Department of Mathematics
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Topology Comprehensive Exam

September 6, 2019, 10-1

Topology I

Problem 1. Let M be a smooth compact manifold, and let $f : M \to \mathbb{R}$ be a smooth function with no critical values in $[0, 1]$. Prove that $f^{-1}(0)$ and $f^{-1}(1)$ are diffeomorphic.

Problem 2. Let $g : M \to M$ be a smooth map from a smooth compact manifold to itself. Prove that there must be a point $y \in M$ with $g^{-1}(y)$ finite.

Problem 3. Let Σ be a compact oriented surface without boundary, let U, V be disjoint open neighbourhoods of points $p, q \in \Sigma$ respectively, and let $\varphi : U \to D(0, 1)$ and $\psi : V \to D(0, 1)$ be orientation-preserving diffeomorphisms to the open unit disc $D(0, 1) \subset \mathbb{R}^2$ sending p and q to the origin.

(1) Determine the rank of the pullback map on de Rham cohomology groups $H^1(\Sigma\backslash\{p\}) \to H^1(U\backslash\{p\})$.
(2) Determine the rank of the following pullback map on de Rham cohomology groups.

$$H^1(\Sigma\backslash\{p, q\}) \to H^1((U \cup V)\backslash\{p, q\})$$

(3) Determine the de Rham cohomology groups of the following surface in terms of the cohomology of Σ:

$$\widetilde{\Sigma} = ((\Sigma\backslash\{p, q\}) \sqcup C)/\sim,$$

where C is the cylinder $(-1, 1) \times S^1$ and the equivalence relation $\sim$ is defined as follows: a point $x \in U \setminus \{p\}$ is equivalent to $(-r, -\theta) \in C$ and a point $y \in V \setminus \{q\}$ is equivalent to $(s, \phi) \in C$, where (r, θ) and (s, ϕ) are the polar coordinates of $\varphi(x)$ and $\psi(y)$, respectively.

Topology II

Problem 4. Find the fundamental group of the Klein bottle with two points removed.

Problem 5. Let $f\colon \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ be a continuous map where $n \geq 1$. Prove that f has a fixed point.

Hint: Look at the induced map $\tilde{f}\colon S^{2n} \to S^{2n}$ and show that there is x such $\tilde{f}(x) = \pm x$.

Problem 6. Let X be obtained from $S^2 \times S^2$ by identifying $S^2 \times \{pt\}$ to a point.

Find $H^*(X, \mathbb{Z})$ including the ring structure.