Department of Mathematics
University of Toronto

MAT1101F **2014**

COMPREHENSIVE ALGEBRA EXAM 2014

Date and Time: Thursday, September 4, 2014, 1:00 - 4:00 p.m. in BA6183.

- No aids are allowed.
- Answer as many problems as you can. The total number of points possible is 100. As an experiment, if you achieve 80% or more on questions (1) through (4) you will not be required to take Algebra I, MAT1100F, the first half of the Algebra core course. If you achieve 70% or more on the whole exam you will not be required to take Algebra I, MAT1100F, nor Algebra II, MAT1101S.

Problems:

(1) (10 POINTS) The dihedral group D_6, of order 12, acts by rotations and reflections on a regular hexagon. Label the vertices of the hexagon $1, 2, 3, 4, 5, 6$. Let $\mathcal{P}$ be the set of ordered pairs of vertices, so the cardinality of $\mathcal{P}$ is 36. Then D_6 acts on $\mathcal{P}$ in the obvious way: For $\sigma \in \mathsf{D}_6, 1 \leqslant i, j \leqslant 6, \sigma \cdot (i, j) = (\sigma(i), \sigma(j))$, where $\sigma(i)$ is the image of the vertex i under the action of σ on the vertices.

(a) Describe the orbits in $\mathcal{P}$ under this action of D_6.

(b) For each of the orbits of D_6 in $\mathcal{P}$, fix an element in the orbit and find the order of its stabilizer in D_6.

(2) (10 POINTS) Let p be an odd prime and S_{2p} the symmetric group on $2p$ letters. Show that a Sylow p–subgroup of S_{2p} is abelian, isomorphic to the direct product of two cyclic groups of order p. How many such Sylow subgroups are there? Verify directly the third Sylow theorem for this case.

(3) (5 POINTS EACH PART)

(a) Write $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/90\mathbb{Z}$ as a product of cyclic groups with the order of each factor a divisor of the order of the next, in the usual way.

(b) Write $\mathbb{Z}/12\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/90\mathbb{Z}$ as a product of cyclic groups with the order of each factor a divisor of the order of the next, in the usual way.

(4) (20 POINTS) Let $(G, \cdot)$ be a multiplicatively written finite group of odd order. Show that any $g \in G$ can be written as a product of squares, $g = g_1^2 \cdots g_n^2$, for some natural number $n \geqslant 1$ and group elements $g_1, ..., g_n \in G$.
Give an example of a finite group of even order, where the statement is false.

(5) (5 POINTS EACH PART) A given endomorphism $\varphi : V \to V$ of a 4–dimensional vector space over a field $\mathbb{F}$ has a 2–dimensional kernel and satisfies $\varphi^2 = \varphi$.

(a) What are the possible minimal polynomials $m_\varphi(x)$ of φ?

(b) What are the possible characteristic polynomials $c_\varphi(x)$ of φ?

(6) (5 POINTS EACH PART) For each of the following statements, either prove it or provide a counterexample (with an explanation why it is a counterexample.)

(a) If R is a principal ideal domain (PID) and $I \subset R$ is a proper prime ideal, then R/I is also a PID.

(b) If R is a PID and $S \subseteq R$ is a subring containing the unit element $1 \in R$, then S is also a PID.

(c) If R is a PID, then so is $R[\![x]\!]$, the formal power series ring in one variable over it.

(d) If R is a Euclidean domain, then it is a Unique Factorization Domain (UFD).

(7) (20 POINTS) Explain why $x^3 + 8x - 6 \in \mathbb{Q}[x]$ is irreducible. What is the degree of a splitting field of this polynomial over $\mathbb{Q}$? What is the Galois group?