DEPARTMENT OF MATHEMATICS
University of Toronto

Real Analysis Comprehensive Exam

2 hours

September, 4 2019

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Make sure to justify all your work. If you make a reference to a result in the textbook, please make sure to carefully quote it (correctly!).

Problem 1

Each of the following questions have equal weight can be solved independently

(a) Let $f_n(x) = \sin(2\pi nx)$ for $n \in \mathbb{N}$. Show that the sequence $\{f_n\}_{n=1}^{\infty}$ has no subsequence which converges Lebesgue-a.e. on $[0, 1]$.

(b) Let μ and ν be two finite positive measures on a measurable space $(X, \mathcal{A})$ so that $\mu \ll \nu$ and $\nu \ll \mu$. Let $\lambda = \mu + \nu$; show that the Radon–Nikodym derivative $d\nu/d\lambda$ satisfies a.e. the following bound:

$$0 < \frac{d\nu}{d\lambda} < 1.$$

Problem 2

(a) State the definition of the space $L^p(\mathbb{R})$ for $1 \le p < \infty$ and for $p = \infty$

(b) Let $1 \le p < \infty$ and $f \in L^p(\mathbb{R})$. Show that:

$$\lim_{h \to 0} \|f(x+h) - f(x)\|_{L^p} = 0.$$

(c) Is the statement true if $p = \infty$? Either prove or find a counterexample.

Problem 3

Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{L}(\mathcal{H}, \mathcal{H})$ denote the space of all bounded linear operators on $\mathcal{H}$.

(a) Let $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$. Define the adjoint T^* of T, quoting necessary theorems on why it's well defined.

(b) Let $V \subset \mathcal{H}$ be a closed linear subspace, and $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$. Suppose that $TV \subset V$ and $T^*V \subset V$, prove that $TV^{\perp} \subset V^{\perp}$ and $T^*V^{\perp} \subset V^{\perp}$.

(c) Suppose $A : \mathcal{H} \to \mathcal{H}$ is a linear mapping (not assumed to be bounded), and suppose $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in \mathcal{H}$, prove that A is bounded and hence self-adjoint. (Hint: closed graph theorem).

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Problem 4

(a) Prove that the Fourier transform $f \mapsto \hat{f}$ is a bounded linear operator from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$.
(b) Let $f, g \in L^1(\mathbb{R})$, show that

$$\int f\hat{g} = \int \hat{f}\, g,$$

including why this expression makes sense.
(c) Let $f_k, f \in L^1(\mathbb{R})$, $k \in \mathbb{N}$, satisfy the following: $\sup_k \|f_k\|_{L^1} < \infty$, and the Fourier transforms $\hat{f}_k \to \hat{f}$ pointwise. Prove that for every Schwartz function φ, we have

$$\lim_{k\to\infty} \int f_k\varphi = \int f\varphi.$$

You may use without proof that the Fourier transform is a bijection on the space of Schwartz functions. You should carefully justify your answer.

— End of the exam —