Real Analysis

Analysis Comprehensive Exam
September 2016

Please be brief but justify your answers, citing relevant theorems.

1. Let $(X,\ \Sigma,\ \mu)$ be a measure space, and $(E_n)_{n\geq 0}$ be a sequence of μ-measurable sets.
   (a) Show that the set
   $$E = \{x \in X \,:\, x \in E_n \text{ for infinitely many } n\}$$
   is μ-measurable.
   (b) (Borel-Cantelli Lemma) Show that if $\sum_{n\geq 0} \mu(E_n) < \infty$, then $\mu(E) = 0$.
   (c) Let $(f_n(x))$ be a sequence of μ-measurable functions such that
   $$\sum_{n\geq 0} \mu\left\{x \,:\, |n^2 f_n(x)| > 2\right\} < \infty.$$
   Prove that the series $\sum_{n\geq 0} f_n(x)$ converges μ-a. e.

2. Let $g_n(x, y)$ be a sequence of Lebesgue measurable functions on the unit square
   $$\{(x, y) \,:\, 0 \leq x \leq 1,\ 0 \leq y \leq 1\}.$$
   Let $f_n^x(y) = g_n(x, y)$.
   (a) Define what it means for the sequence $g_n(x, y)$ to converge to a measurable function $g(x, y)$ *in Lebesgue measure*.
   (b) Show that if for a.e. $x \in [0, 1]$, the sequence of functions $f_n^x(y)$ converges to $f^x(y) = g(x, y)$ in (1-dimensional) Lebesgue measure, then $g_n(x, y)$ converges to $g(x, y)$ in measure.
   (c) Show that the converse is false by constructing a sequence $g_n(x, y)$ converging to 0 in Lebesgue measure such that for no $x \in [0, 1]$, $f_n^x(y)$ converges to zero in Lebesgue measure.
   Hint: First, construct a sequence $h_n(x)$ of functions on $[0, 1]$ which converges to zero in measure, but does not converge pointwise at any point of $[0, 1]$. Use the sequence to easily construct the required sequence $g_n(x, y)$.

3. Let $1 \leq p < q < \infty$. Let
   $$B_n = \left\{ f \in L^p([0,\ 1]) \,:\, \int_0^1 |f(t)|^q \, dt < n \right\}.$$
   (a) Show that B_n has an empty interior in L^p.
   (b) Show that B_n is closed in L^p
   Hint: Use Fatou's Lemma.
   (c) Show that L^q is a subset of the first category in L^p.

(d) Let $\frac{1}{r} + \frac{1}{p} = 1$Assume that g is a measurable function on $[0,\ 1]$ which satisfies the following property:

$$\text{For every } f \in L^p([0,\ 1]),\ f \notin L^q([0,\ 1]),\ \text{we have } fg \in L^1([0,\ 1]).$$

Show that $g \in L^r([0,\ 1])$.
Hint: Use the sequence $g_n(x) = \min\left(|g(x)|, n\right)$, and the Uniform Boundedness Principle.

4. (a) What is the general formula for the Fourier transform of a function f on $\mathbb{R}$? The Fourier inversion formula? The Plancherel formula?

(b) Explain how to make the definition of Fourier transform precise in each of the following cases: f belongs to $L^1(\mathbb{R})$, to $L^2(\mathbb{R})$, to the Schwartz space $\mathcal{S}(\mathbb{R})$, or its dual space $\mathcal{S}'(\mathbb{R})$.

(c) In particular, what does it mean to take the Fourier transform of the function

$$f(x) = 1 + 2x + 3x^2 + \cdots + (n+1)x^n?$$

Calculate it explicitly.