DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

Tuesday, September 29, 2020

The 6 questions on the other side of this page have equal value, but different parts of a question may have different weights.

Good Luck!

Problem 1.

(a) Suppose that $p > q$ are prime numbers and that G is a finite group of order $p^n q$ for some $n \geq 1$. Prove that G contains a unique normal subgroup of index q.

(b) Suppose that G is a finite p-group and $N \neq 1$ a normal subgroup of G. Prove that $N \cap Z(G) \neq 1$, where $Z(G)$ denotes the centre of G.

Problem 2. Suppose that R is a commutative ring that is moreover an integral domain.

(a) If R is finite (as set), show that R is a field.

(b) If R is artinian, show that R is a field. (We say that R is *artinian* if any descending chain $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ of ideals of R stabilises, i.e. $I_k = I_{k+1}$ for all sufficiently large k.)

(c) Give an example of a (commutative) artinian ring that is not a field.

Problem 3.

(a) Determine all $\mathbb{F}_2[x]$-modules M, up to isomorphism, such that $\dim_{\mathbb{F}_2} M = 2$. Your list should not contain any duplicates.

(b) Write $(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) \otimes (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z})$ as direct sum of cyclic groups.

Problem 4.

(a) Prove that $p(x) = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$.

(b) Let θ be a root of $p(x)$. Describe all the elements in the field $\mathbb{Q}(\theta)$ uniquely in terms of θ.

(c) Compute $(1 + \theta)^{-1}$ in the field $\mathbb{Q}(\theta)$.

Problem 5.

(a) State the correspondence between groups and fields given by the Fundamental Theorem of Galois Theory.

(b) Suppose that $K/\mathbb{Q}$ is the splitting field of a monic integral polynomial of degree 3, and that $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$. How many intermediate fields $\mathbb{Q} \subset E \subset K$ are there? For how many of these is K/E Galois, and for how many is $E/\mathbb{Q}$ Galois?

Problem 6.

(a) What is an affine algebraic set, its coordinate ring, and a morphism between affine algebraic sets?

(b) State the Hilbert Nullstellensatz, and describe the correspondence between geometry and algebra that it provides.