On the blow up phenomenon for the L^2 critical nonlinear Schrödinger Equation

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The aim of these notes is to provide a self contained presentation of recent developments concerning the singularity formation for the L^2 critical nonlinear Schrödinger equation

$$(NLS) \quad \begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{N}}u, & (t,x) \in [0,T) \times \mathbf{R}^N \\ u(0,x) = u_0(x), & u_0 : \mathbf{R}^N \to \mathbf{C} \end{cases} \tag{1}$$

with $u_0 \in H^1 = \{u, \nabla u \in L^2(\mathbf{R}^N)\}$ in dimension $N \geq 1$. This equation for $N = 2$ appears in physics as a universal model to describe self trapping of waves propagating in nonlinear media. The physical expectation for large smooth data is the concentration of part of the L^2 mass in finite time corresponding to the focusing of the laser beam. If some explicit examples of this phenomenon are known, and despite a number of both numerical and mathematical works, a general description of blow up dynamics is mostly open.

(NLS) is an infinite dimensional Hamiltonian system with energy space H^1 without any space localization property. It is in this context together with the critical generalized KdV equation the only example where blow up is known to occur. For (NLS), an elementary proof of existence of blow up solutions is known since the 60's but is based on energy constraints and is not constructive. In particular, *no qualitative information of any type on the blow up dynamics is obtained this way.*

The natural questions we address regarding blow up dynamics in the energy space are the following:
-Does there exist a Hamiltonian characterization of blow up solutions, or at least necessary conditions for blow up simply expressed from the Hamiltonian invariants?
-Assuming blow up, does there exist a universal blow up speed, or are there several possible regimes? Among these regimes, which ones are stable?
-Does there exist a universal space time structure for the formation of singularities independent at the first order of the initial data?
We will present precise answers to these issues in the setting of a perturbative analysis close to the exceptional solution to (1): the ground state solitary wave.

These notes are organized as follows. In a first section, we recall main standard results about nonlinear Schrödinger equations. In the second section, we focus onto the critical blow up problem and recall the few known results in the field. The next section is devoted to an exposition of the recent results obtained in collaboration with F.Merle in [20], [21], [22], [23], [24] and [32]. In the last section, we present a detailed proof of the first of these results which is the exhibition of a sharp upper bound on blow up rate for a suitable class of initial data. We expect the presentation to be essentially self contained provided the prior knowledge of standard tools in the study of nonlinear PDE's.

1 Hamiltonian structure and global wellposedness

In this section, we recall main classical facts regarding the global wellposedness in the energy space of nonlinear Schrödinger equations. We will also introduce one of the fundamental objects for the study of (1): the ground state solitary wave.

1.1 Local wellposedness, symmetries and Hamiltonian structure

Let us consider the general nonlinear Schrödinger equation:

$$\begin{cases} iu_t = -\Delta u - |u|^{p-1}u \\ u(0,x) = u_0(x) \in H^1 \end{cases} \tag{2}$$

with

$$1 < p < +\infty \quad \text{for} \quad N = 1, 2, \quad 1 < p < 2^* - 1 \quad \text{for} \quad N \geq 3, \tag{3}$$

where $2^* = \frac{2N}{N-2}$ is the Sobolev exponent. The first fundamental question arising when dealing with a nonlinear PDE like (2) is the existence of a solution locally in time in the given Cauchy space which we have chosen here to be the energy space H^1. This type of result relies on the theory of oscillatory integrals and the well-known Strichartz estimates for the propagator $e^{it\Delta}$ of the linear group. Local wellposedness of (2) in H^1 is in this frame a well known result of Ginibre, Velo, [8]. See also [10]. Thus, for $u_0 \in H^1$, there exists $0 < T \leq +\infty$ such that $u(t) \in \mathcal{C}([0,T), H^1)$. Moreover, the life time of the solution can be proved to be lower bounded by a function depending on the H^1 size of the solution only, $T(u_0) \geq f(\|u_0\|_{H^1})$. A corollary of these techniques is the global wellposedness for small data in H^1. The idea is that small data remain small through the iterative scheme used to construct the solution which may thus be continued up to any arbitrary time.

On the contrary, for large H^1 data, three possibilities may occur:
(i) $T = +\infty$ and $\limsup_{t\to+\infty} |u(t)|_{H^1} < +\infty$, we say the solution is *global and bounded*.
(ii) $T = +\infty$ and $\limsup_{t\to+\infty} |u(t)|_{H^1} = +\infty$, we say the solution*blows up in infinite time.*
(iii) $0 < T < +\infty$, but then from local wellposedness theory:

$$|u(t)|_{H^1} \to +\infty \quad \text{as} \quad t \to T,$$

we say the solution *blows up finite time.*

To prove global wellposedness of the solution, it thus suffices to control the size of the solution in H^1. This is achieved in some cases thanks to the *Hamiltonian structure* of (2). Indeed, (2) admits the following invariants in H^1:

- L^2-norm:
$$\int |u(t,x)|^2 = \int |u_0(x)|^2; \tag{4}$$

- Energy:
$$E(u(t,x)) = \frac{1}{2}\int |\nabla u(t,x)|^2 - \frac{1}{p+1}\int |u(t,x)|^{p+1} = E(u_0); \tag{5}$$

- Momentum:
$$Im\left(\int \nabla u\overline{u}(t,x)\right) = Im\left(\int \nabla u_0\overline{u_0}(x)\right). \tag{6}$$

Note that the growth condition on the nonlinearity (3) ensures from Sobolev embedding that the energy is well defined, and this is why H^1 is referred to as the energy space.

From the Ehrenfest law, these invariants are related to the group of symmetry of (2) in H^1:

- Space-time translation invariance: if $u(t,x)$ solves (2), then so does $u(t+t_0, x+x_0)$, $t_0 \in \mathbf{R}$, $x_0 \in \mathbf{R}^N$.
- Phase invariance: if $u(t,x)$ solves (2), then so does $u(t,x)e^{i\gamma}$, $\gamma \in \mathbf{R}$.
- Scaling invariance: if $u(t,x)$ solves (2), then so does $\lambda^{\frac{2}{p-1}}u(\lambda^2 t, \lambda x)$, $\lambda > 0$.
- Galilean invariance: if $u(t,x)$ solves (2), then so does $u(t, x-\beta t)e^{i\frac{\beta}{2}(x-\frac{\beta}{2}t)}$, $\beta \in \mathbf{R}^N$.

Let us point out that this group of H^1 symmetries is the same as for the linear Schrödinger equation.

This structure allows one to prove global wellposedness of (2) in the subcritical case.

Theorem 1 (Global wellposedness in the subcritical case) *Let $N \geq 1$ and $1 < p < 1+\frac{4}{N}$, then all solutions to (2) are global and bounded in H^1.*

Proof of Theorem 1

The proof is elementary and relies on the Hamiltonian structure and the Gagliardo-Nirenberg interpolation inequality. Indeed, let $u_0 \in H^1$, $u(t)$ the corresponding solution to (2) with $[0,T)$ its maximum time interval existence in H^1, we then have the a priori estimate: there exists $C(u_0)>0$ such that,

$$\forall t\in[0,T),\quad |\nabla u(t)|_{L^2}\leq C(u_0). \tag{7}$$

From the conservation of the L^2 norm, we conclude: $\forall t\in[0,T)$, $|u(t)|_{H^1}\leq C(u_0)$, and this uniform bound on the solution implies global wellposedness from the local Cauchy theory in H^1.
It remains to prove (7) which is a consequence of the conservation of the energy and the Gagliardo-Nirenberg interpolation estimate: let $N=1,2$ and $1\leq p<+\infty$ or $N\geq 3$ and $1\leq p\leq 2^*-1$, then there holds for some universal constant $C(N,p)>0$,

$$\forall v\in H^1,\quad \int|v|^{p+1}\leq C(N,p)\left(\int|\nabla v|^2\right)^{\frac{N(p-1)}{4}}\left(\int|v|^2\right)^{\frac{p+1}{2}-\frac{N(p-1)}{4}}. \tag{8}$$

Applying this with $v=u(t)$, we get from the conservation of the energy and the L^2 norm:

$$\forall t\in[0,T),\quad E_0\geq\frac{1}{2}[\int|\nabla v|^2-C(u_0)\left(\int|\nabla v|^2\right)^{\frac{N(p-1)}{4}}],$$

from which (7) follows from subcriticality assumption $1\leq p<1+\frac{4}{N}$. This concludes the proof of Theorem 1.

The physical meaning of Theorem 1 is that for waves propagating in a too weakly focusing medium, the potential term in the energy is dominated by the kinetic term according to (8) and no focusing can occur. A critical exponent arises from this analysis for which these two terms balance exactly, and we shall concentrate from now on on this case alone which is referred to as *the critical case*:

$$p=1+\frac{4}{N}.$$

1.2 Minimizers of the energy

The criticality of equation (1) may be understood from the exact balance in this case between the kinetic and the potential energy. This may be quantified in a sharp way from the knowledge of the exact constant in the Gagliardo-Nirenberg inequality (8).

Theorem 2 (Minimizers of the energy) *Consider the H^1 functional:*

$$J(v)=\frac{(\int|\nabla v|^2)(\int|v|^2)^{\frac{2}{N}}}{\int|v|^{2+\frac{4}{N}}}. \tag{9}$$

The minimization problem

$$\min_{v\in H^1,\quad v\neq 0}J(v)$$

is attained on the three parameter family:

$$\lambda_0^{\frac{N}{2}} Q(\lambda_0 x + x_0) e^{i\gamma_0}, \quad (\lambda_0, x_0, \gamma_0) \in \mathbf{R}_*^+ \times \mathbf{R}^N \times \mathbf{R},$$

where Q is the unique positive radial solution:

$$\begin{cases} \Delta Q - Q + Q^{1+\frac{4}{N}} = 0 \\ Q(r) \to 0 \quad as \quad r \to +\infty. \end{cases} \tag{10}$$

In particular, there holds the following Gagliardo-Nirenberg inequality with best constant:

$$\forall v \in H^1, \quad E(v) \geq \frac{1}{2} \int |\nabla v|^2 \left(1 - \left(\frac{|v|_{L^2}}{|Q|_{L^2}} \right)^{\frac{4}{N}} \right). \tag{11}$$

The existence of a positive solution to (10) is a result obtained from the theory of calculus of variations by Berestycki-Lions, [1], and lies within the range of the concentration compactness techniques introduced by P.L Lions at the beginning of the 80's, see [13], [14]. An ODE type of approach is also available from [2]. The fact that the positive solution to (10) is necessarily radial is a deep and general result by Gidas, Ni, Nirenberg, [7]. Uniqueness of the ground state in the ODE sense is a result by Kwong, [11]. Last, the fact that the minimization problem is attained is due to Weinstein, [37].

From standard elliptic theory, the ground state Q is C^3_{loc} and exponentially decreasing at infinity in space:

$$Q(r) \leq e^{-C(N)r},$$

and one should think of Q as a smooth well localized bump. In dimension $N = 1$, equation (10) may even be integrated explicitly to obtain:

$$Q(x) = \left(\frac{3}{\operatorname{ch}^2(x)} \right)^{\frac{1}{4}}.$$

In higher dimension on the contrary, equation (10) admits excited solutions $(Q_i)_{i\geq 1}$ with growing L^2 norm: $|Q_i|_{L^2} \to +\infty$ as $i \to +\infty$.

A reformulation of (11) is the following variational characterization of Q which we will mostly use:

Proposition 1 (Variational characterization of the ground state) *Let $v \in H^1$ such that $\int |v|^2 = \int Q^2$ and $E(v) = 0$, then*

$$v(x) = \lambda_0^{\frac{N}{2}} Q(\lambda_0 x + x_0) e^{i\gamma_0},$$

for some parameters $\lambda_0 \in \mathbf{R}_+^*$, $x_0 \in \mathbf{R}^N$, $\gamma_0 \in \mathbf{R}$.

To sum up, the situation is as follows: let $v \in H^1$, then if $|v|_{L^2} < |Q|_{L^2}$ ie for "small" v, the kinetic energy dominates the potential energy and (11) yields $E(v) > C(v) \int |\nabla v|^2$ and the energy is in particular non negative; at the critical mass level $|v|_{L^2} = |Q|_{L^2}$, the only zero energy function, ie for which the kinetic and the potential energies exactly balance, is Q up to the symmetries of scaling, phase and translation which generate the three dimensional manifold of minimizers of (9).

A fundamental generalization of Theorem 1 has been obtained by Weinstein [37]:

Theorem 3 (Global wellposedness for subcritical mass) *Let $u_0 \in H^1$ with $|u_0|_{L^2} < |Q|_{L^2}$, the corresponding solution $u(t)$ to (1) is global and bounded in H^1.*

Proof of Theorem 3

As for the proof of Theorem 1, it suffices from local wellposedness theory to prove a priori estimate (7). But from the conservation of the L^2 norm, $|u(t)|_{L^2} < |Q|_{L^2}$ for all $t \in [0, T)$, and (7) follows from the conservation of the energy and the sharp Gagliardo-Nirenberg inequality (11) applied to $v = u(t)$. This concludes the proof of Theorem 3.

2 General blow up results

Our aim in this section is to recall some known blow up criteria and qualitative properties of the blow up solutions. In contrast with the results in the preceding section which could be extended to more general nonlinearities, we shall now focus onto the very specific algebraic structure of (1).

2.1 Solitary waves and the critical mass blow up

Weinstein's criterion for global solutions given by Theorem 3 is sharp. On the one hand, from (10),

$$W(t,x) = Q(x)e^{it}$$

is a solution to (1) with critical mass $|W|_{L^2} = |Q|_{L^2}$. Note that W keeps its shape in time and *does not disperse*. It is the minimal object in L^2 sense for which dispersion -measured by the kinetic term- and concentration -measured by the potential term- exactly compensate. This exceptional solution is called *the ground state solitary wave*. H^1 symmetries of (1) generate in fact a three parameter family of solitary waves:

$$W_{\lambda_0,x_0,\gamma_0}(t,x) = \lambda_0^{\frac{N}{2}} Q(\lambda_0 x + x_0) e^{i(\gamma_0 + \lambda_0^2 t)}, \quad (\lambda_0, x_0, \gamma_0) \in \mathbf{R}_*^+ \times \mathbf{R}^N \times \mathbf{R}. \tag{12}$$

Now a fundamental remark is the following: in the critical case $p = 1 + \frac{4}{N}$,

all H^1 symmetries of (1) are L^2 isometries.

This is why (1) is called L^2 critical. All the ground states solitary waves (12) thus have critical L^2 mass:

$$|W_{\lambda_0,x_0,\gamma_0}|_{L^2} = |Q|_{L^2}.$$

Moreover, from explicit computation and $E(Q) = 0$, $Im(\int \nabla Q\overline{Q}) = 0$, we have:

$$E(W_{\lambda_0,x_0,\gamma_0}) = 0, \quad Im(\int \nabla W_{\lambda_0,x_0,\gamma_0}\overline{W_{\lambda_0,x_0,\gamma_0}}) = 0.$$

In other words, the L^2 criticality of the equation implies the existence of a three parameters family of solitary waves with arbitrary size in H^1 but frozen Hamiltonian invariants. The consideration of these invariants only is thus no longer enough to estimate the size of the solution nor to separate within these different solitary waves.

In general, the L^2 scaling invariance of the solitary waves is a known criterion of instability, see [35]. In our case, it may be made precise by exhibiting an *explicit blow up solution*. Existence of this object is based on the pseudo-conformal symmetry of (1) which is not in the energy space H^1 but in the so called virial space:

$$\Sigma = H^1 \cap \{xu \in L^2\},$$

and which writes: if $u(t,x)$ is a solution to (1), then so is

$$v(t,x) = \frac{1}{|t|^{\frac{N}{2}}}\overline{u}(\frac{1}{t}, \frac{x}{t})e^{i\frac{|x|^2}{4t}}. \tag{13}$$

An equivalent but more enlightening way of seeing this symmetry is the following: for any parameter $a \in \mathbf{R}$, the solution to (1) with initial data $v_a(0,x) = u(0,x)e^{ia\frac{|x|^2}{4}}$ is

$$v_a(t,x) = \frac{1}{(1+at)^{\frac{N}{2}}}u(\frac{t}{1+at}, \frac{x}{1+at})e^{ia\frac{|x|^2}{4(1+at)}}. \tag{14}$$

Note that this symmetry is also a symmetry of the linear equation. Nevertheless, the fundamental difference between the linear and the nonlinear equation is that all solutions to the linear equation are dispersive and go to zero when time evolves for example in L^2_{loc}, whereas the nonlinear problem admits non dispersive solutions: the solitary waves. *The pseudo-conformal transformation applied to the non dispersive solution now yields a finite time blow up solution*:

$$S(t,x) = \frac{1}{|t|^{\frac{N}{2}}}Q(\frac{x}{t})e^{-i\frac{|x|^2}{4t}+\frac{i}{t}}. \tag{15}$$

This solution should be viewed at the solution to (1) with Cauchy data at $t = -1$:

$$S(-1,x) = Q(x)e^{i\frac{|x|^2}{4}-i}.$$

It blows up at time $T = 0$ with the following explicit properties:

- First observe that pseudo-conformal symmetry (13) is again an L^2 isometry. Thus $|S|_{L^2} = |Q|_{L^2}$ and global wellposedness criterion given by Theorem 3 is sharp.

- From explicit computation, S has non negative energy:
$$E(S) > 0. \tag{16}$$

- The blow up speed, measured by the L^2 norm of the gradient -as the L^2 norm itself is conserved-, is given by:
$$|\nabla u(t)|_{L^2} \sim \frac{1}{|t|}. \tag{17}$$

- The solution leaves the Cauchy space H^1 by forming a Dirac mass in L^2:
$$|S(t)|^2 \rightharpoonup \left(\int Q^2\right) \delta_{x=0} \quad \text{as} \quad t \to 0. \tag{18}$$

Like the solitary wave is a non dispersive global solution, $S(t)$ is a non dispersive blow up solution in the sense that *it accumulates all its L^2 mass into blow up*: no L^2 mass is lost in the focusing process. This property should be understood as a fundamental feature of a critical mass blow up solution, and indeed the critical mass blow up dynamic is very constrained according to the following fundamental classification result by F. Merle, [17]:

Theorem 4 (Uniqueness of the critical mass blow up solution) *Let $u_0 \in H^1$ with $|u_0|_{L^2} = |Q|_{L^2}$, and assume that the corresponding solution $u(t)$ to (1) blows up in finite time $0 < T < +\infty$. Then*
$$u(t) = S(t - T)$$
up to the H^1 symmetries.

2.2 Blow up for large data: the virial identity

Let us now consider super critical mass initial data $u_0 \in H^1$ with $|u_0|_{L^2} > |Q|_{L^2}$, and ask the question of the existence of finite time blow up solutions. The answer is surprisingly simple in the case when the virial law applies. This identity first derived by Zakharov and Shabat, [38], is a consequence of the pseudo-conformal symmetry. Let a data in the virial space $u_0 \in \Sigma$, then the corresponding solution $u(t)$ to (1) on $[0, T)$ satisfies:
$$\forall t \in [0, T), \quad u(t) \in \Sigma \quad \text{and} \quad \frac{d^2}{dt^2} \int |x|^2 |u(t)|^2 = 16 E_0. \tag{19}$$

Let us now observe that if from (11) subcritical mass functions have non negative energy, the sign of the energy is no longer prescribed for super critical mass functions. For example, an explicit computation ensures $\frac{d}{d\eta} E\left((1+\eta)Q\right)_{|\eta=0} < 0$, and from $E(Q) = 0$, any neighborhood of Q in H^1 contains data with non positive energy. Let then $u_0 \in \Sigma$ with

$E_0 < 0$, then from virial law (19) and the conservation of the energy, the positive quantity $\int |x|^2|u(t)|^2$ is an inverted parabola which must thus become negative in finite time, and therefore the solution cannot exist for all time and blows up in finite time $0 < T < +\infty$. Note that this argument can be generalized to the energy space via an H^1 regularization of (19), and we have the following:

Theorem 5 (Virial blow up for $E_0 < 0$) *Let $u_0 \in H^1$ with*

$$E_0 < 0,$$

then:
(i) Ogawa, Tsutsumi, [29]: if $N = 1$, then $0 < T < +\infty$.
(ii) Nawa, [28]: if $N \geq 2$ and u_0 is radial, then $0 < T < +\infty$.

This blow up argument is extraordinary for at least two reasons:
(i) It provides a blow up criterion based on a pure Hamiltonian information $E_0 < 0$ which applies to arbitrarily large initial data in H^1. In particular, it exhibits an open region of the energy space where blow up is known to be a stable phenomenon.
(ii) This argument also applies in Σ to the super critical case $1 + \frac{4}{N} < p < 2^* - 1$ where it is essentially the only known blow up result.

Now this argument has two major weaknesses:
(i) It heavily relies on the pseudo-conformal symmetry, and thus is unstable by perturbation of the equation.
(ii) More fundamentally, *this argument is purely obstructive and says nothing on the singularity formation.*

2.3 The L^2 concentration phenomenon

In the general setting, little is known regarding the description of the singularity formation. This is mainly a consequence of the fact that the virial blow up argument does not provide any insight into the blow up dynamics. Nevertheless, a general result of L^2 concentration obtained by Merle, Tsutsumi, [25], in the radial case, and generalized by Nawa, [28], provides a first description of the singularity formation: at blow up time, the solution leaves the Cauchy space H^1 by forming a Dirac mass in L^2.

Theorem 6 (L^2 concentration phenomenon) *Let $u_0 \in H^1$ such that the corresponding solution $u(t)$ to (1) blows up in finite time $0 < T < +\infty$. Then there exists some continuous function of time $x(t) \in \mathbf{R}^N$ such that:*

$$\forall R > 0, \quad \liminf_{t \to T} \int_{|x-x(t)| \leq R} |u(t,x)|^2 dx \geq \int Q^2. \tag{20}$$

To enlighten the meaning of this theorem, let us first recall from Cazenave, Weissler [6], that the Cauchy problem for (1) is locally wellposed in L^2. This space is the scaling invariant Sobolev space for (1), and in some sense the lowest one in term of Sobolev regularity for which local wellposedness can be expected. In this case, the lifetime of the solution cannot be lower bounded by a function of the size of the data only -which is conserved by the flow-, but fundamentally depends on the full initial profile. In this sense, to understand the way the solution may leave the Cauchy space in the critical space L^2 is a fundamental -and difficult- problem. Theorem 6 implies that a blow up data in H^1 leaves also L^2 when it leaves H^1, and thus the lifetimes in L^2 and H^1 are the same, and this is done by creating a Dirac mass in L^2 with a *minimal universal amount of mass*.

Proof of Theorem 6

We prove the result in the radial case for $N \geq 2$. The general case follows from concentration compactness techniques, see [27].
Let $u_0 \in H^1$ radial and assume that the corresponding solution $u(t)$ to (1) blows up at time $0 < T < +\infty$, or equivalently:

$$\lim_{t \to T} |\nabla u(t)|_{L^2} = +\infty. \tag{21}$$

We need to prove (20) and argue by contradiction: assume that for some $R > 0$ and $\varepsilon > 0$, there holds on some sequence $t_n \to T$,

$$\lim_{n \to +\infty} \int_{|y| \leq R} |u(t_n, y)|^2 \leq \int Q^2 - \varepsilon. \tag{22}$$

Let us rescale the solution by its size and set:

$$\lambda(t_n) = \frac{1}{|\nabla u(t_n)|_{L^2}}, \quad v_n(y) = \lambda^{\frac{N}{2}}(t_n) u(t_n, \lambda(t_n) y),$$

then from explicit computation:

$$|\nabla v_n|_{L^2} = 1 \ \text{ and } \ E(v_n) = \lambda^2(t_n) E(u). \tag{23}$$

First observe that v_n is H^1 bounded and we may assume on a sequence $n \to +\infty$:

$$v_n \rightharpoonup V \ \text{ in } \ H^1.$$

We first claim that V is non zero. Indeed, from (21), (23) and the conservation of the energy for $u(t)$, $E(v_n) \to 0$ as $n \to +\infty$, and thus:

$$\frac{1}{2 + \frac{4}{N}} \int |v_n|^{2+\frac{4}{N}} = \frac{1}{2} \int |\nabla v_n|^2 - E(v_n) = \frac{1}{2} - E(v_n) \to \frac{1}{2} \ \text{ as } \ n \to +\infty.$$

Now from compact embedding of H^1_{radial} to $L^{2+\frac{4}{N}}$, $v_n \to V$ in $L^{2+\frac{4}{N}}$ up to a subsequence, and thus $\frac{1}{2+\frac{4}{N}}\int |V|^{2+\frac{4}{N}} \geq \frac{1}{2}$ and V is non zero. Moreover, from weak H^1 convergence and strong $L^{2+\frac{4}{N}}$ convergence,

$$E(V) \leq \liminf_{n\to+\infty} E(v_n) = 0.$$

Last, we have from (21), (22) and weak H^1 convergence: $\forall A > 0$

$$\begin{aligned}\int_{|y|\leq A} |V(y)|^2 dy &\leq \liminf_{n\to+\infty}\int_{|y|\leq A} |v_n(y)|^2 dy \leq \lim_{n\to+\infty}\int_{|y|\leq \frac{R}{\lambda(t_n)}} |v(t_n,y)|^2 dy \\ &= \lim_{n\to+\infty}\int_{|x|\leq R} |u(t_n,x)|^2 dx \leq \int Q^2 - \varepsilon.\end{aligned}$$

Thus $\int |V|^2 \leq \int Q^2 - \varepsilon$, what together with V non zero and $E(V) \leq 0$ contradicts the sharp Gagliardo-Nirenberg inequality (11). This concludes the proof of Theorem 6.

Remark 1 *The argument above is fundamentally based on the Hamiltonian structure of the equation in H^1. If one restricts itself to pure L^2 data which is a much more difficult situation, a similar concentration result has been proved in dimension $N = 2$ by Bourgain, [3], and then extended by Merle, Vega, [26]. Nevertheless, to obtain in L^2, or even in H^s, $0 \leq s < 1$, the sharp constant $\int Q^2$ of minimal focused mass is open.*

Remark 2 *The non radial case in H^1 is handled by Nawa in [27] using standard concentration compactness techniques to overcome the non compact injection of H^1 into $L^{2+\frac{4}{N}}$. Further refined use of these techniques has also allowed Nawa to more precisely describe the singularity formation by proving in a very weak sense a profile type of decomposition, see [28].*

Two fundamental questions following Theorem 6 are still open in the general case:
(i) Does the function $x(t)$ have a limit as $t \to T$ defining then at least one exact blow up point in space where L^2 concentration takes place?
(i) Which is the exact amount of mass which is focused by the blow up dynamic?

An explicit construction of blow up solutions due to Merle, [16], is the following: given k points $(x_i)_{1\leq i\leq k} \in \mathbf{R}^N$, there exists a blow up solution $u(t)$ which blows up in finite time $0 < T < +\infty$ exactly at these k points and behaves locally near x_i like $S(t)$ given by (15). In particular, it satisfies:

$$|u(t)|^2 \rightharpoonup \Sigma_{1\leq i\leq k}|Q|^2_{L^2}\delta_{x=x_i} \quad \text{as} \quad t \to T,$$

in the sense of measures. Let us observe first that from the construction, one could place at x_i instead of $S(t)$ any pseudo-conformal transformation of an excited ground state solution Q_i solution to (10). The solution focuses then at x_i exactly the mass $|Q_i|_{L^2}$ which is quantized but arbitrarily large. Second, similarly as for $S(t)$, such a solution is non dispersive as it accumulates all its initial L^2 mass into blow up.

A general conjecture concerning L^2 concentration is formulated in [24] and states that a blow up solution focuses a quantized and universal amount of mass at a finite number of points in $\mathbf{R}^N$, the rest of the L^2 mass being purely dispersed. The exact statement is the following:

Conjecture (*): *Let $u(t) \in H^1$ a solution to (1) which blows up in finite time $0 < T < +\infty$. Then there exist $(x_i)_{1\leq i\leq L} \in \mathbf{R}^N$ with $L \leq \frac{\int |u_0|^2}{\int Q^2}$, and $u^* \in L^2$ such that: $\forall R > 0$,*

$$u(t) \to u^* \quad in \quad L^2(\mathbf{R}^N - \bigcup_{1\leq i\leq L} B(x_i, R))$$

$$and \quad |u(t)|^2 \rightharpoonup \Sigma_{1\leq i\leq L} m_i \delta_{x=x_i} + |u^*|^2 \quad with \quad m_i \in [\int Q^2, +\infty).$$

The set M of admissible focused mass m_i for $N \geq 2$ is known to contain the unbounded set of the L^2 masses of excited bound states Q^i solutions to (10) from [16], and these are the only known examples.

2.4 Orbital stability of the ground state

From now on and for the rest of these notes, we restrict ourselves to considering small super critical mass initial data, that is:

$$u_0 \in \mathcal{B}_{\alpha^*} = \{u_0 \in H^1 \text{ with } \int Q^2 \leq \int |u_0|^2 \leq \int Q^2 + \alpha^*\},$$

for some parameter $\alpha^* > 0$ small enough. This situation is conjectured to locally describe the generic blow up dynamic around one blow up point.

In this case, we have a fundamental property which is the so called *orbital stability of the solitary wave*. We will give precise statements later, and we just underline here the main facts. Let $u_0 \in \mathcal{B}_{\alpha^*}$ for some $\alpha^* > 0$ universal and small enough, and assume that the corresponding solution to (1) blows up in finite time $0 < T < +\infty$, then there exist continuous parameters $(x(t), \gamma(t)) \in \mathbf{R}^N \times \mathbf{R}$ such that for t close enough to T, $u(t)$ admits a decomposition:

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q + \varepsilon)(t, \frac{x - x(t)}{\lambda(t)})e^{i\gamma(t)}, \tag{24}$$

where

$$|\varepsilon(t)|_{H^1} \leq \delta(\alpha^*), \quad \delta(\alpha^*) \to 0 \text{ as } \alpha^* \to 0,$$

and

$$\lambda(t) \sim \frac{1}{|\nabla u(t)|_{L^2}}. \tag{25}$$

In other words, finite time blow up solutions to (1) with small super critical mass are close to the ground state in H^1 up to the set of H^1 symmetries. This property is again purely based on the Hamiltonian structure and the variational characterization of Q, and not on refined properties of the flow.

The main point of this nonlinear decomposition is that it now allows a perturbative analysis by studying the equation governing the H^1 small excess of mass $\varepsilon(t)$.

To describe the blow up dynamic is now equivalent to understand in the perturbative regime how to extract from the infinite dimensional dynamic of (1) a finite dimensional and possibly universal dynamic for the evolution of the geometrical parameters $(\lambda(t), x(t), \gamma(t))$ *which is coupled to the dispersive dynamic which drives the small excess of mass* $\varepsilon(t)$.

Indeed, to estimate for example the blow up speed is now equivalent to estimating the size of $\lambda(t)$, or to prove the existence of the blow up point is equivalent to proving the existence of a strong limit $x(t) \to x(T) \in \mathbf{R}^N$ as $t \to T$. Similarly, the structure in space of the singularity relies on the dispersive behavior of ε as t approaches blow up time.

2.5 Explicit construction of blow up solutions

As it allows a perturbative approach of the blow up problem, the existence of the geometrical decomposition (24) is a first step for the construction of blow up solutions to (1). We already mentioned a blow up construction by Merle, [16], which built non L^2 dispersive blow up solutions. There are two other fundamental results of construction of blow up solutions.

A first natural question is the existence in the super critical case of a blow up dynamic similar to the one of the explicit critical mass blow up solution $S(t)$. In [5], Bourgain and Wang construct in dimension $N = 1, 2$ solutions $u(t)$ to (1) which blow up in finite time and behave locally like explicit blow up solution $S(t)$ given by (15). More precisely, given a limiting profile $u^* \in H^1$ sufficiently decaying at infinity -for technical reason- and *flat near zero* -this is not a technical point...- in the sense that for some $A > 0$ large enough,

$$\frac{d^i}{dx^i} u^*(0) = 0, \quad 1 \le i \le A, \tag{26}$$

they build a solution to (1) which blows up in finite time $0 < T < +\infty$ at $x = 0$ and satisfies:

$$u(t) - S(t - T) \to u^* \ \text{ in } \ H^1 \ \text{ as } \ t \to T. \tag{27}$$

Note that flatness assumption (26) is not open in H^1, and this statement ensures thus the stability of the $S(t)$ dynamic on a finite codimensional manifold. The meaning of this flatness assumption is to decouple in space the regular part of the solution which will evolve to u^*, and the singular part which will consist of $S(t)$ only. Recall that the Schrödinger propagator a priori allows infinite speed of propagation for waves, so it is a very non trivial fact to be able to control somehow the decoupling in space of the regular and the singular parts of the solution. As a corollary, these solutions have the same blow up speed like $S(t)$:

$$|\nabla u(t)|_{L^2} \sim \frac{1}{T-t}. \tag{28}$$

Now this rate of blow up turns out not to be the "generic" one. First, we have the following *universal lower bound on the blow up rate* known as the scaling lower bound:

Proposition 2 (Scaling lower bound on blow up rate) *Let $u_0 \in H^1$ such that the corresponding solution $u(t)$ to (1) blows up in finite time $0 < T < +\infty$, then there holds for some constant $C(u_0) > 0$:*

$$\forall t \in [0,T), \quad |\nabla u(t)|_{L^2} \geq \frac{C(u_0)}{\sqrt{T-t}} \tag{29}$$

Proof of Proposition 2

The proof is elementary and based on the scaling invariance of the equation and the local wellposedness theory in H^1. Indeed, consider for fixed $t \in [0,T)$

$$v^t(\tau,z) = |\nabla u(t)|_{L^2}^{-\frac{N}{2}} u\left(t + |\nabla u(t)|_{L^2}^{-2}\tau, |\nabla u(t)|_{L^2}^{-1} z\right).$$

v^t is a solution to (1) by scaling invariance. We have $|\nabla v^t|_{L^2} + |v^t|_{L^2} \leq C$, and so by the resolution of the Cauchy problem locally in time by fixed point argument, there exists $\tau_0 > 0$ independent of t such that v^t is defined on $[0,\tau_0]$. Therefore, $t + |\nabla u(t)|_{L^2}^{-2}\tau_0 \leq T$ which is the desired result. This concludes the proof of Proposition 2.

Because it is related to the scaling symmetry of the problem, on which in other instances formal arguments indeed rely to derive the correct blow up speed, lower bound (29) has long been conjectured to be optimal. Yet, in the mid 80's, numerical simulations, see Landman, Papanicolaou, Sulem, Sulem, [12], have suggested that the correct and stable blow up speed is a slight correction to the scaling law:

$$|\nabla u(t)|_{L^2} \sim \sqrt{\frac{\log|\log(T-t)|}{T-t}}, \tag{30}$$

which is referred to as *the log-log law*. Solutions blowing up with this speed indeed appeared to be stable with respect to perturbation of the initial data. Quite an amount of formal work has been devoted to understanding the exact nature of the double log correction to the scaling estimate. We refer to the excellent monograph by Sulem, Sulem, [34], for further discussions on this subject. In this frame, for $N = 1$, Perelman in [31] rigorously proves the existence of one solution which blows up according to (30) and its stability in some space strictly included in H^1.

These two constructions of blow up solutions thus imply the following: there are at least two blow up dynamics for (1) with two different speeds, one which is a continuation of the explicit $S(t)$ blow up dynamic with the $1/(T-t)$ speed (28), and which is suspected to be unstable because it is not seen numerically; one with the log-log speed (30) which is conjectured to be stable from numerics.

2.6 Structural instability of the log-log law

We have so far exhibited two important features of the blow up dynamics for (NLS):
(i) there exists a critical mass blow up solution;
(ii) there are at least two blow up speeds.

These two facts are somehow fundamental difficulties for the analysis. The existence of the critical mass blow up solution implies that the set of initial data which yields a finite time blow up solution is not open, and thus blow up is not a stable phenomenon a priori. On the contrary, only one blow up regime is from numerics expected to be stable.

These facts are somehow believed to be closely related to the very specific algebraic structure of (1), and in particular to the existence of the pseudo-conformal symmetry.

An important result in this direction is the so called *structural instability* of the log-log law in the following sense. Consider in dimension $N = 2$ the Zakharov system:

$$\left\{ \begin{array}{l} iu_t = -\Delta u + nu \\ \frac{1}{c_0^2} n_{tt} = \Delta n + \Delta |u|^2 \end{array} \right. \tag{31}$$

for some fixed constant $0 < c_0 < +\infty$. This system is the previous step in the asymptotic expansion of Maxwell equations which leads to (1), see [34]. In the limit $c_0 \to +\infty$, we formally recover (1). This system is still a Hamiltonian system and shares much of the variational structure of (1). In particular, a virial law in the spirit of (19) holds and yields finite time blow up for radial non positive energy initial data, see Merle, [19]. Moreover, a one parameter family of blow up solution may be constructed and should be understood as a continuation of the exact $S(t)$ solution for (1), see Glangetas, Merle, [9], and these explicit solutions have blow up speed:

$$|\nabla u(t)|_{L^2} \sim \frac{C(u_0)}{T-t}.$$

They moreover appear to be stable from numerics, see Papanicolaou, Sulem, Sulem, Wang, [30]. Now from Merle, [18], *all finite time blow up solutions to (31) satisfy*

$$|\nabla u(t)|_{L^2} \geq \frac{C(u_0)}{T-t}.$$

In particular, there will be no log-log blow up solutions for (31). This fact suggests that in some sense, the Zakharov system provides a much more stable and robust blow up dynamics than its asymptotic limit (NLS). This fact enlightens the belief that the log-log law heavily relies on the specific algebraic structure of (1), and some nonlinear degeneracy properties will indeed be at the heart of the understanding of the blow up dynamics.

3 Blow up dynamics of small super critical mass initial data

Let us recall that we restrict our study to initial data with small super critical mass

$$u_0 \in \mathcal{B}_{\alpha^*} = \{u_0 \in H^1 \text{ with } \int Q^2 \leq \int |u_0|^2 \leq \int Q^2 + \alpha^*\},$$

for some parameter $\alpha^* > 0$ small enough. We present the results on the blow up dynamics obtained in the series of papers [20], [21], [32], [22], [23], [24] and which allow a precise understanding of the blow up dynamics in this setting. The description of the blow up dynamic involves two different type of questions:

- In [20], [21], we consider non positive energy initial data and address the question of an upper bound on the blow up rate. In [32], the dynamically richer case of non negative energy is addressed together with the issue of the stability of the blow up regimes.
- In [22], using as a starting point the point of view and the estimates in [20], [21], [32], we investigate the question of the shape of the solution in space and the existence of a universal asymptotic profile which attracts blow up solutions. These questions rely on Liouville type of theorems to classify the non dispersive dynamics of solitary waves. Further understanding of these issues will then allow one as in [23], [24], to prove sharp lower bounds on the blow up rate related to the expected log-log law and then quantization results on the focused mass -or equivalently Conjecture (*) for data $u_0 \in \mathcal{B}_{\alpha^*}$-.

First, we introduce for notational purpose the following invariant whose sign is preserved by the H^1 symmetries:

$$E_G(u) = E(u) - \frac{1}{2}\left(\frac{Im(\int \nabla u \overline{u})}{|u|_{L^2}}\right)^2. \tag{32}$$

Next, we will assume in all our results a *Spectral Property* which amounts counting the number of negative directions of an explicit Schrödinger operator $-\Delta + V$ where the well localized potential V is stationary and build from the ground state. This property was proved in dimension $N=1$ in [20] using the explicit formula for the ground state Q, and checked numerically in dimension $N=2,3,4$ to which we will thus restrict ourselves, see Proposition 4. Note that this property is the only part of the proof where the restriction on the dimension is needed.

3.1 Finite time blow up for non positive energy initial data

In this subsection, we address the question of the blow up dynamics for non positive energy solutions. The result is the following:

Theorem 7 ([20],[21]) *Let $N=1,2,3,4$. There exist universal constants $\alpha^*, C^*>0$ such that the following holds true. Given $u_0 \in \mathcal{B}_{\alpha^*}$ with*

$$E_G(u_0)<0,$$

the corresponding solution $u(t)$ to (1) blows up in finite time $0<T<+\infty$ and there holds for t close to T:

$$|\nabla u(t)|_{L^2} \leq C^* \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}. \tag{33}$$

Comments on Theorem 7

1. Galilean invariance: From Galilean invariance, the blow up criterion $E_G(u_0)<0$ is equivalent to

$$E_0<0 \quad \text{and} \quad Im\left(\int \nabla u_0 \overline{u_0}\right)=0. \tag{34}$$

Indeed, let u_0 with $E_G(u_0)<0$ and set

$$(u_0)_\beta = u_0 e^{i\frac{\beta}{2}\cdot x} \quad \text{with} \quad \beta = -2\frac{Im(\int \nabla u_0 \overline{u_0})}{\int |u_0|^2},$$

then from explicit computation, $(u_0)_\beta$ satisfies (34). Now from Galilean invariance, $u_\beta(t,x) = u(t,x-\beta t)e^{i\frac{\beta}{2}\cdot(x-\frac{\beta}{2}t)}$, so that blow behavior of $u(t,x)$ and $u_\beta(t,x)$ are the same.

2. Blow up criterion: The blow up criterion is in H^1 and thus improves the virial result which holds in virial space Σ only -up to results of Theorem 5-. In this region of the energy space, blow up is thus a stable phenomenon. Moreover, the result also holds for $t<0$ by considering $\overline{u}(-t)$ which is also a solution to (1), and thus strictly negative energy solutions blow up in finite time on both sides in time.

3. Instability of $S(t)$: The major fact of Theorem 7 is that it removes for non positive energy solutions the possibility of $S(t)$ type of blow up as log-log upper bound (33) is

below the $1/(T-t)$ speed. We will later prove that there is in this case only one blow up regime with speed given by the exact log-log law (30). Now a fundamental corollary of Theorem 7 obtained using the pseudo-conformal transformation is the instability of $S(t)$ in a strong sense. $S(t)$ is the critical mass blow up solution, so it is unstable in a trivial sense: any H^1 neighborhood of $S(-1)$ contains initial data $u(-1)$ with global in time solution $u(t)$; it suffices to take subcritical mass initial data. We claim a much stronger statement which is that the blow up dynamic of $S(t)$ itself is unstable in the following sense: any H^1 neighborhood of $S(-1)$ contains initial data $u(-1)$ with solution $u(t)$ which blows up in finite time but with the log-log speed.
Indeed, let the initial data at time $t=1$: $u_\eta(1,x)=(1+\eta)Q(x)$ for $\eta>0$ and small, and $u_\eta(t)$ the corresponding solution to (1). From explicit computation, $E(u_\eta)<0$ and thus $u_{0\eta}$ satisfies the hypothesis of Theorem 7. It thus blows up in finite time $1<T_\eta<+\infty$. We now apply the pseudo-conformal symmetry and consider the solutions

$$v_\eta(t)=\frac{1}{|t|^{\frac{N}{2}}}u_\varepsilon\left(\frac{-1}{t},\frac{x}{t}\right)e^{-i\frac{|x|^2}{4t}-i}.$$

First observe that

$$v_\eta(-1)\to S(-1)\ \text{ as }\ \eta\to 0$$

in some strong sense. Next, from its definition, $v_\eta(t)$ blows up in finite time $T'_\eta=\frac{-1}{T_\eta}<0$. Now $T'(\eta)<0$ and the uniform space time bound on $|xu_\varepsilon(t)|_{L^2}$ given by the virial law (19) ensure that $v_\eta(t)$ satisfies upper bound (33) for t close enough to T'_η as desired.

The above example also illustrates a standard feature: upper bound (33) is satisfied asymptotically near blow up time, that is for $t\in[t(u_0),T)$, and the time $t(u_0)$ depends on the full profile of the initial data.

3.2 H^1 stability of the log-log law

Let us now investigate the dynamics for positive energy initial data. In this case, three different dynamics are known to possibly occur:

- $S(t)$ behavior: results in [5] yield existence of finite time blow up solutions $u(t)$ satisfying $u_0\in\mathcal{B}_{\alpha^*}$, $E_0^G>0$ and $|\nabla u(t)|_{L^2}\sim\frac{1}{T-t}$ near blow up time.
- log-log behavior: Using the pseudo-conformal symmetry and Theorem 7, we can easily exhibit strictly positive energy solutions satisfying the log-log upper bound (33). Indeed, for $\eta>0$ small enough, let $u_0(\eta)=(1+\eta)Q$ and $u_\eta(t)$ the corresponding solution to (1). We have $E(u_0(\eta))\sim -C\eta<0$, and thus $u_\eta(t)$ blows up in finite time $T(\eta)$ with upper bound (33). Applying now the pseudo conformal transformation to this solution, we let $v_0(\eta)=u_0(\eta^4)e^{-i\eta\frac{|y|^2}{4}}$ and compute its energy. From:

$$E\left(ve^{-i\eta\frac{|y|^2}{4}}\right)=E(v)-\frac{\eta}{2}Im\left(\int x\cdot\nabla v\overline{v}\right)+\frac{\eta^2}{8}\int|y|^2|v|^2,\tag{35}$$

we have $E(v_0(\eta)) > 0$ for η small enough. Now pseudo-conformal formula (14) yields:

$$v(\eta)(t) = \frac{1}{(1-\eta t)^{\frac{N}{2}}} u_\eta\left(\frac{t}{1-\eta t}, \frac{x}{1-\eta t}\right) e^{-i\eta \frac{|x|^2}{4(1-\eta t)}},$$

so that $v(t)$ is defined on $[0, T_v(\eta))$, $T_v(\eta) = \frac{T_\eta}{1+\eta T_\eta} < \frac{1}{\eta}$, and blows up at $T_v(\eta)$ with upper bound (33) as wanted.

- Global solutions: Given $u(t) \in \Sigma$ a solution to (1) which blows up at $0 < T < +\infty$, pseudo conformal symmetry (14) applied with parameter $a = \frac{1}{T}$ yields a solution $v(t)$ to (1) globally defined on $[0, +\infty)$.

There certainly is a poor understanding in general of which conditions on the initial data are enough to select one of the above dynamics. Nevertheless, we have the following:

Theorem 8 ([32]) *Let $N = 1, 2, 3, 4$. There exist universal constants $C^*, C_1^* > 0$ such that the following is true:*

(i) Rigidity of blow up rate: Let $u_0 \in \mathcal{B}_{\alpha^*}$ with

$$E_G(u_0) > 0,$$

and assume the corresponding solution $u(t)$ to (1) blows up in finite time $T < +\infty$, then there holds for t close to T either

$$|\nabla u(t)|_{L^2} \leq C^* \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}$$

or

$$|\nabla u(t)|_{L^2} \geq \frac{C_2^*}{(T-t)\sqrt{E_G(u_0)}}. \tag{36}$$

(ii) Stability of the log-log law: Moreover, the set of initial data $u_0 \in \mathcal{B}_{\alpha^*}$ such that $u(t)$ blows up in finite time with upper bound (33) is open in H^1.

Comments on Theorem 8

1. Optimal criterion of stability: A slightly more self contained statement is that the set

$$\mathcal{O} = \{u_0 \in \mathcal{B}_{\alpha^*}, \quad \int_0^{T_u} |\nabla u(t)|_{L^2} dt < +\infty\} \text{ is open in } H^1,$$

and that $\mathcal{O}$ is exactly the set of initial data which blow up in finite time with log-log upper bound (33). We will later refer to $\mathcal{O}$ as the open set of log-log blow up.

2. *Size of the log-log set*: $\mathcal{O}$ is known to contain non positive energy initial data from Theorem 7. Then the pseudo-conformal invariance allows one to obtain non negative energy solutions which satisfy log-log upper bound (33). One can prove that this procedure does not describe all $\mathcal{O}$, and that there exist initial data $u_0 \in \Sigma \cap \mathcal{O}$ with non negative energy which cannot be obtained using the pseudo-conformal symmetry from a non positive energy initial data, see [32].

4. *Stability versus instability*: Let us recall that the existence of critical mass blow up solution $S(t)$ implies that the set of initial data which lead to a finite time blow up solution is not open in H^1. In this setting, the fact that the blow up speed is a sufficient criterion of stability in the energy space is a new feature in the nonlinear dispersive setting. Now if stability of the log-log regime is proved, instability in the strong sense of solutions satisfying lower bound (36) is proved only for $S(t)$ itself, see Comment 3 of the previous subsection. Dynamical instability in this sense of these solutions is open. A simpler result would be to prove the strong instability of solutions built in [5], this is also open.

5. *Universal upper bound on the blow up speed*: Upper bound (33) corresponds to the stable blow up dynamic, while lower bound (36) is obtained by the assumption of escaping this stable blow up regime. In this sense, these estimates correspond to two different asymptotic blow up regimes which each require a specific analysis. This is why no general upper bound on blow up rate of any type holds so far. Let us recall that solutions built in [5] satisfy the exact law

$$|\nabla u(t)|_{L^2} \sim \frac{C(u_0)}{T-t},$$

and it seems reasonable to conjecture that this law is sharp. Let us remark that this would imply from the pseudo-conformal symmetry that blow up in Σ always occurs in finite time, this is also an open problem.

We have not addressed so far the question of blow up dynamics of zero energy initial data. This question turns out to be very fundamental but requires different type of ideas. Note that the solitary wave $Q(x)e^{it}$ is a global in time zero energy solution to (1). We will come back later to the issue of classifying this dynamic among the set of zero energy solutions.

3.3 Universality of the blow up profile

We now turn to the question of the dispersive properties of blow up solutions to (1). Recall from existence of the geometrical decomposition (24) that blow up solutions near blow up time may be written:

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q+\varepsilon)(t, \frac{x-x(t)}{\lambda(t)})e^{i\gamma(t)}$$

from some H^1 small excess of mass $\varepsilon(t)$. We ask the question of the dispersive behavior of $\varepsilon(t)$ as $t \to T$. The result is the following:

Theorem 9 ([22]) *Let $N = 1, 2, 3, 4$. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$. Then there exist parameters $\lambda_0(t) = \frac{|\nabla Q|_{L^2}}{|\nabla u(t)|_{L^2}}$, $x_0(t) \in \mathbf{R}^N$ and $\gamma_0(t) \in \mathbf{R}$ such that*

$$e^{i\gamma_0(t)}\lambda_0^{\frac{N}{2}}(t)u(t, \lambda_0(t)x + x_0(t)) \to Q \quad in \quad L^2_{loc} \quad as \quad t \to T.$$

In the variables of the decomposition (24), this means:

$$\varepsilon(t) \to 0 \ \text{ as } \ t \to T \ \text{ in } \ L^2_{loc}.$$

Let us observe that this is the typical dispersive behavior for Schrödinger group: the L^2 mass is conserved, so L^2 convergence to zero is forbidden, but it happens locally in space meaning that the excess of mass is dispersed away. From the geometrical decomposition (24), this theorem thus asserts that in rescaled variables, blow up solutions in $\mathcal{B}_{\alpha^*}$ admit a universal asymptotic profile in space which is given by the ground state Q itself.

This type of questions goes beyond blow up issues and is related to a wide range of problems regarding the asymptotic stability of solitary waves in nonlinear dispersive PDE's. For blow up problems in the nonlinear dispersive setting, the first result in this direction has been obtained by Martel and Merle, [15], for the generalized critical KdV equation:

$$(KdV) \quad \begin{cases} u_t + (u_{xx} + u^5)_x = 0, & (t, x) \in [0, T) \times \mathbf{R}, \\ u(0, x) = u_0(x), \in H^1 & u_0 : \mathbf{R} \to \mathbf{R}. \end{cases} \tag{37}$$

This equation shares a lot of the variational structure of (1), and in particular finite time blow up solutions admit a geometrical decomposition similar to (24). In [15], Martel and Merle also prove the asymptotic stability of Q as the blow up profile. One of the fundamental observation of their proof is to show that this result is essentially equivalent to proving a lower bound on blow up rate which avoids the self similar regime. We similarly have:

Theorem 10 ([22]) *Let $N = 1, 2, 3, 4$. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$. Then:*

$$|\nabla u(t)|_{L^2}\sqrt{T - t} \to +\infty \quad as \quad t \to T. \tag{38}$$

Let us recall that there always holds the scaling lower bound (29):

$$|\nabla u(t)|_{L^2} \geq \frac{C(u_0)}{\sqrt{T - t}}.$$

This lower bound is thus never sharp for data $u_0 \in \mathcal{B}_{\alpha^*}$. Bourgain in [4] conjectured that there indeed are no self similar solutions, that is blowing up with the exact scaling law $|\nabla u(t)|_{L^2} \sim \frac{C(u_0)}{\sqrt{T-t}}$, *in the energy space* H^1.

Now a fundamental fact which will enlighten our further analysis is that there do exist self similar solutions, but they never belong to L^2. More precisely, a standard way of exhibiting self similar solutions is to look for a blow up solution with the form:

$$U_b(t,x) = \frac{1}{(2b(T-t))^{\frac{N}{4}}} Q_b\left(\frac{x}{\sqrt{2b(T-t)}}\right) e^{-i\frac{\log(T-t)}{2b}}$$

for some fixed parameter $b > 0$ and a fixed profile Q_b solving the elliptic radial ODE:

$$\Delta Q_b - Q_b + ib\left(\frac{N}{2}Q_b + y \cdot \nabla Q_b\right) + Q_b|Q_b|^{\frac{4}{N}} = 0. \tag{39}$$

Now from [33], solutions Q_b *never belong to* L^2 from a logarithmic divergence at infinity:

$$|Q_b(y)| \sim \frac{C(b)}{|y|^{\frac{N}{2}}} \quad \text{as} \quad |y| \to +\infty$$

and thus always miss the energy space. Nevertheless, for any given parameter $b > 0$ small enough, one can exhibit a solution to (39) which will be in $\dot{H}^1$ and will satisfy:

$$Q_b \to Q \quad \text{as } b \to 0 \quad \text{in} \quad \dot{H}^1 \cap L^2_{loc}.$$

In other words, one can build self similar solutions to (1) which on compacts sets will look like a smooth solution, but then display an oscillatory behavior at infinity in space which induces *a non* L^2 *tail escaping the soliton core.*

Now to prove Theorems 9 or 10, one needs to understand how to use the information that the solution we consider is in L^2, and one thus needs to exhibit L^2 *dispersive estimates on the solution.* Now recall that L^2 is the scaling invariant space for this equation, and thus any dispersive information in L^2 is in fact a global information in space. Now the only given global information in L^2 is the conservation of the L^2 norm, and somehow the task here is *to be able to use the conservation of the energy in a dynamical way.*

Following the analysis in [15], the strategy used to prove Theorem 9 is by contradiction. Assuming that the blow up profile is not Q, we prove using compactness type of arguments based on the estimates on the blow up dynamic proved in [20], [21], [32], that it implies the existence of a *self similar solution* $v(t)$ in H^1 which is non dispersive in the sense that:

$$|v(t)|^2 \rightharpoonup \left(\int |v(0)|^2\right) \delta_{x=0} \quad \text{as} \quad t \to T \tag{40}$$

in the weak sense of measures. In other words, if the excess mass is not dispersed, one can extract a fully non L^2 dispersive blow up solution in the sense that it accumulates all of its L^2 mass into blow up. A crucial point in this step is the proof of the continuity of the blow up time with respect to the initial data *in the open set* $\mathcal{O}$.

The second step of the analysis is now to classify the non L^2 dispersive solutions. The proof of this step involves the expected new type of dispersive estimates in L^2. The result is the following.

Theorem 11 ([22]) *Let $N = 1, 2, 3, 4$. Let an initial data $v_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution to (1) blows up in finite time $0 < T < +\infty$ and does not disperse in L^2 in the sense that it satisfies (40), then*

$$v(t) = S(t)$$

up to the set of H^1 symmetries of (1).

In other words, the only non dispersive blow up solution in $\mathcal{B}_{\alpha^*}$ is the critical mass blow up solution, which of course cannot lose mass at blow up. This result should be seen as the dispersive super critical version of Theorem 4.

A key in the proof is that non L^2 dispersive information (40) together with the fact that $v(t)$ satisfies the self similar law implies estimates on the solution in the virial space Σ. This allows us to use the pseudo-conformal symmetry, and it then turns out that Theorem 11 is equivalent to classification results of zero energy solutions to (1):

Theorem 12 ([22]) *Let $N = 1, 2, 3, 4$. Let $u_0 \in \mathcal{B}_{\alpha^*} \cap \Sigma$ with*

$$E_G^0 = 0,$$

$u(t)$ the corresponding solution to (1). Assume that $u(t)$ is not a soliton up to the symmetries in H^1, then $u(t)$ blows up in finite time on both sides in time with upper bound (33).

Observe that the solitary wave is a global in time zero energy solution in Σ. In other words, to classify non L^2 dispersive solutions is equivalent from pseudo-conformal symmetry to dynamically classify the solitary wave in the set of zero energy solutions in Σ. This kind of Liouville theorems and dynamical classification is a completely new -and unexpected- feature for (NLS).

3.4 Exact log-log law and the mass quantization conjecture

L^2 dispersive estimates needed for the proof of Theorem 9 are exhibited for the proof of the classification result of Theorem 11. In this sense, these estimates are not proved for a "true" blow up solution but are exhibited as specific properties of a non dispersive blow up solution.

Now a further understanding of these properties in fact allows one to obtain direct dispersive estimates in L^2 on a blow up solution. More specifically, let a finite time blow up solution $u(t) \in \mathcal{O}$. We exhibit a global in space information on the solution by proving that in rescaled variables, the space divides in three specific regions:
(i) on compact sets, the solution looks like Q in a strong sense;
(ii) a radiative regime then takes place where the solution looks like the *non* L^2 *tail* of explicit self similar solutions to (39);
(iii) this regime cannot last forever in space because the tale of self similar solutions is not L^2, while the solution is. We then exhibit a third regime further away in space where a purely linear dispersive dynamic takes place.

Another way of viewing the picture is the following: the nonlinear dynamic represented by Q on compact set is connected to a linear dispersive dynamic at infinity in space by a universal radiative regime given by the tail of explicit self similar solutions. This radiation is the mechanism which takes the L^2 mass out of the soliton core on compact sets to disperse it to infinity in space: this is an "outgoing radiation" process corresponding to the so-called dynamical metastability of self similar profiles Q_b solutions to (39). Now the rate at which the mass is extracted is submitted to one global constraint in time: the conservation of the L^2 norm. Moreover, this mechanism quantifies how the L^2 constraint implies the non persistence of the self similar regime, that is how the radiation is connected to the dispersive dynamic at infinity. And we have seen that this is related to obtaining lower bounds on the blow up rate.

The outcome of this analysis is the following sharp lower bound on blow up rate.

Theorem 13 ([23]) *Let $N = 1, 2, 3, 4$. There exists a universal constants $C_3^* > 0$ such that the following holds true. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$, then one has the following lower bound on blow up rate for t close to T:*

$$|\nabla u(t)|_{L^2} \geq C_3^* \left(\frac{\log |\log(T-t)|}{T-t} \right)^{\frac{1}{2}}. \tag{41}$$

In the log-log regime, the blow up speed may in fact be exactly evaluated according to:

$$\frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}} \left(\frac{T-t}{\log |\log(T-t)|} \right)^{\frac{1}{2}} \to \frac{1}{\sqrt{2\pi}} \quad \text{as} \quad t \to T. \tag{42}$$

In addition, we may extend the dynamical characterization of solitons in the zero energy manifold to the full energy space H^1:

Theorem 14 ([23]) *Let $N = 1, 2, 3, 4$. Let $u_0 \in \mathcal{B}_{\alpha^*}$ with $E_0^G = 0$ and assume u_0 is not a soliton up to fixed scaling, phase, translation and Galilean invariances.*
Then u blows up both for $t < 0$ and $t > 0$, and (42) holds.

It is a surprising fact somehow that the analysis needed to obtain lower and upper bounds in the log-log regime requires different type of informations:
(i) The proof of the upper bound on blow up rate (33) requires only local in space information on the soliton core, and global in $\dot{H}^1$. But nothing is needed in L^2 and indeed explicit self similar profiles solutions to (39) in $\dot{H}^1$ would fit into this analysis.
(ii) The proof of the lower bound on blow up rate (41) requires global in space dispersive information in L^2, that is estimates on the solution in the different regimes in space. One may then estimate the flux of L^2 norm in between these different regimes which is submitted to the L^2 conservation constraint. This yields the exact log-log law.

Moreover, and this certainly is the main motivation to go through the whole log-log analysis, the precise understanding of the L^2 structure in space of the solution in rescaled variables now allows us to investigate the behavior of the solution in the original non rescaled variables.

Indeed, let us make the following simple observation. From the geometrical decomposition (24), a blow up solution $u(t)$ near blow up time may be written:

$$u(t,x) = Q_{sing}(t,x) + \tilde{u}(t,x)$$

with

$$Q_{sing}(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}} Q(t, \frac{x - x(t)}{\lambda(t)}) e^{i\gamma(t)}, \quad \tilde{u}(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}} \varepsilon(t, \frac{x - x(t)}{\lambda(t)}) e^{i\gamma(t)}.$$

Q_{sing} is the singular part of the solution. We address the following natural question: does the excess of mass $\tilde{u}(t,x)$ remain smooth up to blow up time? A first answer to this question has been obtained in rescaled variables. Indeed, Theorem 9 asserts:

$$\varepsilon(t) \to 0 \ \text{ as } \ t \to T \ \text{ in } \ L^2_{loc}.$$

But as $\lambda(t) \to 0$ as $t \to T$, this is very far from obtaining regularity control on $\tilde{u}(t)$. In particular, it does not prevent a priori the excess of mass $\tilde{u}$ from focusing some small mass at blow up time. The regularity of $\tilde{u}$ is thus deeply related both to the shape in space of $\varepsilon(t)$ and the rate at which it is dispersed. Both of these questions are precisely addressed in the proof of Theorem 13. Further use of the obtained estimates then allow one to prove the following result.

Theorem 15 ([24]) *Let $N = 1,2,3,4$. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution to (1) blows up in finite time $0 < T < +\infty$. Then there exist parameters $(\lambda(t), x(t), \gamma(t)) \in \mathbf{R}_+^* \times \mathbf{R}^N \times \mathbf{R}$ and an asymptotic profile $u^* \in L^2$ such that*

$$u(t) - \frac{1}{\lambda(t)^{\frac{N}{2}}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \quad in \quad L^2 \quad as \quad t \to T. \tag{43}$$

Moreover, the blow up point is finite in the sense that

$$x(t) \to x(T) \in \mathbf{R}^N \quad as \quad t \to T.$$

In other words, up to a singular part which has a universal space time structure, blow up solutions remain in L^2 up to blow up time.

A fundamental corollary is the so called *quantization phenomenon* for (1): blow up solutions in $\mathcal{B}_{\alpha^*}$ focus the universal amount of mass $\int Q^2$ into blow up, the rest is purely dispersed, or in other words:

$$|u(t)|^2 \rightharpoonup \left(\int Q^2\right) \delta_{x=x(T)} + |u^*|^2 \quad \text{as} \quad t \to T \quad \text{with} \quad \int |u_0|^2 = \int Q^2 + \int |u^*|^2.$$

This is in contrast with the Zakharov model (31) where explicit blow up solutions build by Glangetas, Merle, [9], accumulate a continuum of mass into blow up.

A second outcome of Theorem 15 is the fact that the formation of the singularity is a well localized in space phenomenon. Indeed, blow up occurs at a well defined blow up point $x(T)$ where a fixed amount of mass is focused, but outside $x(T)$, the solution has a strong L^2 limit. It means in particular that the phase of the solution is not oscillatory outside the blow up point, whereas the phase $\gamma(t)$ of the singularity is known to satisfy $\gamma(t) \to +\infty$ as $t \to T$. This strong regularity of the solution outside the blow up point was not expected. From the proof also, one can prove that the blow up point $x(T)$ and the asymptotic profile u^* are in the log-log regime continuous functions of the initial data.

Observe now that Theorem 15 includes both blow up regimes which would in particular be characterized by a different law for $\lambda(t)$ in the singular part of the solution. We now claim that the difference between the two blow up regimes may be seen on the asymptotic profile u^* which in fact *connects in a universal way depending on the blow up regime the regular and singular parts of the solution.*

Theorem 16 ([24]) *Let $N = 1,2,3,4$. There exists a universal constant $C^* > 0$ such that the following holds true. Let $u_0 \in \mathcal{B}_{\alpha^*}$ and assume that the corresponding solution $u(t)$ to (1) blows up in finite time $0 < T < +\infty$. Let $x(T)$ its blow up point and $u^* \in L^2$*

its profile given by Theorem 15, then for $R > 0$ small enough, we have:
(i) Log-log case: if $u_0 \in \mathcal{O}$, then

$$\frac{1}{C^*(\log|\log(R)|)^2} \leq \int_{|x-x(T)|\leq R} |u^*(x)|^2 dx \leq \frac{C^*}{(\log|\log(R)|)^2}, \tag{44}$$

and in particular:

$$u^* \notin H^1 \quad and \quad u^* \notin L^p \quad for \quad p > 2. \tag{45}$$

(ii) $S(t)$ case: if $u(t)$ satisfies (36), then

$$\int_{|x-x(T)|\leq R} |u^*|^2 \leq C^* E_0 R^2, \tag{46}$$

and

$$u^* \in H^1.$$

The fact that one can separate within the two blow up dynamics and see the different blow up speeds on asymptotic profile u^* is a completely new feature for (NLS) and was not even expected at the formal level. Moreover, this results strengthens our belief that $S(t)$ type of solutions are in some sense on the boundary of the set of finite time blow up solutions:

- The stable log-log blow up scenario is based on the ejection of a radiative mass which strongly couples the singular and the regular parts of the solution and induces singular behavior (44) of the profile at blow up point. The universal singular behavior (44) is the "trace" of the radiative regime in the rescaled variables which couples the blow up dynamic on compact sets to the dispersive dynamic at infinity.

- On the contrary, the $S(t)$ regime corresponds to formation of a minimal mass blow up bubble very decoupled from the regular part which indeed remains smooth in the Cauchy space. This blow up scenario somehow corresponds to the "minimal" blow up configuration. Observe that in this last regime, (46) in dimension $N = 1$ implies $u^*(0) = 0$. Now in [5], Bourgain and Wang construct for a given radial profile u^* smooth with $\frac{d^i}{dr^i}u^*(r)_{|r=0} = 0$, $1 \leq i \leq A$, a solution to (1) with blow up point $x = 0$ and asymptotic profile u^*. In their proof, A is very large, what is used to decouple the regular and the singular parts of the solution. In this sense, estimate (46) proves in general a decoupling of this kind for the $S(t)$ dynamic. It is an open problem to estimate the exact degeneracy of u^*.

4 Log-log upper bound on the blow up rate

This section is devoted to a presentation of the main results needed for the proof of the log-log upper bound on blow up rate in the non positive energy case, that is Theorem 7.

We will in particular focus onto the proof of the key dispersive controls in $\dot{H}^1$ which are at the heart of the control from above on the blow up speed. More detailed proofs are to be found in [20], [21].

The heart of our analysis will be to exhibit as a consequence of dispersive properties of (1) close to Q strong rigidity constraints for the dynamics of non positive energy solutions. These will in turn imply monotonicity properties, that is the existence of a Lyapounov function. The corresponding estimates will then allow us to prove blow up in a dynamical way and the sharp upper bound on the blow up speed will follow.

In the whole section, we consider a data

$$u_0 \in \mathcal{B}_{\alpha^*}$$

for some small universal $\alpha^* > 0$ and let $u(t)$ the corresponding solution to (1) with maximal time interval existence $[0,T)$ in H^1, $0 < T \leq +\infty$. We further assume $E_G(u_0) < 0$. According to Comment 1 after Theorem 7, we equivalently have up to a fixed Galilean Transformation:

$$E_0 < 0 \quad \text{and} \quad Im\left(\int \nabla u_0 \overline{u_0}\right) = 0. \tag{47}$$

For a given function f, we will note

$$f_1 = \frac{N}{2} f + y \cdot \nabla f, \quad f_2 = \frac{N}{2} f_1 + y \cdot \nabla f_1.$$

Note that from integration by parts:

$$(f_1, g) = -(f, g_1).$$

We will not use this notation for ε only.

4.1 Existence of the geometrical decomposition

We recall in this section the orbital stability of the solitary wave which implies the existence of geometrical decomposition (24). The argument is based only on the conservation of the energy and the L^2 norm and the small super critical mass assumption $u_0 \in \mathcal{B}_{\alpha^*}$. The idea is the following. Recall that the ground state minimizes the energy according to Proposition 1: let $v \in H^1$ such that $\int |v|^2 = \int Q^2$ and $E(v) = 0$, then

$$v(x) = \lambda_0^{\frac{N}{2}} Q(\lambda_0 x + x_0) e^{i\gamma_0},$$

for some parameters $\lambda_0 \in \mathbf{R}_+^*$, $x_0 \in \mathbf{R}^N$, $\gamma_0 \in \mathbf{R}$. From standard concentration compactness type of arguments, this implies in particular that functions with negative energy and small super critical mass have a very specific shape, and indeed, are close to the three dimensional minimizing manifold. The result is the following:

Lemma 1 (Orbital stability of the ground state) *There exists a universal constant $\alpha^* > 0$ such that the following is true. For all $0 < \alpha' \leq \alpha^*$, there exists $\delta(\alpha')$ with $\delta(\alpha') \to 0$ as $\alpha' \to 0$ such that $\forall v \in H^1$, if*

$$\int Q^2 \leq \int |v|^2 < \int Q^2 + \alpha' \quad \text{and} \quad E(v) \leq \alpha' \int |\nabla v|^2,$$

then there exist parameters $\gamma_0 \in \mathbf{R}$ and $x_0 \in \mathbf{R}^N$ such that

$$|Q - e^{i\gamma_0} \lambda_0^{\frac{N}{2}} v(\lambda_0 x + x_0)|_{H^1} < \delta(\alpha') \tag{48}$$

with $\lambda_0 = \frac{|\nabla Q|_{L^2}}{|\nabla v|_{L^2}}$.

Proof of Lemma 1

We prove the claim in the radial case for $N \geq 2$. The general case follows from standard concentration compactness techniques.
Arguing by contradiction, we equivalently need to prove the following: let a sequence $v_n \in H^1$ such that:

$$\forall n, \quad \int |\nabla v_n|^2 = \int |\nabla Q|^2, \tag{49}$$

$$\int |v_n|^2 \to \int Q^2 \quad \text{as} \quad n \to +\infty, \tag{50}$$

and

$$\limsup_{n \to +\infty} E(v_n) \leq 0, \tag{51}$$

then there exist $\gamma_n \in \mathbf{R}$ such that

$$e^{i\gamma_n} v_n \to Q \quad \text{in} \quad H^1 \quad \text{as} \quad n \to +\infty. \tag{52}$$

Let us consider $w_n = |v_n|$. First observe from $\int |\nabla w_n|^2 \leq \int |\nabla v_n|^2$ that the sequence w_n is H^1 bounded, thus

$$w_n \rightharpoonup W \quad \text{in} \quad H^1$$

up to a subsequence. We first claim that W is non zero. Indeed, from (49):

$$\frac{1}{2 + \frac{4}{N}} \int |w_n|^{2+\frac{4}{N}} = \frac{1}{2 + \frac{4}{N}} \int |v_n|^{2+\frac{4}{N}} = \frac{1}{2} \int |\nabla v_n|^2 - E(v_n) = \frac{1}{2} \int |\nabla Q|^2 - E(v_n),$$

and thus from (51):

$$\liminf_{n \to +\infty} \int |w_n|^{2+\frac{4}{N}} > 0.$$

Now from compact embedding of H^1_{radial} into $L^{2+\frac{4}{N}}$,

$$\int |w_n|^{2+\frac{4}{N}} \to \int |W|^{2+\frac{4}{N}}, \tag{53}$$

and thus W is non zero. Now from (51),

$$E(W) \leq \liminf_{n\to+\infty} E(w_n) \leq \liminf_{n\to+\infty} E(v_n) \leq 0, \tag{54}$$

and from (49),

$$\int |W|^2 \leq \liminf_{n\to+\infty} \int |w_n|^2 = \int Q^2.$$

Thus W is a non zero negative energy function with subcritical mass, while the sharp Gagliardo-Nirenberg inequality (11) and Proposition 1 characterize the ground state up to fixed scaling and phase invariances. Now W is real so the phase is zero. Moreover, this yields $\int W^2 = \int Q^2$ and thus $\int W_n^2 \to \int W^2$ from (50). Similarly, $E(W) = 0$ and thus from (54), $E(w_n) \to E(W) = 0$, and from (53), $\int |\nabla w_n|^2 \to \int |\nabla W|^2$. This implies from (49) that $W = Q$ and so $w_n \to W = Q$ strongly in H^1. It is now an easy task to conclude to (52). This ends the proof of Lemma 1.

The small critical mass assumption $u_0 \in \mathcal{B}_{\alpha^*}$ and the negative energy assumption now allow us to apply Lemma 1 to $v = u(t)$ for all fixed $t \in [0, T)$, and thus to exhibit parameters $\gamma_0(t) \in \mathbf{R}$, $x_0(t) \in \mathbf{R}^N$ and $\lambda_0(t) = \frac{|\nabla Q|_{L^2}}{|\nabla u(t)|_{L^2}}$ such that $u(t)$ satisfies (48) for all time. Let us observe that this geometrical decomposition is by no mean unique, and the parameters $(\lambda_0(t), \gamma_0(t), x_0(t))$ built from Lemma 1 are a priori no better than continuous functions of time. Nevertheless, one can freeze and regularize this decomposition by choosing a set of orthogonality conditions on the excess of mass: this is the so-called modulation theory which will be examined later on. Let us assume for now that we have a smooth decomposition of the solution: $\forall t \in [0, T)$,

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q+\varepsilon)(t, \frac{x - x(t)}{\lambda(t)})e^{i\gamma(t)}$$

with

$$\lambda(t) \sim \frac{C}{|\nabla u(t)|_{L^2}} \quad \text{and} \quad |\varepsilon(t)|_{H^1} \leq \delta(\alpha^*) \to 0 \quad \text{as} \quad \alpha^* \to 0.$$

To study the blow up dynamic is now equivalent to understanding the coupling between the finite dimensional dynamic which governs the evolution of the geometrical parameters $(\lambda(t), \gamma(t), x(t))$ and the infinite dimensional dispersive dynamic which drives the excess of mass $\varepsilon(t)$.

To enlighten the main issues, let us rewrite (1) in the so-called rescaled variables. Let us introduce the rescaled time:

$$s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)}.$$

It is elementary to check that whatever is the blow up behavior of $u(t)$, one always has:

$$s([0,T)) = \mathbf{R}^+.$$

Let us set:

$$v(s,y) = e^{i\gamma(t)}\lambda(t)^{\frac{N}{2}}u(\lambda(t)x + x(t)),$$

then from direct computation, $u(t,x)$ solves (1) on $[0,T)$ iff $v(s,y)$ solves: $\forall s \geq 0$,

$$iv_s + \Delta v - v + v|v|^{\frac{4}{N}} = i\frac{\lambda_s}{\lambda}\left(\frac{N}{2}v + y\cdot\nabla v\right) + i\frac{x_s}{\lambda}\cdot\nabla v + \tilde{\gamma}_s v, \tag{55}$$

where $\tilde{\gamma} = -\gamma - s$. Now from (48), $v(s,y) = Q + \varepsilon(s,y)$, so we may linearize (55) close to Q. The obtained system has the form:

$$i\varepsilon_s + L\varepsilon = i\frac{\lambda_s}{\lambda}\left(\frac{N}{2}Q + y\cdot\nabla Q\right) + \gamma_s Q + i\frac{x_s}{\lambda}\cdot\nabla Q + R(\varepsilon), \tag{56}$$

$R(\varepsilon)$ formally quadratic in ε, and $L = (L_+, L_-)$ is the matrix linearized operator close to Q which has components:

$$L_+ = -\Delta + 1 - \left(1 + \frac{4}{N}\right)Q^{\frac{4}{N}}, \quad L_- = -\Delta + 1 - Q^{\frac{4}{N}}.$$

A standard approach is to think of equation (56) in the following way: it is essentially a linear equation forced by terms depending on the law for the geometrical parameters. The classical study of this kind of system relies on the understanding of the dispersive properties of the propagator e^{isL} of the linearized operator close to Q. In particular, one needs to exhibit its spectral structure. This has been done by Weinstein, [36], using the variational characterization of Q. The result is the following: L is a non-self-adjoint operator with a generalized eigenspace at zero. The eigenmodes are explicit and generated by the symmetries of the problem:

$$L_+\left(\frac{N}{2}Q + y\cdot\nabla Q\right) = -2Q \quad \text{(scaling invariance)},$$

$$L_+(\nabla Q) = 0 \quad \text{(translation invariance)},$$

$$L_-(Q) = 0 \quad \text{(phase invariance)}, \quad L_-(yQ) = -2\nabla Q \quad \text{(Galilean invariance)}.$$

An additional relation is induced by the pseudo-conformal symmetry:

$$L_-(|y|^2Q) = -4(\frac{N}{2}Q + y\cdot\nabla Q),$$

and this in turns implies the existence of an additional mode ρ solution to

$$L_+\rho = -|y|^2Q.$$

These explicit directions induce "growing" solutions to the homogeneous linear equation $i\partial_S\varepsilon + L\varepsilon = 0$. More precisely, there exists a (2N+3) dimensional space S spanned by the above directions such that $H^1 = M \oplus S$ with $|e^{isL}\varepsilon|_{H^1} \leq C$ for $\varepsilon \in M$ and $|e^{isL}\varepsilon|_{H^1} \sim s^3$

for $\varepsilon \in S$. As each symmetry is at the heart of a growing direction, a first idea is to use the symmetries from modulation theory to a priori ensure that ε is orthogonal to S. Roughly speaking, the strategy to construct blow up solutions is then: chose the parameters λ, γ, x so as to get good a priori dispersive estimates on ε in order to build it from a fixed point scheme. Now the fundamental problem is that one has (2N+2) symmetries, but (2N+3) bad modes in the set S. Both constructions in [5] and [31] develop non trivial strategies to overcome this fundamental difficulty of the problem.

Our strategy will be more nonlinear. On the basis of decomposition (48), we will prove dispersive estimates on ε induced by the virial structure (19). The proof will rely on nonlinear degeneracies of the structure of (1) around Q. Using then the Hamiltonian information $E_0 < 0$, we will inject these estimates into the finite dimensional dynamic which governs $\lambda(t)$ -which measures the size of the solution- and prove rigidity properties of Lyapounov type. This will then allow us to prove finite time blow up together with the control of the blow up speed.

4.2 Choice of the blow up profile

Before exhibiting the modulation theory type of arguments, we present in this subsection a formal discussion regarding explicit solutions of equation (55) which is inspired from a discussion in [34].

First, let us observe that the key geometrical parameters is λ which measures the size of the solution. Let us then set

$$-\frac{\lambda_s}{\lambda} = b$$

and look for solutions to a simpler version of (55):

$$iv_s + \Delta v - v + ib\left(\frac{N}{2}v + y\cdot\nabla v\right) + v|v|^{\frac{4}{N}} \sim 0.$$

Moreover, from orbital stability property, we want solutions which remain close to Q in H^1. Let us look for solutions of the form $v(s,y) = Q_{b(s)}(y)$ where the mappings $b \to Q_b$ and the law for $b(s)$ are the unknown. We think of b remaining uniformly small and $Q_{b=0} = Q$. Injecting this ersatz into the equation, we get:

$$i\frac{db}{ds}\left(\frac{\partial \overline{Q}_b}{\partial b}\right) + \Delta\overline{Q}_{b(s)} - \overline{Q}_{b(s)} + ib(s)\left(\frac{N}{2}\overline{Q}_{b(s)} + y\cdot\nabla\overline{Q}_{b(s)}\right) + \overline{Q}_{b(s)}|\overline{Q}_{b(s)}|^{\frac{4}{N}} = 0.$$

To handle the linear group, we let $\overline{P}_{b(s)} = e^{i\frac{b(s)}{4}|y|^2}\overline{Q}_{b(s)}$ and solve:

$$i\frac{db}{ds}\left(\frac{\partial \overline{P}_b}{\partial b}\right) + \Delta\overline{P}_{b(s)} - \overline{P}_{b(s)} + \left(\frac{db}{ds} + b^2(s)\right)\frac{|y|^2}{4}\overline{P}_{b(s)} + \overline{P}_{b(s)}|\overline{P}_{b(s)}|^{\frac{4}{N}} = 0. \tag{57}$$

A remarkable fact related to the specific algebraic structure of (1) around Q is that (57) admits three solutions:

- The first one is $(b(s), \overline{P}_{b(s)}) = (0, Q)$, that is the ground state itself. This is just a consequence of the scaling invariance.
- The second one is $(b(s), \overline{P}_{b(s)}) = (\frac{1}{s}, Q)$. This non trivial solution is a rewriting of the explicit critical mass blow up solution $S(t)$ and is induced by the pseudo-conformal symmetry.
- The third one is given by $(b(s), \overline{P}_{b(s)}) = (b, \overline{P}_b)$ for some fixed non zero constant b and $\overline{P}_b$ satisfies:
$$\Delta \overline{P}_b - \overline{P}_b + \frac{b^2}{4}|y|^2\overline{P}_b + \overline{P}_b|\overline{P}_b|^{\frac{4}{N}} = 0. \tag{58}$$
Solutions to this nonlinear elliptic equation are those which produce the explicit self similar profiles solutions to (39). A simple way to see this is to recall that we have set $b = -\frac{\lambda_s}{\lambda}$, so if b is frozen, we have from $\frac{ds}{dt} = \frac{1}{\lambda^2}$:
$$b = -\frac{\lambda_s}{\lambda} = -\lambda\lambda_t \ \text{ ie } \ \lambda(t) = \sqrt{2b(T-t)},$$
this is the scaling law for the blow up speed.

Now the fundamental point is, see [33], that solutions to (58) never belong to L^2 from a logarithmic divergence at infinity:
$$|P_b(y)| \sim \frac{C(P_b)}{|y|^{\frac{N}{2}}} \ \text{ as } \ |y| \to +\infty.$$
This behavior is a consequence of the oscillations induced by the linear group after the turning point $|y| \geq \frac{2}{|b|}$. Nevertheless, in the ball $|y| < \frac{2}{|b|}$, the operator $-\Delta + 1 - \frac{b^2|y|^2}{4}$ is coercive, and no oscillations will take place in this zone.

Because we track a log-log correction to the self similar law as an upper bound on the blow up speed, profiles $\overline{Q}_b = e^{-i\frac{b}{4}|y|^2}\overline{P}_b$ are natural candidates as refinements of the Q profile in the geometrical decomposition (24). Nevertheless, as they are not in L^2, we need to build a smooth localized version avoiding the non L^2 tail; according to the above discussion, this may be done in the coercive zone $|y| < \frac{2}{|b|}$.

Proposition 3 (Localized self similar profiles) *There exist universal constants $C > 0$, $\eta^* > 0$ such that the following holds true. For all $0 < \eta < \eta^*$, there exist constants $\nu^*(\eta) > 0$, $b^*(\eta) > 0$ going to zero as $\eta \to 0$ such that for all $|b| < b^*(\eta)$, let*
$$R_b = \frac{2}{|b|}\sqrt{1-\eta}, \quad R_b^- = \sqrt{1-\eta}R_b,$$

$B_{R_b} = \{y \in \mathbf{R}^N, \quad |y| \leq R_b\}$. *Then there exists a unique radial solution* Q_b *to*

$$\begin{cases} \Delta Q_b - Q_b + ib\left(\frac{N}{2}Q_b + y\cdot\nabla Q_b\right) + Q_b|Q_b|^{\frac{4}{N}} = 0, \\ P_b = Q_b e^{i\frac{b|y|^2}{4}} > 0 \quad in \quad B_{R_b}, \\ Q_b(0) \in (Q(0) - \nu^*(\eta), Q(0) + \nu^*(\eta)), \quad Q_b(R_b) = 0. \end{cases}$$

Moreover, introduce a smooth radially symmetric cut-off function $\phi_b(x) = 0$ *for* $|x| \geq R_b$ *and* $\phi_b(x) = 1$ *for* $|x| \leq R_b^-$, $0 \leq \phi_b(x) \leq 1$ *and set*

$$\tilde{Q}_b(r) = Q_b(r)\phi_b(r),$$

then

$$\tilde{Q}_b \to Q \quad as \quad b \to 0$$

in some very strong sense, and $\tilde{Q}_b$ *satisfies*

$$\Delta\tilde{Q}_b - \tilde{Q}_b + ib(\tilde{Q}_b)_1 + \tilde{Q}_b|\tilde{Q}_b|^{\frac{4}{N}} = -\Psi_b$$

with

$$Supp(\Psi_b) \subset \{R_b^- \leq |y| \leq R_b\} \quad and \quad |\Psi_b|_{\mathcal{C}^1} \leq e^{-\frac{C}{|b|}}.$$

The meaning of this proposition is that one can build localized profiles $\tilde{Q}_b$ on the ball B_{R_b} which are a smooth function of b and approximate Q in a very strong sense as $b \to 0$, and these profiles satisfy the self similar equation up to an exponentially small term Ψ_b supported around the turning point $\frac{2}{b}$. The proof of this Proposition uses standard variational tools in the setting of nonlinear elliptic problems. A similar statement is also to be found in [31].

Now one can think of making a formal expansion of $\tilde{Q}_b$ in terms of b, and the first term is non zero:

$$\frac{\partial\tilde{Q}_b}{\partial b}_{|b=0} = -\frac{i}{4}|y|^2 Q.$$

A fundamental degeneracy property is now that the energy of $\tilde{Q}_b$ is degenerated in b at all orders:

$$|E(\tilde{Q}_b)| \leq e^{-\frac{C}{|b|}}, \tag{59}$$

for some universal constant $C > 0$.

The existence of a one parameter family of profiles satisfying the self similar equation up to an exponentially small term and having an exponentially small energy is an algebraic property of the structure of (1) around Q *at the heart of the existence of the log-log regime.*

4.3 Modulation theory

We now are in position to exhibit the sharp decomposition needed for the proof of the log-log upper bound. From Lemma 1 and the proximity of $\tilde{Q}_b$ to Q in H^1, the solution $u(t)$ to (1) is for all time close to the four dimensional manifold

$$\mathcal{M} = \{e^{i\gamma}\lambda^{\frac{N}{2}}\tilde{Q}_b(\lambda y + x), \quad (\lambda, \gamma, x, b) \in \mathbf{R}_+^* \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}\}.$$

We now sharpen the decomposition according to the following Lemma.

Lemma 2 (Nonlinear modulation of the solution close to $\mathcal{M}$) *There exist $\mathcal{C}^1$ functions of time $(\lambda, \gamma, x, b) : [0, T) \to (0, +\infty) \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}$ such that:*

$$\forall t \in [0, T), \quad \varepsilon(t, y) = e^{i\gamma(t)}\lambda^{\frac{N}{2}}(t)u(t, \lambda(t)y + x(t)) - \tilde{Q}_{b(t)}(y) \tag{60}$$

satisfies:
(i)

$$\left(\varepsilon_1(t), (\Sigma_{b(t)})_1\right) + \left(\varepsilon_2(t), (\Theta_{b(t)})_1\right) = 0, \tag{61}$$

$$\left(\varepsilon_1(t), y\Sigma_{b(t)}\right) + \left(\varepsilon_2(t), y\Theta_{b(t)}\right) = 0, \tag{62}$$

$$-\left(\varepsilon_1(t), (\Theta_{b(t)})_2\right) + \left(\varepsilon_2(t), (\Sigma_{b(t)})_2\right) = 0, \tag{63}$$

$$-\left(\varepsilon_1(t), (\Theta_{b(t)})_1\right) + \left(\varepsilon_2(t), (\Sigma_{b(t)})_1\right) = 0, \tag{64}$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $\tilde{Q}_b = \Sigma_b + i\Theta_b$ in terms of real and imaginary parts;

$$\textit{(ii)} \ \ |1 - \lambda(t)\frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}}| + |\varepsilon(t)|_{H^1} + |b(t)| \le \delta(\alpha^*) \quad \textit{with} \quad \delta(\alpha^*) \to 0 \quad \textit{as} \quad \alpha^* \to 0.$$

Let us insist onto the fact that the reason for this precise choice of orthogonality conditions is a fundamental issue which will be addressed in the next section.

Proof of Lemma 2

This Lemma follows the standard frame of modulation theory and is obtained from Lemma 1 using the implicit function theorem.
From Lemma 1, there exist parameters $\gamma_0(t) \in \mathbf{R}$ and $x_0(t) \in \mathbf{R}^N$ such that with $\lambda_0(t) = \frac{|\nabla Q|_{L^2}}{|\nabla u(t)|_{L^2}}$,

$$\forall t \in [0, T), \quad \left|Q - e^{i\gamma_0(t)}\lambda_0(t)^{\frac{N}{2}}u(\lambda_0(t)x + x_0(t))\right|_{H^1} < \delta(\alpha^*)$$

with $\delta(\alpha^*) \to 0$ as $\alpha^* \to 0$. Now we sharpen this decomposition using the fact that $\tilde{Q}_b \to Q$ in H^1 as $b \to 0$, i.e. we chose $(\lambda(t), \gamma(t), x(t), b(t))$ close to $(\lambda_0(t), \gamma_0(t), x_0(t), 0)$ such that

$$\varepsilon(t, y) = e^{i\gamma(t)}\lambda^{1/2}(t)u(t, \lambda(t)y + x(t)) - \tilde{Q}_{b(t)}(y)$$

is small in H^1 and satisfies suitable orthogonality conditions (61), (62), (63) and (64). The existence of such a decomposition is a consequence of the implicit function Theorem. For $\delta>0$, let $V_\delta=\{v\in H^1(\mathbf{C});\quad |v-Q|_{H^1}\leq\delta\}$, and for $v\in H^1(\mathbf{C})$, $\lambda_1>0$, $\gamma_1\in\mathbf{R}$, $x_1\in\mathbf{R}^N$, $b\in\mathbf{R}$ small, define

$$\varepsilon_{\lambda_1,\gamma_1,x_1,b}(y)=e^{i\gamma_1}\lambda_1^{\frac{N}{2}}v(\lambda_1 y+x_1)-\tilde{Q}_b. \tag{65}$$

We claim that there exists $\overline{\delta}>0$ and a unique C^1 map : $V_{\overline{\delta}}\to(1-\overline{\lambda},1+\overline{\lambda})\times(-\overline{\gamma},\overline{\gamma})\times B(0,\overline{x})\times(-\overline{b},\overline{b})$ such that if $v\in V_{\overline{\delta}}$, there is a unique $(\lambda_1,\gamma_1,x_1,b)$ such that $\varepsilon_{\lambda_1,\gamma_1,x_1,b}=(\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1+i(\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2$ defined as in (65) satisfies

$$\rho^1(v)=((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1,(\Sigma_b)_1)+((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2,(\Theta_b)_1)=0,$$

$$\rho^2(v)=((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1,y\Sigma_b)+((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2,y\Theta_b)=0,$$

$$\rho^3(v)=-((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1,(\Theta_b)_2)+((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2,(\Sigma_b)_2)=0,$$

$$\rho^4(v)=((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_1,(\Theta_b)_1)-((\varepsilon_{\lambda_1,\gamma_1,x_1,b})_2,(\Sigma_b)_1)=0.$$

Moreover, there exists a constant $C_1>0$ such that if $v\in V_{\overline{\delta}}$, then $|\varepsilon_{\lambda_1,\gamma_1,x_1}|_{H^1}+|\lambda_1-1|+|\gamma_1|+|x_1|+|b|\leq C_1\overline{\delta}$. Indeed, we view the above functionals $\rho^1,\rho^2,\rho^3,\rho^4$ as functions of $(\lambda_1,\gamma_1,x_1,b,v)$. We first compute at $(\lambda_1,\gamma_1,x_1,b,v)=(1,0,0,0,v)$:

$$\frac{\partial\varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial x_1}=\nabla v,\;\;\frac{\partial\varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial\lambda_1}=\frac{N}{2}v+x\cdot\nabla v,\;\;\frac{\partial\varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial\gamma_1}=iv,\;\;\frac{\partial\varepsilon_{\lambda_1,\gamma_1,x_1,b}}{\partial b}=-\left(\frac{\partial\tilde{Q}_b}{\partial b}\right)_{|b=0}.$$

Now recall that $(\tilde{Q}_b)_{|b=0}=Q$ and $\left(\frac{\partial\tilde{Q}_b}{\partial b}\right)_{|b=0}=-i\frac{|y|^2}{4}Q$. Therefore, we obtain at the point $(\lambda_1,\gamma_1,x_1,b,v)=(1,0,0,0,Q)$,

$$\begin{aligned}
&\frac{\partial\rho^1}{\partial\lambda_1}=|Q_1|_2^2,\quad\frac{\partial\rho^1}{\partial\gamma_1}=0,\quad\frac{\partial\rho^1}{\partial x_1}=0,\frac{\partial\rho^1}{\partial b}=0,\\
&\frac{\partial\rho^2}{\partial\lambda_1}=0,\quad\frac{\partial\rho^2}{\partial\gamma_1}=0,\quad\frac{\partial\rho^2}{\partial x_1}=-\frac{1}{2}|Q|_2^2,\frac{\partial\rho^2}{\partial b}=0,\\
&\frac{\partial\rho^3}{\partial\lambda_1}=0,\quad\frac{\partial\rho^3}{\partial\gamma_1}=-|Q_1|_2^2,\quad\frac{\partial\rho^3}{\partial x_1}=0,\frac{\partial\rho^3}{\partial b}=0,\\
&\frac{\partial\rho^4}{\partial\lambda_1}=0,\quad\frac{\partial\rho^4}{\partial\gamma_1}=0,\quad\frac{\partial\rho^4}{\partial x_1}=0,\frac{\partial\rho^4}{\partial b}=\frac{1}{4}|yQ|_2^2.
\end{aligned}$$

The Jacobian of the above functional is non zero, thus the Implicit Function Theorem applies and conclusion follows. This concludes the proof of Lemma 2.

Let us now write down the equation satisfied by ε in rescaled variables. To simplify notations, we note

$$\tilde{Q}_b=\Sigma+\Theta$$

in terms of real and imaginary parts. We have: $\forall s \in \mathbf{R}_+$, $\forall y \in \mathbf{R}^N$,

$$\begin{aligned} b_s\frac{\partial\Sigma}{\partial b}+\partial_s\varepsilon_1-M_-(\varepsilon)+b\left(\frac{N}{2}\varepsilon_1+y\cdot\nabla\varepsilon_1\right) &= \left(\frac{\lambda_s}{\lambda}+b\right)\Sigma_1+\tilde{\gamma}_s\Theta+\frac{x_s}{\lambda}\cdot\nabla\Sigma \\ &+ \left(\frac{\lambda_s}{\lambda}+b\right)\left(\frac{N}{2}\varepsilon_1+y\cdot\nabla\varepsilon_1\right)+\tilde{\gamma}_s\varepsilon_2+\frac{x_s}{\lambda}\cdot\nabla\varepsilon_1 \\ &+ Im(\Psi)-R_2(\varepsilon), \end{aligned} \tag{66}$$

$$\begin{aligned} b_s\frac{\partial\Theta}{\partial b}+\partial_s\varepsilon_2+M_+(\varepsilon)+b\left(\frac{N}{2}\varepsilon_2+y\cdot\nabla\varepsilon_2\right) &= \left(\frac{\lambda_s}{\lambda}+b\right)\Theta_1-\tilde{\gamma}_s\Sigma+\frac{x_s}{\lambda}\cdot\nabla\Theta \\ &+ \left(\frac{\lambda_s}{\lambda}+b\right)\left(\frac{N}{2}\varepsilon_2+y\cdot\nabla\varepsilon_2\right)-\tilde{\gamma}_s\varepsilon_1+\frac{x_s}{\lambda}\cdot\nabla\varepsilon_2 \\ &- Re(\Psi)+R_1(\varepsilon), \end{aligned} \tag{67}$$

with $\tilde{\gamma}(s) = -s-\gamma(s)$. The linear operator close to $\tilde{Q}_b$ is now a deformation of the linear operator L close to Q and writes $M=(M_+,M_-)$ with

$$M_+(\varepsilon) = -\Delta\varepsilon_1+\varepsilon_1-\left(\frac{4\Sigma^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1-\left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_2,$$

$$M_-(\varepsilon) = -\Delta\varepsilon_2+\varepsilon_2-\left(\frac{4\Theta^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_2-\left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_1.$$

Interaction terms are formally quadratic in ε and write:

$$R_1(\varepsilon) = (\varepsilon_1+\Sigma)|\varepsilon+\tilde{Q}_b|^{\frac{4}{N}}-\Sigma|\tilde{Q}_b|^{\frac{4}{N}}-\left(\frac{4\Sigma^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_1-\left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_2,$$

$$R_2(\varepsilon) = (\varepsilon_2+\Theta)|\varepsilon+\tilde{Q}_b|^{\frac{4}{N}}-\Theta|\tilde{Q}_b|^{\frac{4}{N}}-\left(\frac{4\Theta^2}{N|\tilde{Q}_b|^2}+1\right)|\tilde{Q}_b|^{\frac{4}{N}}\varepsilon_2-\left(\frac{4\Sigma\Theta}{N|\tilde{Q}_b|^2}|\tilde{Q}_b|^{\frac{4}{N}}\right)\varepsilon_1.$$

Two natural estimates may now be performed:

- First, we may rewrite the conservation laws in the rescaled variables and linearize obtained identities close to Q. This will give crucial degeneracy estimates on some specific order one in ε scalar products.
- Next, we may inject orthogonality conditions of Lemma 2 into equations (66), (67). This will compute the geometrical parameters in their differential form $\frac{\lambda_s}{\lambda}, \tilde{\gamma}_s, \frac{x_s}{\lambda}, b_s$ in terms of ε: these are the so called modulation equations. This step requires estimating the nonlinear interaction terms. A crucial point here is to use the fact that the ground state Q is exponentially decreasing in space.

The outcome is the following:

Lemma 3 (First estimates on the decomposition) *We have for all $s \geq 0$:*
(i) Estimates induced by the conservation of the energy and the momentum:

$$|(\varepsilon_1, Q)| \leq \delta(\alpha^*) \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}} + C\lambda^2 |E_0|, \tag{68}$$

$$|(\varepsilon_2, \nabla Q)| \leq C\delta(\alpha^*) \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right)^{\frac{1}{2}}. \tag{69}$$

(ii) Estimate on the geometrical parameters in differential form:

$$\left| \frac{\lambda_s}{\lambda} + b \right| + |b_s| + |\tilde{\gamma}_s| \leq C \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-1|y|} \right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}}, \tag{70}$$

$$\left| \frac{x_s}{\lambda} \right| \leq \delta(\alpha^*) \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-1|y|} \right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}}, \tag{71}$$

where $\delta(\alpha^*) \to 0$ as $\alpha^* \to 0$.

Remark 3 *The exponentially small term in degeneracy estimate (68) is in fact related to the value of $E(\tilde{Q}_b)$, so we use here in a fundamental way nonlinear degeneracy estimate (59).*

Here are two fundamental comments on Lemma 3:

- First, the norm which appears in the estimates of Lemma 3 is essentially a local norm in space. The conservation of the energy indeed relates the $\int |\nabla \varepsilon|^2$ norm with the local norm. These two norms will turn out to play an equivalent role in the analysis. A key is that no global L^2 norm is needed so far.
- Comparing estimates (70) and (71), we see that the term induced by translation invariance is smaller than the ones induced by scaling and phase invariances. This non trivial fact is an outcome of our use of the Galilean transform to ensure the zero momentum condition (47).

4.4 The virial type dispersive estimate

Our aim in this subsection is to exhibit the dispersive virial type inequality at the heart of the proof of the log-log upper bound. This information will be obtained as a consequence of the virial structure of (1) in Σ.

Let us first recall that virial identity (19) corresponds to two identities:

$$\frac{d^2}{dt^2} \int |x|^2 |u|^2 = 4 \frac{d}{dt} Im\left(\int x \cdot \nabla u \overline{u}\right) = 16 E_0. \tag{72}$$

We want to understand what information can be extracted from this dispersive information in the variables of the geometrical decomposition.

To clarify the claim, let us consider an ε solution to the linear homogeneous equation

$$i\partial_s\varepsilon + L\varepsilon = 0 \tag{73}$$

where $L = (L_+, L_-)$ is the linearized operator close to Q. A dispersive information on ε may be extracted using a similar virial law as (19):

$$\frac{1}{2}\frac{d}{ds}Im(\int y\cdot\nabla\varepsilon\overline{\varepsilon}) = H(\varepsilon,\varepsilon), \tag{74}$$

where $H(\varepsilon,\varepsilon) = (\mathcal{L}_1\varepsilon_1,\varepsilon_1) + (\mathcal{L}\varepsilon_2,\varepsilon_2)$ is a Schrödinger type quadratic form decoupled in the real and imaginary parts with explicit Schrödinger operators:

$$\mathcal{L}_1 = -\Delta + \frac{2}{N}\left(\frac{4}{N}+1\right)Q^{\frac{4}{N}-1}y\cdot\nabla Q \quad , \quad \mathcal{L}_2 = -\Delta + \frac{2}{N}Q^{\frac{4}{N}-1}y\cdot\nabla Q.$$

Note that both these operators are of the form $-\Delta + V$ for some smooth well localized time independent potential $V(y)$, and thus from standard spectral theory, they both have a finite number of negative eigenvalues, and then continuous spectrum on $[0,+\infty)$. A simple outcome is then that given an $\varepsilon \in H^1$ which is orthogonal to all the bound states of $\mathcal{L}_1, \mathcal{L}_2$, then $H(\varepsilon,\varepsilon)$ is coercive, that is

$$H(\varepsilon,\varepsilon) \geq \delta_0\left(\int|\nabla\varepsilon|^2 + \int|\varepsilon|^2e^{-|y|}\right)$$

for some universal constant $\delta_0 > 0$. Now assume that for some reason -it will be in our case a consequence of modulation theory and the conservation laws-, ε is indeed for all times orthogonal to the bound states, then injecting the coercive control of $H(\varepsilon,\varepsilon)$ into (74) yields:

$$\frac{1}{2}\frac{d}{ds}Im(\int y\cdot\nabla\varepsilon\overline{\varepsilon}) \geq \delta_0\left(\int|\nabla\varepsilon|^2 + \int|\varepsilon|^2e^{-|y|}\right). \tag{75}$$

Integrating this in time yields a standard dispersive information: a space time norm is controlled by a norm in space.

We want to apply this strategy to the full ε equation. There are two main obstructions.

First, it is not reasonable to assume that ε is orthogonal to the exact bound states of H. In particular, due to the right hand side in the ε equation, other second order terms will appear which will need be controlled. We thus have to exhibit a set of orthogonality conditions which ensures both the coercivity of the quadratic form H and the control of these other second order interactions. Note that the number of orthogonality conditions we can ensure on ε is the number of symmetries plus the one from b. A first key is the following Spectral Property which is precisely the property which has been proved in dimension $N=1$ in [20] using the explicit value of Q and checked numerically for $N=2,3,4$.

Proposition 4 (Spectral Property) *Let $N=1,2,3,4$. There exists a universal constant $\delta_0>0$ such that $\forall\varepsilon\in H^1$,*

$$\begin{aligned} H(\varepsilon,\varepsilon) &\geq \delta_0\left(\int|\nabla\varepsilon|^2+\int|\varepsilon|^2e^{-|y|}\right)-\frac{1}{\delta_0}\Big\{(\varepsilon_1,Q)^2+(\varepsilon_1,Q_1)^2+(\varepsilon_1,yQ)^2 \\ &+ (\varepsilon_2,Q_1)^2+(\varepsilon_2,Q_2)^2+(\varepsilon_2,\nabla Q)^2\Big\}. \end{aligned} \tag{76}$$

To prove this property amounts first counting exactly the number of negative eigenvalues of each Schrödinger operator, and then prove that the specific chosen set of orthogonality conditions, which is not exactly the set of the bound states, is enough to ensure the coercivity of the quadratic form. Both these issues appear to be non-trivial when Q is not explicit.

Then, the second major obstruction is the fact that the right hand side $Im(\int y\cdot\nabla\varepsilon\overline{\varepsilon})$ is an unbounded function of ε in H^1. This is a priori a major obstruction to the strategy, *but an additional nonlinear algebra inherited from virial law (19) rules out this difficulty.*

The formal computation is as follows. Given a function $f\in\Sigma$, we let $\Phi(f)=Im(\int y\cdot\nabla f\overline{f})$. According to (74), we want to compute $\frac{d}{ds}\Phi(\varepsilon)$. Now from (72) and the conservation of the energy:

$$\forall t\in[0,T),\quad \Phi(u(t))=4E_0t+c_0$$

for some constant c_0. The key observation is that the quantity $\Phi(u)$ is scaling, phase and also translation invariant from zero momentum assumption (47). From geometrical decomposition (60), we get:

$$\forall t\in[0,T),\quad \Phi(\varepsilon+\tilde{Q}_b)=4E_0t+c_0.$$

We now expand this according to:

$$\Phi(\varepsilon+\tilde{Q}_b)=\Phi(\tilde{Q}_b)-2(\varepsilon_2,\Sigma_1)+2(\varepsilon_1,\Theta_1)+\Phi(\varepsilon).$$

From explicit computation,

$$\Phi(\tilde{Q}_b)=-\frac{b}{2}|y\tilde{Q}_b|_2^2\sim -Cb$$

for some universal constant $C>0$. Next, from explicit choice of orthogonality condition (64),

$$(\varepsilon_2,\Sigma_1)-(\varepsilon_1,\Theta_1)=0.$$

We thus get using $\frac{dt}{ds} = \lambda^2$:

$$(\Phi(\varepsilon))_s \sim 4\lambda^2 E_0 + Cb_s.$$

In other words, to compute the a priori unbounded quantity $(\Phi(\varepsilon))_s$ *for the full nonlinear equation* is from the virial law equivalent to computing the time derivative of b_s, what of course makes now perfectly sense in H^1.

The virial dispersive structure on $u(t)$ in Σ thus induces a dispersive structure in $L^2_{loc} \cap \dot{H}^1$ on $\varepsilon(s)$ for the full nonlinear equation.

The key dispersive virial estimate is now the following.

Proposition 5 (Local virial estimate in ε) *There exist universal constants $\delta_0 > 0$, $C > 0$ such that for all $s \geq 0$, there holds:*

$$b_s \geq \delta_0 \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - \lambda^2 E_0 - e^{-\frac{C}{|b|}}. \tag{77}$$

Proof of Proposition 5

Using the heuristics, we can compute in a suitable way b_s using the orthogonality condition (64). The computation -see Lemma 5 in [21]- yields:

$$\begin{aligned} & \frac{1}{4}|yQ|_2^2 b_s = H(\varepsilon,\varepsilon) + 2\lambda^2|E_0| - \frac{x_s}{\lambda}\{(\varepsilon_2,(\Sigma_1)_y) - (\varepsilon_1,(\Theta_1)_y)\} \qquad (78) \\ - \quad & \left(\frac{\lambda_s}{\lambda} + b\right)\{(\varepsilon_2,\Sigma_2) - (\varepsilon_1,\Theta_2)\} - \tilde{\gamma}_s\{(\varepsilon_1,\Sigma_1) + (\varepsilon_2,\Theta_1)\} \\ - \quad & (\varepsilon_1, Re(\Psi)_1) - (\varepsilon_2, Im(\Psi)_1) + (l.o.t), \end{aligned}$$

where the lower order terms may be estimated from the smallness of ε in H^1:

$$|l.o.t| \leq \delta(\alpha^*) \left(\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right).$$

We now explain how the choice of orthogonality conditions and the conservation laws allow us to deduce (77).

step 1 Modulation theory for phase and scaling.

The choice of orthogonality conditions (63), (61) has been made to cancel the two second order in ε scalar products in (78):

$$\left(\frac{\lambda_s}{\lambda} + b\right)\{(\varepsilon_2,\Sigma_2) - (\varepsilon_1,\Theta_2)\} + \tilde{\gamma}_s\{(\varepsilon_1,\Sigma_1) + (\varepsilon_2,\Theta_1)\} = 0.$$

step 2 Elliptic estimate on the quadratic form H.

We now need to control the negative directions in the quadratic form as given by Proposition 4. Directions (ε_1, Q_1), (ε_1, yQ), (ε_2, Q_2) and (ε_2, Q_1) are treated thanks to the choice of orthogonality conditions and the closeness of $\tilde{Q}_b$ to Q for $|b|$ small. For example,

$$\begin{aligned}(\varepsilon_2, Q_1)^2 &= |\{(\varepsilon_2, Q_1 - \Sigma_1) + (\varepsilon_1, \Theta_1)\} + (\varepsilon_2, \Sigma_1) - (\varepsilon_1, \Theta_1)|^2 \\ &= |(\varepsilon_2, Q_1 - \Sigma_1) + (\varepsilon_1, \Theta_1)|^2\end{aligned}$$

so that

$$(\varepsilon_2, Q_1)^2 \leq \delta(\alpha^*)(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}).$$

Similarly, we have:

$$(\varepsilon_1, yQ)^2 + (\varepsilon_2, Q_2)^2 + (\varepsilon_1, Q_1)^2 \leq \delta(\alpha^*)(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}). \tag{79}$$

The negative direction $(\varepsilon_1, Q)^2$ is treated from the conservation of the energy which implied (68). The direction $(\varepsilon_2, \nabla Q)$ is treated from the zero momentum condition which ensured (69). Putting this together yields:

$$(\varepsilon_1, Q)^2 + (\varepsilon_2, \nabla Q)^2 \leq \delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} + \lambda^2|E_0|\right) + e^{-\frac{C}{|b|}}.$$

step 3 Modulation theory for translation and use of Galilean invariance.

Galilean invariance has been used to ensure zero momentum condition (47) which in turn led together with the choice of orthogonality condition (62) to degeneracy estimate (71):

$$|\frac{x_s}{\lambda}| \leq C\delta(\alpha^*)(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-1|y|})^{\frac{1}{2}} + e^{-\frac{C}{|b|}}.$$

Therefore, we estimate the term induced by translation invariance in (78) as

$$\left|\frac{x_s}{\lambda}\{(\varepsilon_2, (\Sigma_1)_y) - (\varepsilon_1, (\Theta_1)_y)\}\right| \leq C\delta(\alpha^*)\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) + e^{-\frac{C}{|b|}}.$$

step 4 Conclusion.

Injecting these estimates into the elliptic estimate (76) yields so far:

$$b_s \geq \delta_0\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|}\right) - 2\lambda^2 E_0 - e^{-\frac{C}{|b|}} - \frac{1}{\delta_0}(\lambda^2 E_0)^2.$$

We now use in a crucial way *the sign of the energy* $E_0 < 0$ and the smallness $\lambda^2|E_0| \leq \delta(\alpha^*)$ which is a consequence of the conservation of the energy to conclude. This ends the proof of Proposition 5.

4.5 Monotonicity and control of the blow up speed

Virial dispersive estimate (77) means a control of the excess of mass ε by an exponentially small correction in b in time averaging sense. More specifically, this means that in rescaled variables, the solution writes $\tilde{Q}_b + \varepsilon$ where $\tilde{Q}_b$ is the regular deformation of Q and the rest is in a suitable norm exponentially small in b. This is thus an expansion of the solution with respect to an internal parameter in the problem, b.

This virial control is the first dispersive estimate for the infinite dimensional dynamic driving ε. Observe that it means little by itself if nothing is known about $b(t)$. We shall now inject this information into the finite dimensional dynamic driving the geometrical parameters. The outcome will be *a rigidity property for the parameter $b(t)$ which will in turn imply the existence of a Lyapounov functional in the problem.* This step will again heavily rely on the conservation of the energy. This monotonicity type of results will then allow us to conclude.

We start with exhibiting the rigidity property which is proven using a maximum principle type of argument.

Proposition 6 (Rigidity property for b**)** *$b(s)$ vanishes at most once on $\mathbf{R}_+$.*

Note that the existence of a quantity with prescribed sign in the description of the dynamic is unexpected. Indeed, b is no more then the projection of some a priori highly oscillatory function onto a prescribed direction. It is a very specific feature of the blow up dynamic that this projection has a fixed sign.

Proof of Proposition 6

Assume that there exists some time $s_1 \geq 0$ such that $b(s_1) = 0$ and $b_s(s_1) \leq 0$, then from (77), $\varepsilon(s_1) = 0$. Thus from the conservation of the L^2 norm and $\tilde{Q}_b(s_2) = Q$, we conclude $\int |u_0|^2 = \int Q^2$ which contradicts the strictly negative energy assumption. This concludes the proof of Proposition 6.

The next step is to get the exact sign of b. This is done by injecting virial dispersive information (77) into the modulation equation for the scaling parameter which will yield

$$-\frac{\lambda_s}{\lambda} \sim b. \tag{80}$$

The fundamental monotonicity result is then the following.

Proposition 7 (Existence of an almost Lyapounov functional) *There exists a time $s_0 \geq 0$ such that*

$$\forall s > s_0, \quad b(s) > 0.$$

Moreover, the size of the solution is in this regime an almost Lyapounov functional in the sense that:

$$\forall s_2 \geq s_1 \geq s_0, \quad \lambda(s_2) \leq 2\lambda(s_1). \tag{81}$$

Proof of Proposition 7

step 1 Equation for the scaling parameter.

The modulation equation for the scaling parameter λ inherited from choice of orthogonality condition (61) implied control (70):

$$\left|\frac{\lambda_s}{\lambda} + b\right| \leq C\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-1|y|}\right)^{\frac{1}{2}} + e^{-\frac{C}{|b|}},$$

which implies (80) in a weak sense. Nevertheless, this estimate is not good enough to possibly use the virial estimate (77). We claim using extra degeneracies of the equation that (70) can be improved for:

$$\left|\frac{\lambda_s}{\lambda} + b\right| \leq C\left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 e^{-1|y|}\right) + e^{-\frac{C}{|b|}} \tag{82}$$

step 2 Use of the virial dispersive relation and the rigidity property.

We now inject virial dispersive relation (77) into (82) to get:

$$\left|\frac{\lambda_s}{\lambda} + b\right| \leq Cb_s + e^{-\frac{C}{|b|}}.$$

We integrate this inequality in time to get: $\forall 0 \leq s_1 \leq s_2$,

$$\left|\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) + \int_{s_1}^{s_2} b(s)ds\right| \leq \frac{1}{4} + \int_{s_1}^{s_2} e^{-\frac{C}{|b(s)|}}ds. \tag{83}$$

The key is now to use rigidity property of Proposition 6 to ensure that $b(s)$ has a fixed sign for $s \geq \tilde{s}_0$, and thus: $\forall s \geq \tilde{s}_0$,

$$\left|\int_{s_1}^{s_2} e^{-\frac{C}{|b(s)|}}ds\right| \leq \frac{1}{2}\left|\int_{s_1}^{s_2} b(s)ds\right|. \tag{84}$$

step 3 b is positive for s large enough.

Assume that $\left|\int_0^{+\infty} b(s)ds\right| < +\infty$, then b has a fixed sign for $s \geq \tilde{s}_0$ and $|b_s| \leq C$ is straightforward from the equation, so that we conclude: $b(s) \to 0$ as $s \to +\infty$. Now from (83) and (84), this implies that $|\log(\lambda(s))| \leq C$ as $s \to +\infty$, and in particular

$\lambda(s) \geq \lambda_0 > 0$ for s large enough. Injecting this into virial control (77) for s large enough yields:

$$b_s \geq \frac{1}{2}|E_0|\lambda_0^2.$$

Integrating this on large time intervals contradicts the uniform boundedness of b. Here we have used again assumption $E_0 < 0$.
We thus have proved: $\left|\int_0^{+\infty} b(s)ds\right| = +\infty$. Now assume that $b(s) < 0$ for all $s \geq \tilde{s}_1$, then from (83) and (84) again, we conclude that $\log(\lambda(s)) \to 0$ as $s \to +\infty$. Now from $\lambda(t) \sim \frac{1}{|\nabla u(t)|_{L^2}}$, this yields $|\nabla u(t)|_{L^2} \to 0$ as $t \to T$. But from Gagliardo-Nirenberg inequality and the conservation of the energy and the L^2 mass, this implies $E_0 = 0$, contradicting again the assumption $E_0 < 0$.

step 4 Almost monotonicity of the norm.

We now are in position to prove (81). Indeed, injecting the sign of b into (83) and (84) yields in particular: $\forall s_0 \leq s_1 \leq s_2$,

$$\frac{1}{4} + \frac{1}{2}\int_{s_1}^{s_2} b(s)ds \leq -\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) \leq \frac{1}{4} + 2\int_{s_1}^{s_2} b(s)ds, \tag{85}$$

and thus:

$$\forall s_0 \leq s_1 \leq s_2, \quad -\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) \geq \frac{1}{4},$$

what yields (81). This concludes the proof of Proposition 7.

Note that from the above proof, we have obtained $\int_0^{+\infty} b(s)ds = +\infty$, and thus from (85):

$$\lambda(s) \to 0 \quad \text{as} \quad s \to \infty, \tag{86}$$

that is finite or infinite time blow up. *In contrast with the virial argument, the blow up proof is no longer obstructive but completely dynamical, and relies mostly on the rigidity property of Proposition 6.*

Let us conclude these notes by finishing the proof of Theorem 7. We need to prove finite time blow up together with the log-log upper bound (33) on blow up rate. The proof goes as follows.

step 1 Lower bound on $b(s)$.

We claim: there exist some universal constant $C > 0$ and some time $s_1 > 0$ such that $\forall s \geq s_1$,

$$Cb(s) \geq \frac{1}{\log|\log(\lambda(s))|}. \tag{87}$$

Indeed, first recall (77). Now that we know the sign of $b(s)$ for $s \geq s_0$, we may view this inequality as a differential inequality for b for $s > s_0$:

$$b_s \geq -e^{-\frac{C}{b}} \geq -b^2 e^{-\frac{C}{2b}} \quad \text{ie} \quad -\frac{b_s}{b^2} e^{\frac{C}{2b}} \leq 1.$$

We integrate this inequality from the non vanishing property of b and get for $s \geq \tilde{s}_1$ large enough:

$$e^{\frac{C}{b(s)}} \leq s + e^{\frac{C}{b(1)}} \leq 2Cs \quad \text{ie} \quad b(s) \geq \frac{C}{\log(s)}. \tag{88}$$

We now recall (85) on the time interval $[\tilde{s}_1, s]$:

$$\frac{1}{2}\int_{\tilde{s}_1}^{s} b \leq -\log(\frac{\lambda(s)}{\lambda(\tilde{s}_1)}) + \frac{1}{4} \leq -2\log(\lambda(s))$$

for $s \geq \tilde{s}_2$ large enough from $\lambda(s) \to 0$ as $s \to +\infty$. Inject (88) into the above inequality, we get for $s \geq \tilde{s}_3$

$$C\frac{s}{\log(s)} \leq \int_{\tilde{s}_2}^{s} \frac{Cd\tau}{\log(\tau)} \leq \frac{1}{4}\int_{\tilde{s}_2}^{s} b \leq -\log(\lambda(s)) \quad \text{ie} \quad |\log(\lambda(s))| \geq C\frac{s}{\log(s)}$$

for some universal constant $C > 0$, and thus for s large

$$\log|\log(\lambda(s))| \geq \log(s) - \log(\log(s)) \geq \frac{1}{2}\log(s)$$

and conclusion follows from (88). This concludes the proof of (87).

step 2 Finite time blow up and control of the blow up speed.

We first use the finite or infinite time blow up result (86) to consider a sequence of times $t_n \to T \in [0, +\infty]$ defined for n large such that

$$\lambda(t_n) = 2^{-n}.$$

Let $s_n = s(t_n)$ the corresponding sequence and $\bar{t}$ such that $s(\bar{t}) = s_0$ given by Proposition 7. Note that we may assume $n \geq \overline{n}$ such that $t_n \geq \bar{t}$. Remark that $0 < t_n < t_{n+1}$ from (81), and so $0 < s_n < s_{n+1}$. Moreover, there holds from (81)

$$\forall s \in [s_n, s_{n+1}], \quad 2^{-n-1} \leq \lambda(s) \leq 2^{-(n-1)}. \tag{89}$$

We now claim that (33) follows from a control from above of the size of the intervals $[t_n, t_{n+1}]$ for $n \geq \overline{n}$.
Let $n \geq \overline{n}$. (87) implies

$$\int_{s_n}^{s_{n+1}} \frac{ds}{\log|\log(\lambda(s))|} \leq C\int_{s_n}^{s_{n+1}} b(s)ds.$$

(85) with $s_1 = s_n$ and $s_2 = s_{n+1}$ yields:

$$\frac{1}{2}\int_{s_n}^{s_{n+1}} b(s) \le \frac{1}{4} - |yQ|_{L^2}^2 \log(\frac{\lambda(s_{n+1})}{\lambda(s_n)}) \le C.$$

Therefore,

$$\forall n \ge \overline{n}, \quad \int_{s_n}^{s_{n+1}} \frac{ds}{\log|\log(\lambda(s))|} \le C.$$

Now we change variables in the integral at the left of the above inequality according to $\frac{ds}{dt} = \frac{1}{\lambda^2(s)}$ and estimate with (89):

$$C \ge \int_{s_n}^{s_{n+1}} \frac{ds}{\log|\log(\lambda(s))|} = \int_{t_n}^{t_{n+1}} \frac{dt}{\lambda^2(t)\log|\log(\lambda(t))|} \ge \frac{1}{10\lambda^2(t_n)\log|\log(\lambda(t_n))|}\int_{t_n}^{t_{n+1}} dt$$

so that

$$t_{n+1} - t_n \le C\lambda^2(t_n)\log|\log(\lambda(t_n))|.$$

From $\lambda(t_n) = 2^{-n}$ and summing the above inequality in n, we first get

$$T < +\infty$$

and

$$\begin{aligned}
C(T - t_n) &\le \sum_{k\ge n} 2^{-2k}\log(k) = \sum_{n\le k\le 2n} 2^{-2k}\log(k) + \sum_{k\ge 2n} 2^{-2k}\log(k)\\
&\le C2^{-2n}\log(n) + 2^{-4n}\log(2n)\sum_{k\ge 0} 2^{-2k}\frac{\log(2n+k)}{\log(2n)}\\
&\le C2^{-2n}\log(n) + C2^{-4n}\log(n) \le C2^{-2n}\log(n) \le C\lambda^2(t_n)\log|\log(\lambda(t_n))|.
\end{aligned}$$

From the monotonicity of λ (81), we extend the above control to the whole sequence $t \ge \overline{t}$. Let $t \ge \overline{t}$, then $t \in [t_n, t_{n+1}]$ for some $n \ge \overline{n}$, and from $\frac{1}{2}\lambda(t_n) \le \lambda(t) \le 2\lambda(t_n)$, we conclude

$$\lambda^2(t)\log|\log(\lambda(t))| \ge C\lambda^2(t_n)\log|\log(\lambda(t_n))| \ge C(T - t_n) \ge C(T - t).$$

Now remark that the function $f(x) = x^2\log|\log(x)|$ is non decreasing in a neighborhood at the right of $x = 0$, and moreover

$$f\left(\frac{C}{2}\sqrt{\frac{T-t}{\log|\log(T-t)|}}\right) = \frac{C^2}{4}\frac{(T-t)}{\log|\log(T-t)|}\log\left|\log\left(C\sqrt{\frac{T-t}{\log|\log(T-t)|}}\right)\right| \le C(T-t)$$

for t close enough to T, so that we get for some universal constant C^*:

$$f(\lambda(t)) \ge f\left(C^*\sqrt{\frac{T-t}{\log|\log(T-t)|}}\right) \quad \text{ie} \quad \lambda(t) \ge C^*\sqrt{\frac{T-t}{\log|\log(T-t)|}}$$

and (33) is proved.

This concludes the proof of Theorem 7.

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