Department of Mathematics
University of Toronto

Topology qualifying exam
September 2014

Do all questions. No aids allowed.

(1) (**15 points**) Prove that $SU(2)$ is a smooth manifold, and find its dimension. Recall it is defined as

$$SU(2) = \{A \in M_{2,2}(\mathbb{C}) \mid AA^\dagger = \text{id}, \quad \det(A) = 1\}$$

where $A^\dagger$ is the conjugate transpose of A and $M_{22}(\mathbb{C})$ denotes the complex 2×2 matrices. Here id is the identity matrix.

(2) [**10 points**] In the following question, you should choose coordinate systems on S^1 and S^3 and write the vector fields explicitly in terms of these coordinates.

(a) Exhibit a nowhere zero vector field on S^1.
(b) Exhibit a nowhere zero vector field on S^3.
(c) Can there be a vector field on S^2 which is nonzero everywhere? If so, give a construction of such a vector field; if not, give a proof that such a vector field does not exist.

(3) [**15 points**]

(a) Let $\omega = xdy \wedge dz + zdx \wedge dy + ydz \wedge dx$ be a 2-form on $S^2 = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$. Let X be the vector field $X(x,y,z) = (-y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, 0)$. Prove that $L_X\omega = 0$, where L_X is the Lie derivative with respect to the vector field X. You may use Cartan's formula for the Lie derivative:

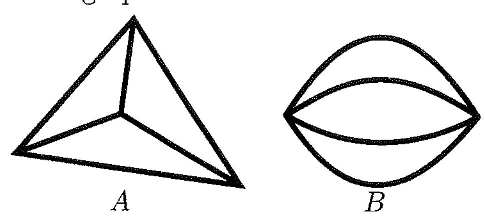
$$L_X = di_X + i_X d$$

Here i_X is the interior product with respect to X.

(b) Show that $i_X\omega = df$ for some function $f : S^2 \to \mathbb{R}$.

(4) (a) State the fundamental theorem of covering spaces (aka. the lifting criterion), which gives necessary and sufficient conditions for the existence of a lift of a map $f : Y \to X$.
(b) Suppose that Y is path-connected and locally path-connected and that $\pi_1(Y)$ is finite. Show that any map $Y \to S^1$ is homotopic to a constant map.

(5) Let A and B be the graphs below:



(a) Compute the fundamental groups $\pi_1(A)$ and $\pi_1(B)$.
(b) Show that A and B are homotopy equivalent.

(6) (a) Give the n-sphere S^n the structure of a CW-complex and calculate its cellular homology.
(b) Calculate the homology of S^n using the long exact sequence of the pair $(D^n, \partial D^n)$.