$, this is a minimal mass blowup solution in $H^1$ with $T^* = 0$. F. Merle \cite{7}, \cite{8} has shown that $u_0 \in H^1$, $\|u_0\|_{L^2} = \|Q\|_{L^2}$, $T^*(u_0) < \infty$ implies that $u = S$ up to an explicit solution symmetry of the equation (1.1).

A direct calculation using the equation (1.1) establishes the variance identity$^1$:
\[(3.2) \quad \partial_t^2 \int_{\mathbb{R}^2} |u(t, x)|^2 dx = 8H[u_0].\]

$^1$This identity is also called the virial identity.
Initial data with negative Hamiltonian and finite variance collapse with variance going to zero and conserved $L^2$ mass in finite time. Note that the variance of positive energy solutions is upwardly convex with time.

- $H^1$ Theory of Mass Concentration: F. Merle and Y. Tsutsumi have proved [11] that $H^1 \cap \{ \text{radial} \} \ni u_0 \liminf_{T^* < \infty} \int_{|x| < (T^* - t)^{1/2}} |u(t, x)|^2 \mathrm{d}x \geq \|Q\|_{L^2}^2$.

Thus, any radial initially $H^1$ solution concentrates at least the mass of the ground state on an essentially parabolic concentration window as $t \uparrow T^*$. In contrast, the explicit blowup solution (3.1) concentrates all of its mass on the tighter linear window $|x| < (T^* - t)$. Subsequent work has removed the radial assumption and provided further extensions of this result for $H^1$ blowup solutions.

- $L^2$ Theory of Mass Concentration: By exploiting delicate refinements of the Strichartz estimate, J. Bourgain has proved that $L^2 \ni u_0 \limsup_{T^* < \infty} \sup_{\text{cubes } I, \text{side}(I) \leq (T^* - t)^{1/2}} \int_I |u(t, x)|^2 \mathrm{d}x \geq \|u_0\|_{L^2}^{-M}$.

for some large $M$. Thus, nontrivial mass concentration occurs at blowup time. The result is also valid for the defocusing analog of (1.1). Note that the dependence of the concentrated mass upon $\|u_0\|_{L^2}$ allows for the possibility that huge initial data could possibly concentrate a tiny amount of mass at blowup. Work by F. Merle & L. Vega [10] and recent preprints by S. Keraani and P. Bégout & A. Vargas have improved our knowledge of the blowup of $L^2$ solutions of $L^2$-critical NLS.

- Fantastic progress on the blowup of $H^1$ solutions having mass slightly larger than $\|Q\|_{L^2}$ has been made recently in work by G. Perelman and in a series of articles by P. Raphaël and F. Merle. In particular, precise asymptotic profile properties and norm explosion rates, the existence of two distinct blowup regimes, and regularity properties of the solution outside the blowup have been established.

4. Conjectures/Open Questions

- Scattering Below the Ground State Mass:

(4.1) $\|u_0\|_{L^2} < \|Q\|_{L^2} \implies ??? u_0 \liminf ??? u_0 \mapsto u$ with $\|u\|_{L^4} < \infty$.

(Also, $L^2$ solutions of the defocusing analog of (1.1) on $\mathbb{R}^2$ satisfy $\|u\|_{L^4} < \infty$.)

- Minimal Mass Blowup Characterization:

(4.2) $\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \liminf ??? u_0 \mapsto u, T^* < \infty \implies ??? u = S,$
modulo a solution symmetry. An intermediate step would be to extend Merle’s characterization of the minimal mass blowup solutions in $H^s$ for $s < 1$.

- Concentrated mass amounts are quantized: The explicit blowups constructed by pseudoconformally transforming time periodic solutions with ground and excited state profiles are the only asymptotic profiles. (See [9] for further discussion on the quantization conjectures.)

- Are there any general upper bounds?

5. Partial Results

- With S. Raynor, C. Sulem and J.D. Wright, we have shown: For $0.86 \sim \frac{1}{T}(1 + \sqrt{T}) < s < 1$, $H^s \cap \{radial\} \ni u_0 \mapsto u, T^* < \infty \Rightarrow$

\[
\liminf_{t \uparrow T^*} \int_{|x| < (T^* - t)^{s/2}} |u(t, x)|^2 \, dx \geq \|Q\|_{L^2}^2.
\]

Thus, for a certain interval of $s < 1$, $H^s$ finite time blowup solutions concentrate at least the mass of the ground state on a disk shrinking slower than parabolic. This result is based on an imitation of a proof of (3.3) using the almost conserved quantity $H[Iu]$ from [4] in place of the Hamiltonian. Note also the lim inf in (5.1) versus the lim sup appearing in (3.3).

- Recently, T. Hmidi and S. Keraani have shown an $H^s$ analog of an $H^1$ result of M. Weinstein for the interval of $s$ where (5.1) holds: \[\|u_0\|_{L^2} = \|Q\|_{L^2}, u_0 \in H^s, \sim 0.86 < s < 1, T^* < \infty \Rightarrow \text{there exists } t_n \uparrow T^* \text{ such that } u(t_n) \rightarrow Q \text{ in } H^{\tilde{s}(s)} (\text{mod symmetries along the sequence}). \]

For $L^2$ blowups with $\|u_0\|_{L^2} > \|Q\|_{L^2}, u(t_n) \rightarrow V \in H^1$ (mod symmetries along the sequence). Weinstein’s result was a precursor to Merle’s characterization of the minimal mass $H^1$ blowup solutions so these developments may forecast an extension of the minimal mass blowup characterization to the $H^s$ setting.

- Work in progress with S. Roudenko investigates how lower bounds on the divergence rate of $\|u\|_{L^4_{[0, t]L^4}}$ as $t \uparrow T^*$ influence the rate the concentration window shrinks and vice versa.

- With J. Holmer and W. Staubach, we are considering ideas toward proving (4.1).

A recent preprint of S. Keraani establishes:

- There exists a minimal mass blowup threshold $\delta_0 \geq \mu_0 > 0$ for $L^2$ solutions.

- There exists a minimal mass blowup solution $L^2 \ni u_0 \mapsto u, T^* < \infty$ with $\|u_0\|_{L^2} = \delta_0$.

- For all $t_n \uparrow T^*$ there is a subsequence $t_n$ along which $u(t_n) \rightarrow \psi$ (mod symmetries along the sequence) in $L^2$. Thus, there exists a ”rescaled asymptotic object” along subsequences.)
Suppose $\delta_0 < \|Q\|_{L^2}$ for contradiction. Assume the radial case. There exists a subsequence $t_n \uparrow T^*$ such that $u(t_n) = \rho_n^{-1} \phi(x\rho_n^{-1} ) + z_n, \rho_n \downarrow 0$ for some asymptotic object $\phi \in L^2$ and $\|z_n\|_{L^2} = \epsilon_n \downarrow 0$. Since $u(t_n)$ is asymptotically represented as a dilation of $\phi$, any localized variance of $u(t_n)$ should shrink to zero. Meanwhile, all good $H^1$ approximators $w$ of $u(t_n)$ are mass subcritical: $\|w\|_{L^2} = \delta_0 < \|Q\|_{L^2} \Rightarrow H[w] > 0$. Thus, by (3.2), good $H^1$ approximators have global variance exploding to infinity. So the variance appears to shrink and also explode and this may yield a contradiction.

6. $H^{1/2}$-Critical Focusing Cubic NLS on $\mathbb{R}^3$

There exists a similar well-posedness theory with $L^2(\mathbb{R}^2)$ systematically replaced by $H^{1/2}(\mathbb{R}^3)$ and other associated adjustments. For example the relevant spacetime norm is $L^5_{t,x} \in \mathbb{R}^3$ instead of $L^4_{t,x} \in \mathbb{R}^2$. For the rest of the talk, we will consider $u$ to be a solution of (1.1) on $\mathbb{R}^3$.

No explicit blowup solutions are known. An adjusted variance identity still implies that there exists many blowup solutions. How do these blowup solutions behave? I learned the following argument recently from C. Sulem. If $H^1 \cap \{ \text{radial} \} \ni u_0 \mapsto u,T^* < \infty$ then

$$\|\nabla u(t)\|^2_{L^2_{\{x\mid x\mid < a\}}} \uparrow \infty \text{ as } t \uparrow T^*.$$  

Thus, radial solutions must explode at the origin.

Proof. By Hamiltonian conservation and an easy decomposition,

$$\|\nabla u(t)\|^2_{L^2} = H[u_0] + \frac{1}{2} \|u(t)\|^4_{L^4_{\{x\mid x\mid < a\}}} + \frac{1}{2} \|u(t)\|^4_{L^4_{\{x\mid x\mid > a\}}}.$$  

The potential energy in the interior region $|x| < a$ is estimated using Gagliardo-Nirenberg by $C(Mass,a)\|\nabla u(t)\|^3_{L^3_{\{x\mid x\mid < a\}}}$. The potential energy in the exterior region is estimated by pulling out two factors in $L^\infty_x$ and then using the radial Sobolev estimate of W. Strauss. Using $ab < \epsilon a^2 + \epsilon^{-1}b^2$ for an appropriate $\epsilon$ allows one to absorb the exterior kinetic energy contribution into the left side to obtain

$$\|\nabla u(t)\|^2_{L^2} \lesssim C(a,Mass[u_0],H[u_0]) + C(a,Mass[u_0])\|\nabla u(t)\|^3_{L^3_{\{x\mid x\mid < a\}}}.$$  

Since the constants in (6.3) are time-independent, we see that the explosion of the total kinetic to infinity drives the kinetic energy on the interior region to infinity.

We conclude the talk with four remarks.

• Consider the preceding argument for a $p$-th power generalization of (1.1) with nonlinearity $|u|^{p-1}u$. The needed property of the interior potential energy estimate remains valid if the equation is energy subcritical. The exterior potential energy estimate remains valid.
provided $p < 5$. Thus, energy-subcritical subquintic radial focusing NLS blowup solutions must blowup at the origin.

- Very recently, a preprint of P. Raphaël establishes: There exists a $H^1 \cap \{\text{radial}\} \ni v_0 \mapsto v, T^*(v_0) < \infty$ solving quintic focusing NLS on $\mathbb{R}^2$ (this is also $H^{1/2}$-critical) which blows up precisely on a circle.

- Based on numerical evidence and heuristic arguments about the blowup profile, it is conjectured that $\|u(t)\|_{L^3_x} \uparrow \infty$ as $t \uparrow T^*$. This conjecture is analogous to a recent result [6] on the Navier-Stokes equation: finite time blowup solutions of Navier-Stokes on $\mathbb{R}^3$ must blowup in $L^3_x$.

- Note that an $H^{1/2}$ self-similar dilation to blowup with fixed profile has an asymptotically shrinking $L^2$ norm. Thus, we should not expect self-similar blowup solutions in the setting of cubic NLS on $\mathbb{R}^3$. With Sulem, Raynor and Wright, we are investigating a proof by contradiction strategy toward proving the divergence of the $L^3_x$ and $H^{1/2}$ norms of $u(t)$ as $t \uparrow T^*$. We suppose for contradiction that $\|u(t)\|_{H^{1/2}_x} < \Lambda < \infty$ for all times $t \in [0, T^*)$. This is a scaling invariant proxy for the $L^2$ upper bound implied by $L^2$ conservation in the $L^2$ critical blowup problem. We hope to show, based on $L^2$ conservation and the assumed $H^{1/2}$ upper bound that $u(t)$ contains nontrivial low and high frequency components which asymptotically decouple as $t \uparrow T^*$. The high frequency part of the blowup then behaves essentially like the self-similar solution which we know does not exist since it contradicts mass conservation.

References


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