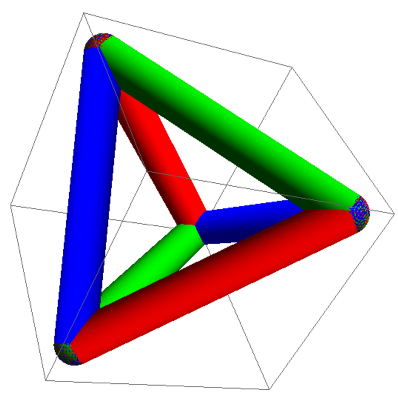
DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

Thursday, September 10, 2015, 1-4 PM

The 6 questions on the other side of this page have equal value, but different parts of a question may have different weights.



a surjection $S_4 \to S_3$

Good Luck!

Problem 1. Let G be a group and let $Z(G)$ denote its centre.

1. Show that if $G/Z(G)$ is cyclic then G is Abelian.

2. Prove that if the group $\mathrm{Aut}(G)$ of automorphisms of G is cyclic, then G is Abelian.

Problem 2. Define a "Principal Ideal Domain (PID)" and a "Unique Factorization Domain (UFD)" and show that every PID is a UFD. If you need to use the lemma that an increasing chain of ideals in a PID must become constant at some point (i.e., that a PID is "Noetherian"), prove it.

Problem 3. Let $V = F^n$ be an n-dimensional vector space over some field F, let $T: V \to V$ be a linear transformation, let $R = F[x]$ denote the ring of polynomials in a variable x with coefficients in F, and consider V as an R-module by setting $xv = Tv$ for any $v \in V$. Let $\pi: R^n \to F^n$ be the morphism of R-modules defined by mapping the standard basis elements e_i of R^n to their obvious counterparts in F^n. Propose a set of n generators r_i of $\ker \pi$ and prove in detail that your proposed r_i indeed generate $\ker \pi$.

Problem 4. Let $\bar{\mathbb{Q}}$ be an algebraic closure of the rational numbers. Suppose that $\alpha, \beta \in \bar{\mathbb{Q}}$ are such that $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are Galois extensions of $\mathbb{Q}$.

1. Prove that $\mathbb{Q}(\alpha, \beta)$ is a Galois extension of $\mathbb{Q}$.

2. Suppose that $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ and $\mathrm{Gal}(\mathbb{Q}(\beta)/\mathbb{Q})$ are Abelian. Prove that $\mathrm{Gal}(\mathbb{Q}(\alpha, \beta)/\mathbb{Q})$ is abelian.

3. If $F \subset \bar{\mathbb{Q}}$ is a finite extension of $\mathbb{Q}$, we say that F is *Abelian* over $\mathbb{Q}$ if F is Galois over $\mathbb{Q}$ and $\mathrm{Gal}(F/\mathbb{Q})$ is Abelian. Prove that if $K \subset \mathbb{Q}$ is a finite Galois extension of $\mathbb{Q}$, there exists a maximal Abelian subfield F of K: that is, F is abelian over $\mathbb{Q}$ and contains any subfield K' of K that is Abelian over $\mathbb{Q}$.

Note: In your solution to this question, please do not quote theorems about Galois groups of composites of field extensions. Feel free to use other results from Galois theory.

Problem 5. Let G be a finite group and let $\chi = \chi_\rho$ be the character of an irreducible complex representation $\rho: G \to GL(V)$ of G. (In other words, V is a simple $\mathbb{C}[G]$-module.) Let H be a proper subgroup of G and let $\chi_1, \ldots, \chi_r$ be the characters of the distinct isomorphism (that is, equivalence) classes of irreducible complex representations of H. Let χ_H be the restriction of χ to H.

1. Show that $\chi_H = \sum_{j=1}^r n_j \chi_j$ for nonnegative integers $n_1, \ldots, n_r$.

2. Show that $\sum_{j=1}^r n_j^2 \leq |G|/|H|$.

3. Show that if H is a normal subgroup of G, then all of the nonzero n_j's are equal.

Problem 6. Let I be an ideal in a commutative ring R with 1. The *radical* of I, denoted by $\mathrm{Rad}(I)$ is equal to $\{c \in R \mid c^n \in I \text{ for some n} \geq 1\}$. The ideal I is *primary* if whenever a, $b \in R$ satisfy $ab \in I$, then $a \in I$ or $b \in \mathrm{Rad}(I)$. Prove that I is prime if and only if $I = \mathrm{Rad}(I)$ and I is primary.

Good Luck!