Improved Interaction Morawetz Inequalities for the Cubic Nonlinear Schrödinger Equation on $\mathbb{R}^2$

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We prove global well-posedness for low regularity data for the $L^2$-critical defocusing nonlinear Schrödinger equation (NLS) in 2D. More precisely, we show that a global solution exists for initial data in the Sobolev space $H^s(\mathbb{R}^2)$ and any $s > \frac{2}{5}$. This improves the previous result of Fang and Grillakis where global well-posedness was established for any $s \geq \frac{1}{2}$. We use the $I$-method to take advantage of the conservation laws of the equation. The new ingredient is an interaction Morawetz estimate similar to one that has been used to obtain global well-posedness and scattering for the cubic NLS in 3D. The derivation of the estimate in our case is technical since the smoothed out version of the solution $Iu$ introduces error terms in the interaction Morawetz inequality. A by-product of the method is that the $H^s$ norm of the solution obeys polynomial-in-time bounds.
1 Introduction

In this paper we study the \( L^2 \)-critical Cauchy problem

\[
\begin{aligned}
&iu_t + \Delta u - |u|^2 u = 0, \quad x \in \mathbb{R}^2, \quad t \in \mathbb{R}, \\
u(x, 0) = u_0(x) \in H^s(\mathbb{R}^2).
\end{aligned}
\] (1.1)

The problem is known to be locally well-posed for any \( s > 0 \). The local well-posedness definition that we use here reads as follows: for any choice of initial data \( u_0 \in H^s \), there exists a positive time \( T = T(\|u_0\|_{H^s}) \) depending only on the norm of the initial data, such that a solution to the initial value problem exists on the time interval \([0, T]\), it is unique in a certain Banach space of functions \( X \subset C([0, T], H^s_x) \), and the solution map from \( H^s_x \) to \( C([0, T], H^s_x) \) depends continuously on the initial data on the time interval \([0, T]\). If the time \( T \) can be proved to be arbitrarily large, we say that the Cauchy problem is globally well-posed. A local solution also exists for \( L^2 \) initial data but the time of existence depends not only on the \( L^2 \) norm of the initial data but also on the profile of \( u_0 \). For all the above results the reader can refer to \([2], [3], \) and \([17]\). Local in-time solutions enjoy mass conservation

\[
\|u(\cdot, t)\|_{L^2(\mathbb{R}^2)} = \|u_0(\cdot)\|_{L^2(\mathbb{R}^2)}.
\] (1.2)

Moreover, \( H^1 \) solutions enjoy conservation of the energy

\[
E(u)(t) = \frac{1}{2} \int_{\mathbb{R}^2} \|
abla u(t)\|^2 dx + \frac{1}{4} \int_{\mathbb{R}^2} |u(t)|^4 dx = E(u)(0),
\] (1.3)

which, together with (1.2) and the local theory, immediately yields global-in-time well-posedness for (1.1) with initial data in \( H^1 \). The \( L^2 \) local well-posedness of the Initial Value Problem (IVP) (1.1) in an interval \([0, T]\) and the conservation of the \( L^2 \) norm, cannot be immediately used to give global-in-time solutions as in the case of the finite energy data. As we said, in this case \( T = T(u_0) \) and the lifetime of the local-in-time result can approach zero for fixed \( L^2 \) norm. For the focusing case it is known that large mass solutions can blow up in finite time. Nevertheless, in the defocusing case no blowup solutions are known and thus it is conjectured that (1.1) is globally well-posed and scatters for \( L^2 \) initial data. In other words, we expect the solution of the nonlinear equation to scatter to a free solution \( e^{it\Delta} u_\pm \) as \( t \to \pm \infty \) for some \( u_\pm \in L^2_x(\mathbb{R}^2) \) in the sense that

\[
\lim_{t \to \pm \infty} \|u(t) - e^{it\Delta} u_\pm\|_{L^2_x(\mathbb{R}^2)} = 0.
\]
Conversely, given any \( u_0 \) in \( L^2_2(\mathbb{R}^2) \) there exists a solution which scatters to it in the sense above, thus giving rise to well-defined wave and scattering operators. This conjecture is known to be true in the case that the initial data has sufficiently small mass. For details see, [3].

For solutions below the energy threshold, the first result was established by Bourgain, [1]. Bourgain decomposed the initial data into low frequencies and high frequencies and estimated separately the evolution of low and high frequencies. He showed that the solution is globally well-posed with initial data in \( H^s(\mathbb{R}^2) \) for any \( s > \frac{3}{5} \). Moreover if we denote with \( S_t \) the nonlinear flow and with \( S(t) = e^{it\Delta}u_0 \) the linear group, Bourgain’s method shows in addition that \( (S_t - S(t))u_0 \in H^1(\mathbb{R}^2) \) for all times provided \( u_0 \in H^s, s > \frac{3}{5} \). Further improvements (but without the \( H^1 \) proximity of the linear and nonlinear flows) were given in [7] and [12], where the authors used the “I-method” that we describe below. Recently, Tao, Visan, and Zhang proved [19] global well-posedness and scattering for the \( L^2 \)-critical problem in all dimensions \( n \geq 3 \), assuming radially symmetric initial data. They used the reductions in [18] to eliminate blowup solutions that are almost periodic modulo scaling. As in [9] they obtained a frequency-localized Morawetz estimate and excluded a “mass evacuation scenario” in order to conclude the argument. Their argument cannot be extended easily in low dimensions or without the radial assumption on the initial data. We, on the other hand, consider the general problem in 2D with general initial data but we only relax the regularity requirements of the initial data, being unable so far to prove the result for initial data in \( L^2 \).

We use the I-method and we follow closely the argument in [12] (see also [7], [8]) which is based on the almost conservation of a certain modified energy functional. The idea is to replace the conserved quantity \( E(u) \) which is no longer available for \( s < 1 \), with an “almost conserved” variant \( E(Iu) \) where \( I \) is a smoothing operator of order \( 1 - s \) which behaves like the identity for low frequencies and like a fractional integral operator for high frequencies. Thus the operator \( I \) maps \( H^s \) to \( H^1 \). Notice that \( Iu \) is not a solution to (1.1) and hence we expect an energy increment. This increment is in fact quantifying \( E(Iu) \) as an “almost conserved” energy. The key is to prove that on intervals of fixed length, where local well-posedness is satisfied, the increment of the modified energy \( E(Iu) \) decays with respect to a large parameter \( N \). (For the precise definition of \( I \) and \( N \) we refer the reader to Section 2.) This requires delicate estimates on the commutator between \( I \) and the nonlinearity. In dimensions 1 and 2, where the nonlinearity is algebraic, one can write the commutator explicitly using the Fourier
transform, and control it by multilinear analysis and bilinear estimates. The statement of our main result is:

**Theorem 1.1.** The initial value problem (1.1) is globally well-posed in $H^s(\mathbb{R}^2)$, for any $1 > s > \frac{5}{2}$. Moreover, the solution satisfies

$$
\sup_{[0,T]} \| u(t) \|_{H^s(\mathbb{R}^2)} \leq C(1 + T)^{\frac{3(1-s)}{4s-2}}
$$

where the constant $C$ depends only on the index $s$, $\| u_0 \|_{L^2}$ and $\| u_0 \|_{H^s}$.

**Remark 1.2.** We view this result as another incremental step towards the conjecture that $L^2$ initial data $u_0$ evolves under (1.1) to a global-in-time solution with $\| u \|_{L^4(\mathbb{R}^4 \times \mathbb{R}^2)} < C(\| u_0 \|)$. Recent work [10], based upon the second modified energy and certain angular refinements of the bilinear Strichartz estimate, has improved the energy increment quantification (see (3.8)) from $N^{-3/2^+}$ to $N^{-2^+}$. This improvement is applied in [10] following the globalizing scheme in [7] (which does not rely upon any monotonicity or Morawetz-type inputs) to prove that (1.1) is globally well-posed in $H^s(\mathbb{R}^2)$ for $s > \frac{1}{2}$. Similarities between the proofs of the almost conservation estimate (3.8) and the almost Morawetz increment estimate (4.5) suggest that angle refinements as in [10] could possibly improve the decay in (4.5) below $N^{-1^+}$, possibly to $N^{-3/2^+}$. If true, this would improve the global well-posedness range to $s > \frac{4}{13}$.

The basic ingredient in our proof is an a priori interaction Morawetz-type estimate for the “approximate solution” $Iu$ to the initial value problem

$$
\begin{align*}
\begin{cases}
    iIu_t + \Delta Iu - I(|u|^2u) = 0 & x \in \mathbb{R}^2, \ t \in \mathbb{R} \\
    Iu(x,0) = Iu_0(x) & \in H^1(\mathbb{R}^2).
\end{cases}
\end{align*}
$$

(1.4)

For the original system (1.1) it has been shown in [12] that solutions satisfy the following a priori bound

$$
\| u \|^4_{L^4_t L^4_x} \lesssim T^4 \sup_{t \in [0,T]} \| u(t) \|^2_{H^1} \| u_0 \|^2_{L^2}.
$$

This estimate follows from a “two-particle” Morawetz inequality. “Two-particle” Morawetz estimates first appeared in three dimensions in [8]. Roughly speaking, we refer to Morawetz inequalities as monotonicity formulae that take advantage of the conservation
of momentum
\[ \tilde{p}(t) = \int_{\mathbb{R}^n} \mathcal{J}(\tilde{u}(x) \nabla u(x)) \, dx. \]

In dimensions \( n \geq 3 \) the classical Morawetz inequality relies ultimately on the fact that the tempered distribution \( \Delta \Delta |x| \) is well-defined and non-negative. In particular for \( n = 3 \) we have \( \Delta \Delta |x| = \delta_0 \). The case \( n = 2 \) is more delicate and this is the novelty of the approach in [12]. In this paper we improve the above inequality to

\[ \| u \|_{L_t^2 L_x^4}^4 \lesssim T^+ \sup_{t \in [0,T]} \| u(t) \|_{L_x^2}^2 \| u_0 \|_{L_x^2}^2 + C T^+ \| u_0 \|_{L_x^4}^4 \]

where \( C \) is a large constant. For initial data below \( H^\perp \) both of these estimates are not useful. This is mainly the limitation of the result in [12]. To avoid this difficulty we can introduce the \( I \)-operator and hope to get an apriori estimate of the form

\[ \| Iu \|_{L_t^2 L_x^4}^4 \lesssim T^+ \sup_{t \in [0,T]} \| Iu(t) \|_{H^s} \| Iu_0 \|_{L_x^2}^2 + C T^+ \| u_0 \|_{L_x^4}^4 + \text{Error}. \]  

Then the restriction \( s \geq \frac{1}{2} \) is not present and, in principle, one can improve the result in [12]. Of course, we have to show that the \( \text{Error} \) terms are negligible in some sense. More precisely we show that on the local well-posedness time interval the error terms are very small. The proof of this fact relies on multilinear harmonic analysis estimates (similar to those used to prove almost conservation bounds in [8]) and is given in Section 4. Before we outline the general method of the paper we define the following Banach space:

\[ Z_I(J) := \sup_{(q,r) \text{ admissible}} \| \langle \nabla \rangle^s Iu \|_{L_t^q L_x^r (J \times \mathbb{R}^2)} \]

where a pair \( (q,r) \) is said to be admissible if \( \frac{1}{q} + \frac{1}{r} = \frac{1}{2}, 2 \leq r, q \leq \infty \) and \( (q,r) \neq (2, \infty) \). We will sometimes write \( Z_I(t) \) to denote \( Z_I([0,t]) \).

Now fix a large value of time \( T_0 \). If \( u \) is a solution to (1.1) on the time interval \([0, T_0]\), then \( u^\lambda(x) = \frac{1}{\lambda} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right) \) is a solution to the same equation on \([0, \lambda^2 T_0]\). We choose the parameter \( \lambda > 0 \) so that \( E(Iu_0^\lambda) = O(1) \). Using Strichartz estimates we show that if \( J = [a,b] \) and \( \| Iu^\lambda \|_{L_t^2 L_x^4 (J \times \mathbb{R}^2)} < \mu \), where \( \mu \) is a small universal constant, then

\[ \sup_{(q,r) \text{ admissible}} \| \langle \nabla \rangle^s Iu^\lambda \|_{L_t^q L_x^r (J \times \mathbb{R}^2)} \lesssim \| Iu^\lambda(a) \|_{H^s} \lesssim 1. \]
Moreover, in this same time interval where the problem is well-posed, we can prove the “almost conservation law”

\[ |E(Iu^\lambda)(b) - E(Iu^\lambda)(a)| \lesssim N^{-\frac{1}{2}} \|Iu^\lambda(a)\|_{H^1}^4 + N^{-2} \|Iu^\lambda(a)\|_{H^1}^6 \lesssim N^{-\frac{1}{2}}. \quad (1.6) \]

For the arbitrarily large interval \([0, \lambda^2 T_0]\), we do not have that

\[ \|Iu^\lambda\|_{L^4_tL^4_x([0,\lambda^2 T_0] \times \mathbb{R}^2)} < \mu. \]

But we can partition the arbitrarily large interval \([0, \lambda^2 T_0]\) into \(L\) intervals where the local theory uniformly applies. \(L = L(N, T)\) is finite and defines the number of the intervals in the partition that will make the Strichartz \(L^4_tL^4_x\) norm of \(Iu\) less than \(\mu\). Since \(E(Iu^\lambda)\) controls the \(H^1\) norm of \(Iu\), we have by (1.6) that

\[ \|Iu^\lambda\|_{H^1}^2 \lesssim LN^{-\frac{1}{2}}. \]

To maintain the bound \(\|Iu^\lambda\|_{H^1} \lesssim 1\) we choose

\[ L(N, T) \sim N^{\frac{1}{4}} \]

and this condition will ultimately lead to the requirement \(s > \frac{3}{4}\). For the detailed proof, see Section 5.

The rest of the paper is organized as follows. In Section 2 we introduce some notation and state important propositions that we will use throughout the paper. In Section 3 we prove the local well-posedness theory for \(Iu\), and the main estimates that we use to prove the decay of the increment of the modified energy. The decay itself is obtained in Section 3. In Section 4 we prove the “almost Morawetz” inequality which is the heart of our argument. Finally in Section 5 we give the details of the proof of global well-posedness stated in Theorem 1.1.

2 Notation

In what follows we use \(A \lesssim B\) to denote an estimate of the form \(A \leq CB\) for some constant \(C\). If \(A \lesssim B\) and \(B \lesssim A\) we say that \(A \sim B\). We write \(A \ll B\) to denote an estimate of the form \(A \leq cB\) for some small constant \(c > 0\). In addition \(\langle a \rangle := 1 + |a|\) and \(a^{\pm} := a \pm \epsilon\) with \(0 < \epsilon \ll 1\).
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We use $\mathcal{S}$ to denote the Schwartz class. We use $L^r_x$ to denote the Lebesgue space of functions $f : \mathbb{R}^2 \to \mathbb{C}$ whose norm
$$\|f\|_{L^r_x} := \left( \int_{\mathbb{R}^2} |f(x)|^r \, dx \right)^{\frac{1}{r}}$$
is finite, with the usual modification in the case $r = \infty$. We also define the space-time spaces $L^q_t L^r_x$ by
$$\|u\|_{L^q_t L^r_x} := \left( \int_J \|u\|_{L^r_x}^q \, dt \right)^{\frac{1}{q}}$$
for any space-time slab $J \times \mathbb{R}^2$, with the usual modification when either $q$ or $r$ is infinity. When $q = r$ we abbreviate $L^q_t L^r_x$ by $L^q_t$. Sometimes we will write $L^q_T$ to denote $L^q_t \in [0,T]$.

Definition 2.1. A pair of exponents $(q, r)$ is called admissible in $\mathbb{R}^2$ if
$$\frac{1}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 \leq q, r \leq \infty, \quad \text{and} \quad (q, r) \neq (2, \infty).$$

We recall the Strichartz estimates [13], [14] (and the references therein).

Proposition 2.2. Let $(q, r)$ and $(\tilde{q}, \tilde{r})$ be any two admissible pairs. Suppose that $u$ is a solution to
$$\begin{cases}
i \partial_t u + \Delta u - G(x, t) = 0, \quad (t, x) \in J \times \mathbb{R}^2 \\
u(x, 0) = u_0(x).
\end{cases}$$

Then we have the estimate
$$\|u\|_{L^q_t L^r_x(J \times \mathbb{R}^2)} \lesssim \|u_0\|_{L^2(\mathbb{R}^2)} + \|G\|_{L^{\tilde{q}}_t L^{\tilde{r}}_x(J \times \mathbb{R}^2)}$$
with the prime exponents denoting Hölder dual exponents. \hfill \Box

We define the Fourier transform of $f(x) \in L^1_x$ by
$$\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i \xi \cdot x} f(x) \, dx.$$ 

For an appropriate class of functions the following Fourier inversion formula holds:
$$f(x) = \int_{\mathbb{R}^2} e^{2\pi i \xi \cdot x} \hat{f}(\xi) \, d\xi.$$ 

Moreover we know that the following identities are true:
We also define the fractional differentiation operator $D^\alpha$ for any real $\alpha$ by
\[
\hat{D}^\alpha u(\xi) := |\xi|^\alpha \hat{u}(\xi)
\]
and analogously
\[
\hat{(D)}^\alpha u(\xi) := (\xi)^\alpha \hat{u}(\xi).
\]
We then define the inhomogeneous Sobolev space $H^s$ and the homogeneous Sobolev space $\dot{H}^s$ by
\[
\|u\|_{H^s} = \|\langle D \rangle^s u\|_{L^2_x}; \quad \|u\|_{\dot{H}^s} = \|D^s u\|_{L^2_x}.
\]

## 3 The I-method and the proof of Theorem 1.1
We shall also need some Littlewood-Paley theory, [16]. The reader must have in mind that wherever in this paper we restrict the functions in frequency we do it in a smooth way using the Littlewood-Paley projections. In particular, let $\eta(\xi)$ be a smooth bump function supported in the ball $|\xi| \leq 2$, which is equal to 1 on the unit ball. Then, for each dyadic number $M$ we define the Littlewood-Paley operators
\[
\begin{align*}
\hat{P}_{\leq M} \hat{f}(\xi) &= \eta(\xi/M) \hat{f}(\xi), \\
\hat{P}_{> M} \hat{f}(\xi) &= (1 - \eta(\xi/M)) \hat{f}(\xi), \\
\hat{P}_M \hat{f}(\xi) &= (\eta(\xi/M) - \eta(2\xi/M)) \hat{f}(\xi).
\end{align*}
\]
Similarly, we can define $P_{< M}, P_{\geq M}$. The Littlewood-Paley decomposition we write, at least formally, is $u = \sum_M P_M u$. For convenience we abbreviate the Littlewood-Paley operator $P_M$ by $u_M$ or even $u_j$ when its meaning is clear from the context. We can write $u = \sum u_j$ and obtain bounds on each piece separately or by examining the interactions of the several pieces. We can recover information for the original function $u$ by applying
the Cauchy-Schwartz inequality and using the Littlewood-Paley Theorem or the cheap Littlewood-Paley inequality
\[ \|P_N u\|_{L^p} \lesssim \|u\|_{L^p} \]
for any \(1 \leq p \leq \infty\). Since this process is fairly standard we will omit the details of the argument throughout the paper. We also recall the definition of the operator \(I\). For \(s < 1\) and a parameter \(N \gg 1\) let \(m(\xi)\) be the following smooth monotone multiplier:
\[ m(\xi) := \begin{cases} 1 & \text{if } |\xi| < N \\ \left(\frac{|\xi|}{N}\right)^s & \text{if } |\xi| > 2N. \end{cases} \]

We define the multiplier operator \(I : H^s \to H^1\) by
\[ \hat{I}u(\xi) = m(k)\hat{u}(\xi). \]

The operator \(I\) is smoothing of order \(1 - s\) and we have that:
\[ \|u\|_{H^s} \lesssim \|Iu\|_{H^s+1-s} \lesssim N^{1-s}\|u\|_{H^s} \tag{3.1} \]
for any \(s_0 \in \mathbb{R}\).

We set
\[ E^1(u) = E(Iu), \tag{3.2} \]
where
\[ E(u)(t) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(t)|^2 dx + \frac{1}{4} \int_{\mathbb{R}^2} |u(t)|^4 dx = E(u)(0) = E(u_0). \]

We refer to \(E^1(u)\) as the first modified energy.

3.1 Modified Local Well-Posedness

**Proposition 3.1.** Define the quantity \(\mu([0, T]) = \int_0^T \int_{\mathbb{R}^2} |Iu(x, t)|^4 dx dt\). If \(\mu([0, T]) < \mu_0\) where \(\mu_0\) is a universal constant, then for any \(s > 0\) the initial value problem \((1.4)\) is locally well-posed and we have that
\[ Z_I([0, T]) := \sup_{(q, r) \text{ admissible}} \| \langle D \rangle \|_{L^q_x L^r_t([0, T] \times \mathbb{R}^2)} \| \langle D \rangle Iu_0 \|_{L^2} \lesssim \| \langle D \rangle Iu_0 \|_{L^2}. \]
Proof. By standard well-posedness theory, see for example [12], it is enough to show
\[ Z_I([0, T]) := \sup_{(q,r) \text{ admissible}} \|\langle D \rangle Iu\|_{L_q^2 L_r^\alpha([0,T] \times \mathbb{R}^3)} \lesssim \|\langle D \rangle Iu_0\|_{L^2}. \]

By the Duhamel formula we have an equivalent representation for the solution of (1.4) and is given by
\[ Iu(x,t) = e^{it\Delta} Iu_0 - i \int_0^T e^{i(t-s)\Delta} I(\|u\|^4 u)(s) ds. \]

Applying the \(\langle D \rangle\) operator in Equation (1.4) and using the Strichartz estimates (2.1) we have that
\[ Z_I \lesssim \|Iu_0\|_{H^s} + \|\langle D \rangle I(\|u\|^4 u)\|_{L_t^4 L_{\alpha}^\frac{4}{\alpha}} \lesssim \|Iu_0\|_{H^s} + \|\langle D \rangle Iu\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}}. \]

Note that we have used Leibnitz’s rule for fractional derivatives in the previous step. Indeed the multiplier \(\langle D \rangle I\) is increasing for any \(s \geq 0\). Using this fact one can modify the proof of the usual Leibnitz rule for fractional derivatives and prove it also for \(\langle D \rangle I\). Thus,
\[ Z_I \lesssim \|Iu_0\|_{H^s} + Z_I \|u\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}}^2. \]  

(3.3)

Now recall the definition of \(I\). We write \(u = u_{<N} + \sum_{j=1}^\infty u_k\), where \(u_{<N}\) has spatial frequency support on \(\langle \xi \rangle \leq N\) and the \(u_k\) have support \(\langle \xi_j \rangle = N_j = 2^k\) where \(k_j\) are consecutive integers starting with \(|\log N|\) indexed by \(j = 1, 2, \ldots\). Note that
\[ \|u_k\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}} \lesssim N_j^{1-s} N_j^{-1} \|Iu_k\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}}, \quad \text{if } j = 1, 2, \ldots \]

By the triangle inequality we have
\[ \|u\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}} \leq \|u_{<N}\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}} + \sum_{j=1}^\infty \|u_k\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}} = \|Iu_{<N}\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}} + \sum_{j=1}^\infty \|u_k\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}} \|u_k\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}}. \]

By the definition of the \(u_j\)’s the following estimates are true:
\[ \|Iu_{<N}\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}} \lesssim \|Iu\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}} \]  

(3.4)
\[ \|u_k\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}} \lesssim N_j^{1-s} N_j^{-1} \|Iu_k\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}}, \quad j = 1, 2, \ldots \]  

(3.5)
\[ \|u_k\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}} \lesssim N_j^{1-s} N_j^{-1} \|\langle D \rangle Iu_k\|_{L_t^4 L_{\frac{4}{\alpha}}^\frac{4}{\alpha}}, \quad j = 1, 2, \ldots \]  

(3.6)
Combining all these estimates we get

\[ \|u\|_{L_t^4L_x^4} \lesssim \|u_0\|_{L_x^4} + \sum_{j=1}^{\infty} N_j^{-s+\epsilon} N_j^{-1} \|\langle D \rangle u_{k_j}\|_{L_t^4L_x^4}^{1-s} \|\langle D \rangle u_{k_j}\|_{L_t^4L_x^4}^s \]

and after eliminating \(N_j^{-1} \leq 1\) we have

\[ \|u\|_{L_t^4L_x^4} \lesssim \|u_0\|_{L_x^4} + \sum_{j=1}^{\infty} N_j^{-s+\epsilon} \|\langle D \rangle u_{k_j}\|_{L_t^4L_x^4}^{1-s} \|\langle D \rangle u_{k_j}\|_{L_t^4L_x^4}^s. \]

Now if we apply the cheap Littlewood-Paley inequality

\[ \|u_k\|_{L^p} \lesssim \|u\|_{L^p} \]

for any \(1 \leq p \leq \infty\) and sum, we have that for any \(s > \epsilon\) we get

\[ \|u\|_{L_t^4L_x^4} \lesssim \|u_0\|_{L_x^4} + \|\langle D \rangle u\|_{L_t^4L_x^4}^{1-s} \|\langle D \rangle u\|_{L_t^4L_x^4}^s. \]

Putting all these into Equation (3.3) we have

\[ Z_i \lesssim \|u_0\|_{\dot{H}^s} + Z_i \|u\|_{L_t^4L_x^4}^2 + Z_i^{3-2\epsilon} \|\langle D \rangle u\|_{L_t^4L_x^4}^{2\epsilon} \]

Now if we pick a time \(T\) such that \(\mu([0,T]) < \mu_0 \ll 1\) we have

\[ Z_i([0,T]) := \sup_{(q,r) \text{ admissible}} \|\langle D \rangle u\|_{L_t^4L_x^4([0,T] \times \mathbb{R}^2)} \lesssim \|\langle D \rangle u_0\|_{L^2}. \]

We finish this section with the almost conservation of the modified energy.

**Proposition 3.2.** If \(H^s \ni u_0 \mapsto u(t)\) with \(\frac{1}{2} > s > \frac{1}{2}\) solves (1.1) for all \(t \in [0,T_{\text{lwp}}]\) where \(T_{\text{lwp}}\) is the time that Proposition 3.1 applies. Then

\[ \sup_{t \in [0,T_{\text{lwp}}]} |E[I_N u(t)]| \leq |E[I_N u(0)]| + CN^{-\frac{1}{2}} \|I_N(D)u(0)\|_{L_x^4}^2 + CN^{-2} \|I_N(D)u(0)\|_{L_x^4}^{6}. \]  

(3.7)

In particular, when \(\|I_N(D)u(0)\|_{L_x^4} \lesssim 1\) we have

\[ \sup_{t \in [0,T_{\text{lwp}}]} |E[I_N u(t)]| \leq |E[I_N u(0)]| + CN^{-\frac{1}{2}} \lesssim N^{-\frac{1}{2}}. \]  

(3.8)

Proof. This is Proposition 3.7 in [6]. The restriction for \(s > \frac{1}{2}\) appears in Case 3 in the proof. \(\blacksquare\)
4 The almost Morawetz estimate

For what follows we sometimes abbreviate $u_i = u(x_i)$ where $u_i$ is a solution to

$$iu_t + \Delta u = |u|^2 u, \ (x_i, t) \in \mathbb{R}^2 \times [0, T]. \quad (4.1)$$

Here $x_i \in \mathbb{R}^n$, not a coordinate. In this section we wish to prove the ‘almost Morawetz’ estimate. For this consider $a : \mathbb{R}^n \to \mathbb{R}$, a convex and locally integrable function of polynomial growth.

**Theorem 4.1.** Let $u \in L^\infty_{[0,T]} S_x$ be a solution to the NLS

$$iu_t + \Delta u = |u|^2 u, \ (x, t) \in \mathbb{R}^2 \times [0, T] \quad (4.2)$$

and $Iu \in L^\infty_{[0,T]} S_x$ be a solution to the $I$-NLS

$$iIu_t + \Delta Iu = I(|u|^2 u), \ (x, t) \in \mathbb{R}^2 \times [0, T]. \quad (4.3)$$

Then,

$$\|Iu\|_{L^4 T L^4}^4 \lesssim T^{\frac{4}{3}} \sup_{[0,T]} \|Iu\|_{L^1}^\frac{1}{4} \|Iu\|_{L^2}^\frac{3}{4} + T^{\frac{4}{3}} \|u_0\|_{L^2}^4 + \quad (4.4)$$

$$T^{\frac{4}{3}} \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla a \cdot \{\tilde{N}_{bad}, Iu(x_1, t)Iu(x_2, t)\}_p dx_1 dx_2 dt.$$ 

where

$$\tilde{N}_{bad} = \sum_{i=1}^2 \left( I(|u_i|^2 u_i) - |u_i|^2 Iu_i \right) \prod_{j=1, j \neq i}^2 Iu_j$$

and $\{,\}_p$ is the momentum bracket defined by

$$\{f, g\}_p = \Re(f \nabla g - g \nabla f).$$

In particular, on a time interval $J_k$ where the local well-posedness Proposition 3.1 holds we have

$$\int_{J_k} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla a \cdot \{N_{bad}, Iu(x_1, t)Iu(x_2, t)\}_p dx_1 dx_2 dt \lesssim \frac{1}{N^{\frac{4}{3}}} Z_{J_k}^6.$$  

$\square$
Improved Interaction Morawetz Inequalities for Cubic NLS on $\mathbb{R}^2$

Toward this aim, we recall the idea of the proof of the interaction Morawetz estimate for the defocusing nonlinear cubic Schrödinger equation in three space dimensions [8]. We present the result using a different argument involving a tensor of Schrödinger solutions that emerged from a conversation between Andrew Hassell and Terry Tao. We will establish the ‘almost Morawetz’ estimate that we need in this paper, along the lines of this new point of view. In all of our arguments we will assume smooth solutions. This will simplify the calculations and will enable us to justify the steps in the subsequent proofs. The local well-posedness theory and the perturbation theory [3] that has been established for this problem can be then applied to approximate the $H^s$ solutions by smooth solutions and conclude the proofs. For most of the calculations in this section the reader can consult [9], [17].

Let start with a solution to the NLS

$$iu_t + \Delta u = \tilde{N}(u), \quad (x, t) \in \mathbb{R}^n \times [0, T]$$  \hspace{1cm} (4.6)

with $u$ Schwartz-class-in-space and $\tilde{N}$ such that there exist a defocusing potential $G$, (meaning $G$ positive) such that

$$\{\tilde{N}, u\}_p = -\partial_j G.$$

Let us define also the momentum density

$$T_{0j} = 2\Im(\overline{\dot{u}} \partial_j u)$$

for $j = 1, 2, \ldots, n$, and the linearized momentum current

$$L_{jk} = -\partial_j \partial_k (|u|^2) + 4\Re(\overline{\partial_j u} \partial_k u).$$

A computation shows that

$$\partial_t T_{0j} + \partial_k L_{jk} = 2\{\tilde{N}, u\}_p^j$$

where we have adopted Einstein’s summation convention. Notice also that in our case where $\tilde{N} = |u|^2 u$ we have $\{\tilde{N}, u\}_p^j = -\partial_j G$, where $G = \frac{1}{2}|u|^4$. By integrating in space we have the total momentum conserved in time,

$$\int_{\mathbb{R}^n} T_{0j}(x, t) dx = C.$$

We recall the generalized virial identity [15].
Proposition 4.2. If \( a \) is convex and \( u \) is a smooth solution to \((4.6)\) on \([0, T] \times \mathbb{R}^n\) with a defocusing potential \( G \). Then, the following inequality holds:

\[
\int_0^T \int_{\mathbb{R}^n} \left(-\Delta \Delta a\right) \|u(x, t)\|^2 \, dx \, dt \lesssim \sup_{[0, T]} |M_a(t)|,
\]

where \( M_a(t) \) is the Morawetz action and is given by

\[
M_a(t) = 2 \int_{\mathbb{R}^n} \nabla a(x) \cdot \Im(u(x) \nabla u) \, dx.
\]

Proof. We can write the Morawetz action as

\[
M_a(t) = \int_{\mathbb{R}^n} \partial_j a T_{0j}.
\]

Then

\[
\partial_t M_a(t) = \int_{\mathbb{R}^n} \partial_j a \partial_k T_{0j} = \int_{\mathbb{R}^n} \partial_j a \left(-\partial_k L_{jk} + 2 \{\tilde{N}, u\}_p\right)
\]

\[
= \int_{\mathbb{R}^n} \partial_j a \left(-\partial_k L_{jk} - 2\partial_j G\right) = \int_{\mathbb{R}^n} (\partial_j \partial_k a) L_{jk} \, dx + 2 \int_{\mathbb{R}^n} \Delta a G \, dx
\]

where in the last equality we used integration by parts. By the definition of \( L_{jk} \) we have

\[
\partial_t M_a(t) = \int_{\mathbb{R}^n} (\partial_j \partial_k a) \partial_j \partial_k (|u|^2) \, dx + 4 \int_{\mathbb{R}^n} (\partial_j \partial_k a) \Re (\partial_j \tilde{u} \partial_k u) \, dx + 2 \int_{\mathbb{R}^n} \Delta a G \, dx.
\]

Performing the summations, we record the generalized virial identity

\[
\partial_t M_a(t) = -\int_{\mathbb{R}^n} (\Delta \Delta a) |u|^2 \, dx + 2 \int_{\mathbb{R}^n} \Delta a G \, dx + 4 \int_{\mathbb{R}^n} (\partial_j \partial_k a) \Re (\partial_j \tilde{u} \partial_k u) \, dx.
\]

But since \( a \) is convex, we have

\[
4(\partial_j \partial_k a) \Re (\partial_j \tilde{u} \partial_k u) \geq 0
\]

and the trace of the Hessian of \( \partial_j \partial_k a \), which is \( \Delta a \), is positive. Thus,

\[
-\int_{\mathbb{R}^n} (\Delta \Delta a) |u|^2 \, dx \leq \partial_t M_a(t)
\]

and by the fundamental theorem of calculus we have

\[
\int_0^T \int_{\mathbb{R}^n} \left(-\Delta \Delta a\right) \|u(x, t)\|^2 \, dx \, dt \lesssim \sup_{[0, T]} |M_a(t)|.
\]
In the case of a solution to an equation with a nonlinearity which is not associated to a defocusing potential, we immediately obtain the following corollary.

**Corollary 4.3.** Let \( a : \mathbb{R}^d \to \mathbb{R} \) be convex and \( u \) be a smooth solution to the equation

\[
iu_t + \Delta u = \tilde{N}, \quad (x, t) \in \mathbb{R}^d \times [0, T].
\]

Then, the following inequality holds

\[
\int_0^T \int_{\mathbb{R}^n} (\Delta \Delta a) |u(x,t)|^2 \, dx \, dt + 2 \int_0^T \int_{\mathbb{R}^d} \nabla a \cdot \{ \tilde{N}, u \}_p \, dx \, dt \lesssim |M_a(T) - M_a(t)|,
\]

where \( M_a(t) \) is the Morawetz action corresponding to \( u \). □

### 4.1 Interaction Morawetz inequality in three dimensions

Now we consider the interaction Morawetz inequality. Let \( u_i, \tilde{N}_i \) be solutions to (4.6) in \( n_i \)-spatial dimensions and suppose we have as before momentum conservation with a defocusing potential. Define the tensor product \( u := (u_1 \otimes u_2)(t,x) \) for \( x \) in

\[
\mathbb{R}^{n_1+n_2} = \{ (x_1, x_2) : x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2} \}
\]

by the formula

\[
(u_1 \otimes u_2)(t,x) = u_1(x_1, t)u_2(x_2, t).
\]

It can be easily verified that if \( u_1 \) solves (4.6) with forcing term \( \tilde{N}_1 \) and \( u_2 \) solves (4.6) with forcing term \( \tilde{N}_2 \), then \( u_1 \otimes u_2 \) solves (4.6) with forcing term \( \tilde{N} = \tilde{N}_1 \otimes u_2 + \tilde{N}_2 \otimes u_1 \).

Since

\[
\{ \tilde{N}_1 \otimes u_2 + \tilde{N}_2 \otimes u_1, u_1 \otimes u_2 \}_p = \left( \{ \tilde{N}_1, u_1 \}_p \otimes |u_2|^2, \{ \tilde{N}_2, u_2 \}_p \otimes |u_1|^2 \right)
\]

we have the important fact that the tensor product of the defocusing semilinear Schrödinger equation is also defocusing in the sense that

\[
\{ \tilde{N}_1 \otimes u_2 + \tilde{N}_2 \otimes u_1, u_1 \otimes u_2 \}_p = -\nabla G
\]
where $G = G_1 \otimes |u_2|^2 + G_1 \otimes |u_1|^2$ and $\nabla = (\nabla_{x_1}, \nabla_{x_2})$. Thus $G \geq 0$. Since $u_1 \otimes u_2$ solves (4.6) and obeys momentum conservation with a defocusing potential, we can apply the Proposition 4.2 and obtain for a convex functions $\alpha$:

$$
\int_0^T \int_{\mathbb{R}^n_1 \times \mathbb{R}^n_2} (-\Delta u)(x, t) dx dt \leq \sup_{[0,T]} |M_{\alpha}^{\otimes 2}(t)|
$$

(4.12)

where $\Delta = \Delta_{x_1} + \Delta_{x_2}$ is the Laplacian in $\mathbb{R}^{n_1+n_2}$ and $M_{\alpha}^{\otimes 2}(t)$ is the Morawetz action that corresponds to $u_1 \otimes u_2$ and thus

$$
M_{\alpha}^{\otimes 2}(t) = 2 \int_{\mathbb{R}^n_1 \times \mathbb{R}^n_2} \nabla a(x) \cdot \Im (\bar{u_1} \otimes u_2(x) \nabla (u_1 \otimes u_2(x))) dx.
$$

Now we pick $a(x) = a(x_1, x_2) = |x_1 - x_2|$ where $(x_1, x_2) \in \mathbb{R}^3 \times \mathbb{R}^3$. Then an easy calculation shows that $-\Delta \Delta a(x_1, x_2) = C\delta(x_1 - x_2)$. Applying Equation (4.12) with this choice of $a$ and choosing $u_1 = u_2$ we get

$$
\int_0^T \int_{\mathbb{R}^3} |u(x, t)|^4 dx \leq \sup_{[0,T]} |M_{\alpha}^{\otimes 2}(t)|.
$$

It can be shown using Hardy’s inequality (for details see [8]) that in 3D

$$
\sup_{[0,T]} |M_{\alpha}^{\otimes 2}(t)| \lesssim \sup_{[0,T]} \|u(t)\|_{H^2}^2 \|u(t)\|_{L^2}^2
$$

and thus

$$
\int_0^T \int_{\mathbb{R}^3} |u(x, t)|^4 dx \lesssim \sup_{[0,T]} \|u(t)\|_{H^2}^2 \|u(t)\|_{L^2}^2
$$

which is the 3D interaction Morawetz estimate that appears in [8].

Remark 4.4. Note that although we start with different solutions $u_1$ and $u_2$ at the end, we specialize to $u_1 = u_2 = u$. We will omit this step because the notation can be confused with the abbreviation $u_i = u(x_i)$. The meaning will always be clear from the context and thus we avoid introducing a notation that will read $u_i(x_i) := u_i^i$ for different solutions taking values in $\mathbb{R}^{n_i}$.

4.2 Interaction Morawetz inequality in two dimensions

For $n = 2$ (in that case $(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2$) we proceed as follows:
Let $f : [0, \infty) \to [0, \infty)$ be such that

$$f(x) := \begin{cases} 
\frac{1}{2M} x^2 (1 - \log \frac{M}{x}) & \text{if } |x| < \frac{M}{\sqrt{e}} \\
100x & \text{if } |x| > M
\end{cases}$$

smooth and convex for all $x$.

and $M$ is a large parameter that we will choose later. It is obvious that the functions $\frac{1}{2M} x^2 (1 - \log \frac{M}{x})$ and $100x$ are convex in their domain, and the graph of either function lies strictly above the tangent lines of the other. Thus one can construct a function with the above properties. If we apply Proposition 4.2 with the weight $a(x_1, x_2) = f(|x_1 - x_2|)$, and tensoring again two functions we conclude

$$\int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} (-\Delta \Delta a(x_1, x_2)) u(x_1, t) u(x_2, t)^2 dx_1 dx_2 dt \lesssim 2 \sup_{[0, T]} |M a_2(t)|$$

But for $|x_1 - x_2| < \frac{M}{\sqrt{e}}$ we have $\Delta a(x_1, x_2) = \frac{2}{M} \log \left( \frac{M}{|x_1 - x_2|} \right)$ and thus

$$- \Delta \Delta a(x_1, x_2) = \frac{4\pi}{M} \delta_{[x_1, x_2]}.$$ 

On the other hand, for $|x_1 - x_2| > M$ we have

$$- \Delta \Delta a(x_1, x_2) = O \left( \frac{1}{|x_1 - x_2|^3} \right) = O \left( \frac{1}{M^3} \right).$$

We have a similar bound in the region in between just because $a(x_1, x_2)$ is smooth, so all in all, we have

$$- \Delta \Delta a(x_1, x_2) = \frac{4\pi}{M} \delta_{[x_1, x_2]} + O \left( \frac{1}{M^3} \right).$$

Thus

$$\int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} (-\Delta \Delta a(x_1, x_2)) u(x_1, t)^2 u(x_2, t) dx_1 dx_2 dt$$

$$= \frac{4\pi}{M} \int_0^T \int_{\mathbb{R}^2} u(x, t)^4 dx dt + O \left( \frac{1}{M^3} \right) \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} u(x_1, t) u(x_2, t)^2 dx_1 dx_2 dt.$$

By Fubini’s Theorem

$$\int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} |u(x_1, t)|^2 |u(x_2, t)|^2 dx_1 dx_2 dt \lesssim \frac{C}{M^3} \|u\|_{L^4_{T \times \mathbb{R}^2}}^4.$$ (4.13)
On the other hand,

$$\sup_{[0,T]} |M_u^{\alpha}(t)| \lesssim \sup_{[0,T]} \|u\|_{L^\infty L^2_x}^3 \|u\|_{L^\infty H^1_t}.$$ 

Thus by applying Proposition 4.2

$$\frac{4\pi}{M} \int_0^T \int_{\mathbb{R}^2} |u(t,x)|^4 \, dx \, dt \lesssim \sup_{[0,T]} \|u\|_{L^\infty L^2_x}^3 \|u\|_{L^\infty H^1_t} + \frac{CT}{M^\gamma} \|u\|_{L^\infty L^2_x}^4.$$ 

Multiplying the above equation by $M$ and balancing the two terms on the right hand side by picking $M \sim T^\gamma$, we get a better estimate than was obtained in [12]

$$\|u\|_{L^\infty L^2_x}^4 \lesssim T^\gamma \sup_{[0,T]} \|u\|_{L^\infty L^2_x}^3 \|u\|_{L^\infty H^1_t} + T^\gamma \|u\|_{L^\infty L^2_x}^4.$$ 

We note that $\nabla a \in L^\infty(\mathbb{R}^2)$, an observation that will be used strongly later.

### 4.3 A new a priori Strichartz estimate in one dimension

We can combine the calculations that we did in the two-dimensional case with the work in [5] and obtain the following estimate in one dimension,

$$\|u\|_{L^6_{x;[0,T]} L^2_x}^6 \lesssim T^\gamma \sup_{[0,T]} \|u\|_{L^\infty L^2_x}^5 \|u\|_{L^\infty H^1_t} + T^\gamma \|u\|_{L^\infty L^2_x}^6.$$ (4.14)

We can derive this estimate by considering $(x_1, x_2, x_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, tensoring three solutions, and using an orthonormal change of variables $z = Ax$ where $A$ is an orthonormal matrix. Then we pick the convex weight function to be

$$a(z_1, z_2, z_3) = \begin{cases} \frac{1}{2M} (z_2^2 + z_3^2) \left(1 - \log \frac{(z_2^2 + z_3^2)^{1/2}}{M}\right) & \text{if } (z_2^2 + z_3^2)^{1/2} < \frac{M}{\sqrt{e}} \\ 100(z_2^2 + z_3^2)^{1/2} & \text{if } (z_2^2 + z_3^2)^{1/2} > M \\ \text{smooth and convex for all } z \in \mathbb{R}^3 \end{cases}$$

But then $\Delta_x = \Delta z$, and an explicit calculation shows that

$$\Delta_x a(z_1, z_2, z_3) = C a(z_1, z_2, z_3) \frac{1}{M} \delta_{z_2 - z_3} + O \left(\frac{1}{M^2}\right).$$

Thus balancing the two terms as in the two-dimensional case and going back to the original variables, for details see [5], we obtain (4.14).
Remark 4.5. A similar estimate can be obtained if one interpolates the one-dimensional estimate

\[ \|u\|_{L^8_t L^8_x} \lesssim \sup_{[0,T]} \|u\|_{L^{\infty}_x L^2_t} \|u\|_{L^{\infty}_t \dot{H}^1_x} \]

that was proved in [5] and the trivial estimate

\[ \|u\|_{L^2_t L^2_x} \lesssim T^{1/3} \sup_{[0,T]} \|u\|_{L^{\infty}_x L^2_t} \lesssim T^{1/2} \|u_0\|_{L^2_x} \]

where we used Hölder’s inequality in time and the conservation of mass. This kind of estimate has been already used in [11] to improve the known global well-posedness results for the quintic defocusing nonlinear Schrödinger equation in one dimension.

4.4 Interaction Morawetz inequality in two dimensions for the I-NLS equation and the proof of Theorem 4.1

We now proceed to prove Theorem 4.1. For motivational purposes let us consider the solution $Iu$ of

\begin{equation}
    iIu_t + \Delta Iu = I(|u|^2u), \ (x,t) \in \mathbb{R}^2 \times [0,T].
\end{equation}

If $Iu$ would solve not (4.15) but the nonlinear Schrödinger equation

\begin{equation}
    iIu_t + \Delta Iu = |Iu|^2Iu, \ (x,t) \in \mathbb{R}^2 \times [0,T],
\end{equation}

then the calculations that we did above in two dimensions would reveal that

\[ \|Iu\|_{L^4_t L^4_x}^4 \lesssim T^{1/2} \sup_{[0,T]} \|Iu\|_{L^{\infty}_x L^2_t}^3 \|Iu\|_{L^{\infty}_t \dot{H}^1_x} + T \|Iu\|_{L^2_x}^4. \]

Of course this is not the case. But we can rewrite Eqn (4.15) as

\begin{equation}
    iIu_t + \Delta Iu = |Iu|^2Iu + I(|u|^2u) - |Iu|^2Iu = F(Iu) + (IF(u) - F(Iu)).
\end{equation}

Then if we repeat the calculations, the commutator $IF(u) - F(Iu)$ will introduce an error term while the term $F(Iu)$ again gives rise to a defocusing potential. Thus by Corollary 4.3
we get
\[
\int_0^T \int_{\mathbb{R}^2} (-\Delta \Delta a) |Iu(x,t)|^2 \, dx \, dt \lesssim \sup_{0,T} \int_{\mathbb{R}^2} \nabla a(x) \cdot \Im (Iu(x) \nabla Iu) \, dx \\
+ \left| \int_0^T \int_{\mathbb{R}^2} \nabla a \cdot \{IF(u) - F(Iu), Iu(x,t)\}_p \, dx \, dt \right|.
\]

The second term on the right hand side of the inequality is what we call an Error in (1.5).

We now turn to the details. The conjugates will play no crucial role in the upcoming argument. Let us set
\[
IU(x,t) = I \otimes I (u(x_1,t) \otimes u(x_2,t)) = \prod_{j=1}^2 Iu(x_j,t).
\]

If \(u\) solves (4.6) for \(n = 2\), then we observe that \(IU\) solves (4.6) for \(n = 4\), with right hand side \(\tilde{N}_I\) given by
\[
\tilde{N}_I = \sum_{i=1}^2 \left( I(\tilde{N}_i) \prod_{j=1,j \neq i}^2 Iu_j \right).
\]

Now let us decompose,
\[
\tilde{N}_I = \tilde{N}_{\text{good}} + \tilde{N}_{\text{bad}} = \sum_{i=1}^2 \left( \tilde{N}_i(Iu) \prod_{j=1,j \neq i}^2 Iu_j \right) + \sum_{i=1}^2 \left( I(\tilde{N}_i) - \tilde{N}_i(Iu_i) \right) \prod_{j=1,j \neq i}^2 Iu_j
\]

The first summand creates a defocusing potential like in the applications before. Thus after integration by parts, it creates a positive term that we can ignore. The term we call \(\tilde{N}_{\text{bad}}\) produces the Error term. Repeating the calculations above with \(Iu\) instead of \(u\) we have the bound:
\[
||IU||_{L^4_t L^4_x} \lesssim T \sup_{0,T} ||IU||_{L^2_t} ||IU||_{L^2} + T \sup_{0,T} ||u_0||_{L^2}^4 \\
+ T \left| \int_0^T \int_{\mathbb{R}^4} \nabla a \cdot \{\tilde{N}_{\text{bad}}, Iu(x_1,t)Iu(x_2,t)\}_p \, dx_1 \, dx_2 \, dt \right|. \quad (4.18)
\]

Note that we also used the fact that
\[
||IU||_{L^4} \lesssim ||u||_{L^4} = ||u_0||_{L^2}.
which follows by the definition of the $I$-operator and conservation of mass. Note that
the third term of (4.18) comes from the momentum bracket term in the proof of Proposition
4.2. We also remark that $\nabla a$ is real valued, thus
\[
\nabla a \cdot \Re( f \nabla \tilde{g} - g \nabla \tilde{f}) = \Re \left( \nabla a \cdot ( f \nabla \tilde{g} - g \nabla \tilde{f}) \right)
\]
and that $\nabla = (\nabla_{x_1}, \nabla_{x_2})$. We now wish to compute the dot product under the integral in
(4.18), that is
\[
\Re \left\{ \sum_{i=1}^{2} \nabla_{x_i} a \left( \tilde{N}_{bad}(\nabla_{x_i}(Iu_1Iu_2)) - Iu_1Iu_2\nabla_{x_i}\tilde{N}_{bad} \right) \right\}.
\]
We start by computing the first summand. Recall that
\[
\tilde{N}_{bad} = \sum_{i=1}^{2} \left( I(\tilde{N}_i) - \tilde{N}_i(Iu_i) \right) \prod_{j=1, j \neq i}^{2} Iu_j.
\]
Using the definition of $\tilde{N}_{bad}$, and the fact that $\nabla_{x_i}$ acts only on $Iu_1$ a direct calculation
shows
\[
\tilde{N}_{bad}(\nabla_{x_1}(Iu_1Iu_2)) - Iu_1Iu_2(\nabla_{x_1}\tilde{N}_{bad}) =
\]
\[
\left[ (I(\tilde{N}_1) - \tilde{N}(Iu_1))\nabla_{x_1}\overline{Iu_1} - \nabla_{x_1}(I(\tilde{N}_1) - \tilde{N}(Iu_1))Iu_1 \right] |Iu_2|^2.
\]
Hence the first summand is given by,
\[
\Re \left\{ \nabla_{x_1} a \left[ (I(\tilde{N}_1) - \tilde{N}(Iu_1))\nabla_{x_1}\overline{Iu_1} - \nabla_{x_1}(I(\tilde{N}_1) - \tilde{N}(Iu_1))Iu_1 \right] |Iu_2|^2 \right\}.
\]
Analogously, one can see that the second summand is given by:
\[
\Re \left\{ \nabla_{x_2} a \left[ (I(\tilde{N}_2) - \tilde{N}(Iu_2))\nabla_{x_2}\overline{Iu_2} - \nabla_{x_2}(I(\tilde{N}_2) - \tilde{N}(Iu_2))Iu_2 \right] |Iu_1|^2 \right\}.
\]
Thus, our error term
\[
E = \int_0^T \int_{\mathbb{R}^4} \nabla a \cdot \left( \tilde{N}_{bad}(\nabla(Iu_1Iu_2)) - \sum_{j=1}^{2} Iu_j \nabla\tilde{N}_{bad} \right)
\]
reduces to

\[ E = \Re \left\{ \int_0^T \int_{\mathbb{R}^4} \sum_{i=1}^2 \left\{ \nabla_x a \left[ (I(\tilde{N}_i) - \tilde{N}(Iu_i))\nabla_x \overline{Iu_i} - \nabla_x (I(\tilde{N}_i) - \tilde{N}(Iu_i))Iu_i \right] \right. \right. \\
\left. \left. \times \left| \prod_{j-1,j\neq i}^{|Iu_j|^2} \right| dx_1 dx_2 dt \right\}. \]

Hence, by symmetry,

\[ |E| \lesssim |E|, \quad (4.19) \]

where

\[ E = \int_0^T \int_{\mathbb{R}^4} \left\{ \nabla_x a \left[ (I(\tilde{N}_1) - \tilde{N}(Iu_1))\nabla_x \overline{Iu_1} - \nabla_x (I(\tilde{N}_1) - \tilde{N}(Iu_1))Iu_1 \right] \right. \right. \\
\left. \left. \times |Iu_2|^2 \right\} dx_1 dx_2 dt. \]

We have,

\[ |E| \leq E_1 + E_2 \quad (4.20) \]

where

\[ E_1 = \int_0^T \int_{\mathbb{R}^4} |\nabla_x a||I(\tilde{N}_1) - \tilde{N}(Iu_1)||\nabla_x \overline{Iu_1}||Iu_2|^2 dx_1 dx_2 dt \]

and

\[ E_2 = \int_0^T \int_{\mathbb{R}^4} |\nabla_x a||\nabla_x (I(\tilde{N}_1) - \tilde{N}(Iu_1))||Iu_1||Iu_2|^2 dx_1 dx_2 dt. \]

Since \(|\nabla_x a| \lesssim 1\), applying Fubini’s theorem we have

\[ E_1 \leq \left( \int_0^T \int_{\mathbb{R}^2} |I(\tilde{N}_1) - \tilde{N}(Iu_1)||\nabla_x \overline{Iu_1}| dx_1 dt \right) ||Iu||^2_{L^\infty L^2} \]

and

\[ E_2 \leq \left( \int_0^T \int_{\mathbb{R}^2} |\nabla_x (I(\tilde{N}_1) - \tilde{N}(Iu_1))||Iu_1| dx_1 dt \right) ||Iu||^2_{L^\infty L^2}. \]
Since the pair \((\infty, 2)\) is admissible and by renaming \(x_1 = x\) we have

\[
E_1 \leq \left( \int_0^T \int_{\mathbb R^2} |I(\tilde{N}_1) - \tilde{N}(Iu_1)| \|
abla Iu_1\| \, dx \, dt \right)^2 Z_i^2
\]

and

\[
E_2 \leq \left( \int_0^T \int_{\mathbb R^2} |\nabla (I(\tilde{N}_1) - \tilde{N}(Iu_1))| \|Iu_1\| \, dx \, dt \right)^2 Z_i^2.
\]

Therefore,

\[
E_1 \leq \|I(\tilde{N}) - \tilde{N}(Iu)\|_{L^1_t L^2_x} \|\nabla Iu\|_{L^{\infty}_t L^2_x} Z_i^2
\]

and

\[
E_2 \leq \|\nabla (I(\tilde{N}) - \tilde{N}(Iu))\|_{L^1_t L^2_x} \|Iu\|_{L^{\infty}_t L^2_x} Z_i^2.
\]

Again, since \((\infty, 2)\) is admissible we obtain:

\[
E_1 \leq \|I(\tilde{N}) - \tilde{N}(Iu)\|_{L^1_t L^2_x} Z_i^2
\]

and

\[
E_2 \leq \|\nabla_x (I(\tilde{N}) - \tilde{N}(Iu))\|_{L^1_t L^2_x} Z_i^2.
\]

Therefore, from (4.20) and the bounds above, we deduce that

\[
|E| \leq \left( \|I(\tilde{N}) - \tilde{N}(Iu)\|_{L^1_t L^2_x} + \|\nabla_x (I(\tilde{N}) - \tilde{N}(Iu))\|_{L^1_t L^2_x} \right) Z_i^2. \tag{4.21}
\]

We proceed to estimate \(\|\nabla (I(\tilde{N}) - \tilde{N}(Iu))\|_{L^1_t L^2_x}\), which is the hardest of the two terms. Toward this aim, let us observe that since \(\tilde{N}(u) = |u|^2 u\), we will be able to work on the Fourier side to estimate the commutator \(I(\tilde{N}) - \tilde{N}(Iu)\).

We compute,\(^1\)

\[
\nabla_x (I(\tilde{N}) - \tilde{N}(Iu))(\xi) = \int_{\mathbb R^3} i\xi [m(\xi) - m(\xi_1) m(\xi_2) m(\xi_3)] \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \, d\xi_1 d\xi_2 d\xi_3
\]

\(^1\)We ignore complex conjugates, since our computations are not affected by conjugation.
We decompose \( u \) into a sum of dyadic pieces \( u_j \) localized around \( N_j \). Note that the actual decay of the error term is of order \( O\left(\frac{1}{N_j^{1.2.3}}\right) \). This is because we have to keep a factor of size \( \max_{j=1,2,3} N_j^{1.2.3} \) in order to sum the different Littlewood–Paley pieces. For simplicity of argument we omit this technicality that does not affect the final result. Then,

\[
\|\nabla (I(\tilde{N}) - \tilde{N}(Iu))\|_{L^2_s} = \|\nabla (I(\tilde{N}) - \tilde{N}(Iu))\|_{L^2_s} = \sum_{N_1, N_2, N_3} \left| \int_{|\xi| + |\xi_2| + |\xi_3| = N} \left| \frac{m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1)m(\xi_2)m(\xi_3)}{m(\xi_1)m(\xi_2)m(\xi_3)} \right| \right|
\]

Without loss of generality, we can assume that the \( N_j \)'s are rearranged so that

\[ N_1 \geq N_2 \geq N_3. \]

Set,

\[ \sigma(\xi_1, \xi_2, \xi_3) = |\xi_1 + \xi_2 + \xi_3| \frac{m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1)m(\xi_2)m(\xi_3)}{m(\xi_1)m(\xi_2)m(\xi_3)}. \]

Then,

\[ \sigma(\xi_1, \xi_2, \xi_3) = \sum_{j=1}^{4} \chi_j(\xi_1, \xi_2, \xi_3)\sigma(\xi_1, \xi_2, \xi_3) = \sum_{j=1}^{4} \sigma_j(\xi_1, \xi_2, \xi_3), \]

where \( \chi_j(\xi_1, \xi_2, \xi_3) \) is a smooth characteristic function of the set \( \Omega_j \) defined as follows:

- \( \Omega_1 = \{ |\xi_i| \sim N_i, i = 1, 2, 3; N_1 \ll N \} \).
- \( \Omega_2 = \{ |\xi_i| \sim N_i, i = 1, 2, 3; N_1 \gg N \gg N_2 \} \).
- \( \Omega_3 = \{ |\xi_i| \sim N_i, i = 1, 2, 3; N_1 \geq N \gg N \gg N_3 \} \).
- \( \Omega_4 = \{ |\xi_i| \sim N_i, i = 1, 2, 3; N_1 \geq N_2 \gg N_3 \gg N \} \).

Hence, from (4.22) we get,

\[
\|\nabla (I(\tilde{N}) - \tilde{N}(Iu))\|_{L^2_s} \lesssim \sum_{N_1, N_2, N_3} \sum_{j=1}^{4} \int_{|\xi| + |\xi_2| + |\xi_3|} \sigma_j(\xi_1, \xi_2, \xi_3) \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 d\xi_1 d\xi_2 d\xi_3 \]

We proceed to analyze the contribution of each of the integrals \( L_j \).
The application of the multiplier theorem is justified by the fact that the symbol $L$ is of order zero. Indeed, the contribution to the sum above.

**Contribution of $L_1$.** Since $\sigma_1$ is identically zero when $N \geq 4N_1$, $L_1$ gives no contribution to the sum above.

**Contribution of $L_2$.** We have,

$$
\| \int_{\xi - \xi_1 + \xi_2 + \xi_3} \sigma_2(\xi_1 + \xi_2 + \xi_3) \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 d\xi_1 d\xi_2 d\xi_3 \|_{L_1 L_2^2}
$$

$$
= \frac{1}{N} \| \int_{\xi - \xi_1 + \xi_2 + \xi_3} \frac{N}{\xi_1 \xi_2} \sigma_2(\xi_1, \xi_2, \xi_3) \nabla \tilde{u}_1 \nabla \tilde{u}_2 \tilde{u}_3 d\xi_1 d\xi_2 d\xi_3 \|_{L_1 L_2^2}
$$

$$
\lesssim \frac{1}{N} \| \nabla \tilde{u}_1 \|_{L_2^2} \| \nabla \tilde{u}_2 \|_{L_2^2} \| \tilde{u}_3 \|_{L_1 L_2^2}
$$

where in the last line we used the Coifman-Meyer multiplier theorem, [4], and Hölder in time. The application of the multiplier theorem is justified by the fact that the symbol

$$a_2(\xi_1, \xi_2, \xi_3) = \frac{N}{\xi_1 \xi_2} \sigma_2(\xi_1, \xi_2, \xi_3)
$$

is of order zero. The $L^\infty$ bound follows after an application of the mean value theorem. Indeed,

$$|a_2(\xi_1, \xi_2, \xi_3)| \leq \frac{N}{N_1 N_2} |\xi_1 + \xi_2 + \xi_3| \frac{\nabla m(\xi_1)(\xi_2 + \xi_3)}{m(\xi_1)} \lesssim \frac{N}{N_1 N_2} \frac{N_2}{N_1} \lesssim 1.$$

**Contribution of $L_3$.** We have,

$$
\| \int_{\xi - \xi_1 + \xi_2 + \xi_3} \sigma_3(\xi_1 + \xi_2 + \xi_3) \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 d\xi_1 d\xi_2 d\xi_3 \|_{L_1 L_2^2}
$$

$$
= \frac{1}{N} \| \int_{\xi - \xi_1 + \xi_2 + \xi_3} \frac{N}{\xi_1 \xi_2} \sigma_3(\xi_1, \xi_2, \xi_3) \nabla \tilde{u}_1 \nabla \tilde{u}_2 \tilde{u}_3 d\xi_1 d\xi_2 d\xi_3 \|_{L_1 L_2^2}
$$

$$
\lesssim \frac{1}{N} \| \nabla \tilde{u}_1 \|_{L_2^2} \| \nabla \tilde{u}_2 \|_{L_2^2} \| \tilde{u}_3 \|_{L_1 L_2^2}
$$

where in the last line we used the Coifman-Meyer multiplier theorem, and Hölder in time. The application of the multiplier theorem is justified by the fact that the symbol

$$a_3(\xi_1, \xi_2, \xi_3) = \frac{N}{\xi_1 \xi_2} \sigma_3(\xi_1, \xi_2, \xi_3)
$$

is of order zero. The $L^\infty$ bound follows from the following chain of inequalities:

$$|a_3(\xi_1, \xi_2, \xi_3)| \lesssim \frac{N}{N_1 N_2} |\xi_1 + \xi_2 + \xi_3| \left( \frac{m(\xi_1 + \xi_2 + \xi_3)}{m(\xi_1)} + 1 \right)
$$

$$\lesssim \frac{N}{N_1 N_2} \left( \frac{N_1}{m(N_2)} + N_1 \right) \lesssim 1,$$
where we have used the fact that $|\xi|m(\xi)$ is monotone increasing for any $s > 0$ and thus

$$|\xi_1 + \xi_2 + \xi_3|m(\xi_1 + \xi_2 + \xi_3) \lesssim |\xi_1|m(\xi_1).$$

**Contribution of $L_4$.** We continue as above using the $L^3_t L^6_x$ Strichartz norms and get

$$\left\| \int_{\xi = \xi_1 + \xi_2 + \xi_3} \sigma_4(\xi_1 + \xi_2 + \xi_3) \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \partial_1 \xi_1 d\xi_1 d\xi_2 d\xi_3 \right\|_{L^1_t L^6_x} \lesssim \frac{1}{N^2} \prod_{j=1}^3 \|\nabla Iu_j\|_{L^2_t L^2_x},$$

where in this case the symbol to which we apply the multiplier theorem is:

$$a_4(\xi_1, \xi_2, \xi_3) = \frac{N^2}{\xi_1 \xi_2 \xi_3} \sigma_4(\xi_1, \xi_2, \xi_3).$$

In all the cases above, we proved the $L^\infty$ bound for the symbols $a_i(\xi_1, \xi_2, \xi_3)$, $i = 2, 3, 4$. The reader can easily verify the conditions of the Coifman-Meyer theorem for the higher-order derivatives.

Finally, since the pair $(3, 6)$ is admissible, we obtain that in all the cases above

$$\left\| \int_{\xi = \xi_1 + \xi_2 + \xi_3} \sigma_1(\xi_1 + \xi_2 + \xi_3) \tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \partial_1 \xi_1 d\xi_1 d\xi_2 d\xi_3 \right\|_{L^1_t L^6_x} \lesssim \frac{1}{N^3} Z^3_i.$$

Therefore, we deduce from (4.23) that

$$\|\nabla(I(\tilde{N}) - \tilde{N}(Iu))\|_{L^1_t L^2_x} \lesssim \frac{1}{N^3} Z^3_i.$$

Analogously,

$$\|I(\tilde{N}) - \tilde{N}(Iu)\|_{L^1_t L^2_x} \lesssim \frac{1}{N^3} Z^3_i.$$

Hence, in view of (4.21) we obtain the following estimate for the error term,

$$|E| \lesssim \frac{1}{N^3} Z^3_i.$$

Thus, (4.19) implies

$$\left| \int_0^T \int_{\mathbb{R}^d} \nabla a \cdot (\tilde{N}_{bad}(\nabla(Iu_1 Iu_2) - Iu_1 Iu_2 \nabla \tilde{N}_{bad})) \right| \lesssim \frac{1}{N^3} Z^3_i.$$ 

This completes the proof of Theorem 4.1.
5 Proof of the main Theorem and comments on further refinements

5.1 Proof of the main Theorem

Proof. Suppose that \( u(t, x) \) is a global-in-time solution to (1.1) with initial data \( u_0 \in \mathcal{C}_0^\infty(\mathbb{R}^2) \). We will prove that \( \|u(t)\|_{H^s} \) obeys polynomial-in-time upper bounds with the implied constants not depending upon the extra decay and regularity properties of \( u_0 \). A familiar density argument then establishes that (1.1) is globally well-posed for \( H^s \) initial data in the range of \( s \) for which we prove the polynomial bounds, namely for \( s > \frac{2}{5} \).

Set \( u^\lambda(x) = \frac{1}{\lambda} u(\frac{x}{\lambda}, \frac{t}{\lambda^2}) \). We choose the parameter \( \lambda \) so that \( \|Iu^\lambda\|_{L^4_t L^4_x([0,t] \times \mathbb{R}^2)} = O(1) \), that is

\[
\lambda \sim N^{\frac{1}{2s - 2}}. \tag{5.1}
\]

Next, let us pick a time \( T_0 \) arbitrarily large, and let us define

\[
S := \{ 0 < t < \lambda^2 T_0 : \|Iu^\lambda\|_{L^4_t L^4_x([0,t] \times \mathbb{R}^2)} \leq KN^{\frac{1}{2} T^\frac{3}{2}} \}, \tag{5.2}
\]

with \( K \) a constant to be chosen later. Notice that this choice of the set \( S \) is dictated by the a priori estimate of the \( L^4_t L^4_x \) norm of \( Iu \). See also Equation (5.12).

We claim that \( S \) is the whole interval \([0, \lambda^2 T_0]\). Indeed, assume by contradiction that it is not so; then, since

\[
\|Iu^\lambda\|_{L^4_t L^4_x([0,t] \times \mathbb{R}^2)}
\]

is a continuous function of time, there exists a time \( T \in [0, \lambda^2 T_0] \) such that

\[
\|Iu^\lambda\|_{L^4_t L^4_x([0,T] \times \mathbb{R}^2)} > KN^\frac{1}{2} T^\frac{3}{2} \tag{5.3}
\]
\[
\|Iu^\lambda\|_{L^4_t L^4_x([0,T] \times \mathbb{R}^2)} \leq 2KN^\frac{1}{2} T^\frac{3}{2}. \tag{5.4}
\]

We now split the interval \([0, T]\) into subintervals \( J_k, k = 1, \ldots, L \) in such a way that

\[
\|Iu^\lambda\|_{L^4_t L^4_x([0,J_k] \times \mathbb{R}^2)} \leq \mu_0, \tag{5.5}
\]

with \( \mu_0 \) as in Proposition 3.1. This is possible because of (5.4). Then, the number \( L \) of possible subintervals must satisfy

\[
L \sim \frac{(2KN^\frac{1}{2} T^\frac{3}{2})^4}{\mu_0} = \frac{(2K)^4 N^\frac{1}{2} T^\frac{3}{2}}{\mu_0}. \tag{5.6}
\]
From Propositions 3.1 and 3.2, we know that, for any \( \frac{1}{4} < s < \frac{1}{2} \)

\[
\sup_{[0, T]} E(Iu^\lambda(t)) \lesssim E(Iu_0^\lambda) + \frac{L}{N^s},
\]

(5.7)

and by our choice of \( \lambda \), \( E(Iu_0^\lambda) \lesssim 1 \). Note that \( \frac{s}{1} > \frac{1}{4} \) and we can apply the previous Propositions. Hence, in order to guarantee that

\[ E(Iu^\lambda) \lesssim 1 \]

(5.8)

holds for all \( t \in [0, T] \) we need to require that

\[ L \lesssim N^{\frac{s}{2}}. \]

Since \( T \leq \lambda^2 T_0 \), according to (5.1), this is fulfilled as long as

\[ (2K)^4 \mu_0^4 (\lambda^2 T_0)^{\frac{1}{2}} \sim N^{\frac{s}{2}}. \]

(5.9)

From our choice (5.1) of \( \lambda \), the expression (5.9) implies that

\[ T_0^\lambda (2K)^4 \mu_0^4 \sim N^{2 \mu_0^4}. \]

(5.10)

If \( s > \frac{2}{5} \), we have that \( T_0 \) is arbitrarily large if we send \( N \) to infinity.

We now use the energy control in the Morawetz estimate to show that (5.3) fails to hold. Recall the a priori estimate (4.18)

\[
\|Iu\|_{L^3_t L^6_x}^3 \lesssim T^\frac{3}{4} \sup_{[0, T]} \|Iu\|_{H^s}^3 \|Iu\|_{L^6_x}^2 + T^\frac{1}{2} \|u_0\|_{L^6_x}^4,
\]

(5.11)

\[ T^\frac{3}{4} \int_0^T \int_{\mathbb{R}^4} \nabla a \cdot \{\tilde{N}_{bad}, Iu_1 Iu_2\}_p dx_1 dx_2 dt. \]

Let us define

\[ \text{Error}(t) := \int_{\mathbb{R}^4} \nabla a \cdot \{\tilde{N}_{bad}, Iu_1 Iu_2\}_p dx_1 dx_2. \]

By Theorem 4.1 and Proposition 3.1 we know that on each interval \( J_k \) we have

\[ \int_{J_k} \text{Error}(t) dt \lesssim \frac{1}{N} \|a\|_6^6 \lesssim \frac{1}{N} \|Iu^\lambda(a)\|_{H^s}^6 \lesssim \frac{1}{N} \]

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and summing all the $J_k$’s we have
\[
\int_0^T \text{Error}(t) \, dt \lesssim \frac{N}{N} - \frac{N^2}{N} - N^\frac{1}{2}
\]
But then we have
\[
\|I^2 u\|_{L^4_t L^2_x} \lesssim \frac{N}{N} \sup_{[0,T]} \|I^2 u\|_{L^2_t \|I^2 u\|_{L^2_x}} + T^\frac{1}{2} \|u^\lambda\|_{L^2_x}^\frac{4}{N}
\]
\[
+ T^\frac{1}{2} \int_0^T \text{Error}(t) \, dt \lesssim \frac{N}{N} + T^\frac{1}{2} N^\frac{1}{2} \lesssim N^\frac{1}{2} T^\frac{1}{2}.
\]
(5.12)
This estimate contradicts (5.3) for an appropriate choice of $K$. Hence $S = [0, \lambda^2 T_0]$, and $T_0$ can be chosen arbitrarily large. In addition, we have also proved that for $s > \frac{2}{5}$
\[
\|I^2 u\|_{H^s} \|u\|_{H^s} = O(1).
\]
But then,
\[
\|u(T_0)\|_{H^s} \lesssim \|u(T_0)\|_{L^2} + \|u(0)\|_{H^s} + \lambda^s \|u^\lambda(\lambda^2 T_0)\|_{H^s},
\]
\[
\lesssim \lambda^s \|I^2 u\|_{H^s} \|u\|_{H^s} \lesssim \lambda^s \lesssim N^{1-s} \lesssim T_0^{\frac{2(1-s)}{5}}.
\]
Since $T_0$ is arbitrarily large, the a priori bound on the $H^s$ norm concludes the global well-posedness of the the Cauchy problem (1.1).

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\section*{References}


