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# Inverse Spectral problem for the Sturm-Liouville Operator with Eigenvalue Parameter Dependent Boundary Conditions

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**Abstract.** To solve the inverse problem for the Sturm-Liouville operator with eigenvalue parameter dependent boundary conditions we reconstruct the spectral distribution function from two spectra of the boundary-value problems with equal $\Theta(\lambda)$ and different real constants in the boundary conditions. The well-known results of A.V. Strauss [5] concerning the connection between the eigenvalue problems with the spectral parameter in the boundary conditions and the theory of generalized resolvents is used.

**I.** Let us consider the regular differential equation

$$-y'' + q(x)y = \lambda y, \tag{1}$$

where $q(x)$ $(0 \leq x \leq l)$ is a real-valued continuous function and two boundary-value problems:

$$y'(0) - h_1 y(0) = 0, \quad y(l) = \Theta(\lambda)y'(l), \tag{2}$$

$$y'(0) - h_2 y(0) = 0, \quad y(l) = \Theta(\lambda)y'(l), \tag{3}$$

where $h_1$ and $h_2$ are finite real numbers, $h_1 \neq h_2$ and $\Theta(\lambda)$ is a rational function for which $\operatorname{Im}\Theta(\lambda)\operatorname{Im}\lambda \leq 0$. We write $\Theta(\lambda) = \frac{\Theta_1(\lambda)}{\Theta_2(\lambda)}$, where $\Theta_1(\lambda)$ and $\Theta_2(\lambda)$ are relatively prime polynomials. If the boundary conditions (2), (3) do not contain the eigenvalue parameter $\lambda$, then the solution of the inverse problem from two spectra is given by M.G. Krein, B.M. Levitan, I.M. Gelfand [1,2,3].

Let $u_1(x,\lambda)$ and $u_2(x,\lambda)$ be the solutions of (1) satisfying the following initial conditions:

$$u_1(0,\lambda) = 1, \quad u_1'(0,\lambda) = h_1,$$

$$u_2(0,\lambda) = 1, \quad u_2'(0,\lambda) = h_2.$$

Then the spectra of the boundary-value problems (1), (2) and (1), (3) are the zero-sequences $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ of the entire functions:

$$\Phi_1(\lambda) = \Theta_2(\lambda)u_1(l,\lambda) - \Theta_1(\lambda)u_1'(l,\lambda),$$

$$\Phi_2(\lambda) = \Theta_2(\lambda)u_2(l,\lambda) - \Theta_1(\lambda)u_2'(l,\lambda).$$

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As the spectra of these problems are bounded from below we can number the eigenvalues so that $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$; $\mu_0 < \mu_1 < \cdots < \mu_n < \cdots$.

The spectral distribution function $\rho(\lambda)$ of the problem (1), (2) is a jump function [4,5] and it is determined by equality

$$\rho(\lambda) = \sum_{\lambda_n < \lambda} \frac{1}{\alpha_n}.$$

The numbers $\alpha_n$ are known as the normalizing numbers [3]. In the classical case ($\Theta(\lambda) = const$) the equality $\alpha_n = \int_0^l |u_1(x, \lambda_n)|^2 dx$ takes place. Next we will show how to use $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ to obtain $\alpha_n$, the normalizing numbers of eigenfunctions of the first boundary-value problem.

Let $A_1$ be the differential operator of the second order acting in $L^2(0, l)$ which is defined by the expression

$$l[f] = -f'' + q(x) f$$

and the boundary conditions

$$f_1'(0) = h_1 f_1(0), \quad f_1'(l) = f_1(l) = 0.$$

Let $m$ be the number of poles $\Theta(\lambda)$. The selfadjoint extension $\tilde{A}$ of the differential operator is in some sense a coupling of the operators $A_1$ and $A_2$ ($A_2$ is a difference operator) acting in the orthogonal spaces $L^2(0, l)$ and $C^m$, respectively [4,6]. Eigenfunctions of the selfadjoint operator $\tilde{A}$ are discretely extended into the space $C^m$. Therefore we can write

$$\alpha_n = \alpha_{n1} + \alpha_{n2},$$

where

$$\alpha_{n1} = \int_0^l |u_1(x, \lambda_n)|^2 dx, \quad \alpha_{n2} = -\Theta'(\lambda_n) |u_1'(l, \lambda_n)|^2.$$

**Example 1.** *Let us consider the boundary-value problem*

$$-y'' + q(x) y = \lambda y,$$
$$y(0) = \lambda y'(0), \quad y'(l) = h y(l).$$

*Let $\tilde{A}$ be the selfadjoint operator*

$$D(\tilde{A}) = \{\langle f, f'(0)\rangle : f(x) \in L^2(0, l),\ f'(l) = h f(l)\},$$

$$\tilde{A}\langle f(x), f'(0)\rangle = \langle -f''(x) + q(x) f(x), f(0)\rangle.$$

*The eigenfunction system $u(x, \lambda_n)$ is complete in $L^2(0, l)$, but not minimal.*

$$\tilde{A} u = \lambda u, \quad u(x, \lambda_n) = \langle u_2(x, \lambda_n) + \alpha(\lambda) u_1(x, \lambda_n), -1\rangle.$$

Denote by

$$f(x,\lambda) = u_2(x,\lambda) + m(\lambda)u_1(x,\lambda),$$

where the Weyl function $m(\lambda)$ is defined by the condition

$$f(l,\lambda) = \Theta(\lambda)f'(l,\lambda)$$

and hence

$$m(\lambda) = \frac{\Theta_2(\lambda)u_2(l,\lambda) - \Theta_1(\lambda)u_2'(l,\lambda)}{\Theta_1(\lambda)u_1'(l,\lambda) - \Theta_2(\lambda)u_1(l,\lambda)} = -\frac{\Phi_1(\lambda)}{\Phi_2(\lambda)}.$$

The formula shows that $m(\lambda)$ is a meromorphic function, whose poles and zero-sequences are the spectra of the appropriate boundary-value problems (1), (2) and (1), (3) respectively. In addition, by the Green's formula, we have

$$(\lambda-\overline{\mu})\int_0^l f(x,\lambda)\overline{f(x,\mu)}dx = (h_1-h_2)(m(\lambda)-\overline{m(\mu)})+(\Theta(\lambda)-\overline{\Theta(\mu)})f'(l,\lambda)\overline{f'(l,\mu)}.$$

If $\mu = \lambda$, then, because $\operatorname{Im}\Theta(\lambda)\operatorname{Im}\lambda \leq 0$,

$$(h_1 - h_2)\frac{\operatorname{Im}\, m(\lambda)}{\operatorname{Im}\,\lambda} = \int_0^l |f(x,\lambda)|^2 dx - \frac{\operatorname{Im}\,\Theta(\lambda)}{\operatorname{Im}\,\lambda}|f'(l,\lambda)|^2 \geq 0.$$

Hence, in the case $h_1 > h_2$, $m(\lambda)$ is a Nevanlinna function and in the case $h_1 < h_2$, $-m(\lambda)$ is a Nevanlinna function [8]. We have thus proved

**Theorem 1.** *The spectra of boundary-value problems with equal $\Theta(\lambda)$ and different real constants in boundary conditions are alternating.*

Let us use Green's formula once more

$$(\lambda - \lambda_n)\int_0^l f(x,\lambda)u_1(x,\lambda_n)dx = h_2 - h_1 + (\Theta(\lambda) - \Theta(\lambda_n))f'(l,\lambda)u_1'(l,\lambda_n),$$

$$\begin{aligned}
(\lambda - \lambda_n)&\int_0^l f(x,\lambda)u_1(x,\lambda_n)dx \\
&= (\lambda - \lambda_n)\int_0^l u_2(x,\lambda)u_1(x,\lambda_n)dx - (\lambda - \lambda_n)\frac{\Phi_2(\lambda)}{\Phi_1(\lambda)}\int_0^l u_1(x,\lambda)u_1(x,\lambda_n)dx \\
&= h_2 - h_1 + (\Theta(\lambda) - \Theta(\lambda_n))u_2'(l,\lambda)u_1'(l,\lambda_n) \\
&- (\Theta(\lambda) - \Theta(\lambda_n))\left(\frac{\Phi_2(\lambda)}{\Phi_1(\lambda)}\right)u_1'(l,\lambda)u_1'(l,\lambda_n).
\end{aligned}$$

Change the formula in the form of

$$\begin{aligned}\int_0^l u_1(x,\lambda)u_1(x,\lambda_n)dx - \frac{\Theta(\lambda)-\Theta(\lambda_n)}{\lambda-\lambda_n}u_1'(l,\lambda)u_1'(l,\lambda_n)\\ = \frac{\Phi_1(\lambda)}{\Phi_2(\lambda)}\int_0^l u_2(x,\lambda)u_1(x,\lambda_n)dx - (h_2-h_1)\frac{\Phi_1(\lambda_n)}{\Phi_2(\lambda_n)(\lambda-\lambda_n)}\\ - \frac{\Theta(\lambda)-\Theta(\lambda_n)}{\lambda-\lambda_n}\frac{\Phi_1(\lambda_n)}{\Phi_2(\lambda_n)}u_2'(l,\lambda)u_1'(l,\lambda_n).\end{aligned}$$

But now as $\lambda \to \lambda_n$ we see that

$$\int_0^l |u_1(x,\lambda_n)|^2 dx - \Theta'(\lambda_n)|u_1'(l,\lambda_n)|^2 = -(h_2-h_1)\frac{\Phi_1'(\lambda_n)}{\Phi_2(\lambda_n)},$$

since $\Phi_1(\lambda_n)=0$.

$$\alpha_n = -(h_2-h_1)\frac{\Phi_1'(\lambda)}{\Phi_2(\lambda)}. \tag{4}$$

Using (4) we find that

$$\alpha_n = \frac{h_2-h_1}{\mu_n-\lambda_n}{\prod_{k=0}^{\infty}}' \frac{\lambda_k-\lambda_n}{\mu_k-\lambda_n}. \tag{5}$$

$\Phi_1(\lambda)$ and $\Phi_2(\lambda)$ are entire functions which behave as $O(1/|\lambda|^{1/2})$ [1,2]. We can write the infinite product

$$\Phi_1(\lambda) = c_1\prod_{k=0}^{\infty}\left(1-\frac{\lambda}{\lambda_k}\right), \quad \Phi_2(\lambda) = c_2\prod_{k=0}^{\infty}\left(1-\frac{\lambda}{\mu_k}\right),$$

where $c_1$ and $c_2$ are constants. It follows from the statement above and (4) that

$$\alpha_n = (h_1-h_2)\frac{c_1}{c_2}\frac{1}{\lambda_n}\frac{\prod_{k=0}^{\prime\infty}\left(1-\frac{\lambda_n}{\lambda_k}\right)}{\prod_{k=0}^{\infty}\left(1-\frac{\lambda_n}{\mu_k}\right)} = \frac{h_2-h_1}{\mu_n-\lambda_n}\frac{c_1}{c_2}\prod_{k=0}^{\infty}\frac{\mu_k}{\lambda_k}{\prod_{k=0}^{\infty}}'\frac{\lambda_k-\lambda_n}{\mu_k-\lambda_n}.$$

Therefore, to obtain (5) we must show that

$$\frac{c_1}{c_2}\prod_{k=0}^{\infty}\frac{\mu_k}{\lambda_k} = 1.$$

Let us consider two cases.

**(a)** In this case $\deg\Theta_1(\lambda) \geq \deg\Theta_2(\lambda)$ [5]. Using the classical asymptotic formulas for the solutions of Sturm-Liouville equation to obtain the limit

$$\begin{aligned}\lim_{\lambda\to-\infty}\frac{\Phi_1(\lambda)}{\Phi_2(\lambda)} &= \lim_{\lambda\to-\infty}\frac{\Theta_2(\lambda)u_1(l,\lambda)-\Theta_1(\lambda)u_1'(l,\lambda)}{\Theta_2(\lambda)u_2(l,\lambda)-\Theta_1(\lambda)u_2'(l,\lambda)}\\ &= \lim_{\lambda\to-\infty}\frac{Hu_1(l,\lambda)-u_1'(l,\lambda)}{Hu_2(l,\lambda)-u_2'(l,\lambda)},\end{aligned}$$

where $H = \lim_{\lambda\to-\infty} \frac{\Theta_2(\lambda)}{\Theta_1(\lambda)}$.

$$\lim_{\lambda\to-\infty} \frac{c_1}{c_2} \prod_{k=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right) \left(1 - \frac{\lambda}{\mu_k}\right)^{-1} = \frac{c_1}{c_2} \prod_{k=0}^{\infty} \frac{\mu_k}{\lambda_k} \lim_{\lambda\to-\infty} \prod_{k=0}^{\infty} \frac{\lambda_k - \lambda}{\mu_k - \lambda} = 1.$$

We must show that

$$\lim_{\lambda\to-\infty} \prod_{k=0}^{\infty} \frac{\lambda_k - \lambda}{\mu_k - \lambda} = \lim_{\lambda\to-\infty} \prod_{k=0}^{\infty} \left(1 + \frac{\lambda_k - \mu_k}{\mu_k - \lambda}\right) = 1. \tag{6}$$

The asymptotic behavior of eigenvalues (2), (3) is given by formulas [6,7]

$$\lambda_k = (k-m)^2 + O(1), \quad \mu_k = (k-m)^2 + O(1).$$

Therefore $\lambda_k - \mu_k = O(1)$, i.e., the spectra of boundary-value problems are asymptotically close and the series $\sum_{k=0}^{\infty} \left| \frac{\lambda_k - \mu_k}{\mu_k - \lambda} \right|$ is uniformly convergent in the neighborhood of the point $\lambda = -\infty$. This allows us to write

$$\lim_{\lambda\to-\infty} \prod_{k=0}^{\infty} \left(1 + \frac{\lambda_k - \mu_k}{\mu_k - \lambda}\right) = 1.$$

This shows that (5) holds.

**(b)** In this case $\deg \Theta_1(\lambda) < \deg \Theta_2(\lambda)$.

$$\lim_{\lambda\to-\infty} \frac{\Phi_1(\lambda)}{\Phi_2(\lambda)} = \lim_{\lambda\to-\infty} \frac{u_1(l,\lambda) - \frac{\Theta_1(\lambda)}{\Theta_2(\lambda)} u_1'(l,\lambda)}{u_2(l,\lambda) - \frac{\Theta_1(\lambda)}{\Theta_2(\lambda)} u_2'(l,\lambda)} = 1.$$

Moreover, $\lambda_k = (k - m + 1/2) + O(1)$ and $\mu_k = (k - m + 1/2) + O(1)$, therefore $\lambda_k - \mu_k = O(1)$, i.e., the linear term of $\Theta(\lambda)$ does not affect the asymptotic closeness of the spectra of boundary-value problems.

Using formula (5) we can reconstruct the spectral distribution function. If $q(x)$ is a sufficiently differentiable (twice differentiable) function then asymptotic formulas take place

$$\sqrt{\lambda_{n+m}} = n + \frac{a_0}{n} + \frac{a_1}{n^3} + o\left(\frac{1}{n^3}\right), \tag{7}$$

$$\sqrt{\mu_{n+m}} = n + \frac{a_0'}{n} + \frac{a_1'}{n^3} + o\left(\frac{1}{n^3}\right), \tag{8}$$

where

$$a_0 = (h_1 + H)/l + (2l)^{-1} \int_0^l q(x)dx,$$

$$a_0' = (h_2 + H)/l + (2l)^{-1} \int_0^l q(x)dx,$$

and therefore

$$a_0 - a_0' = (h_2 - h_1)/l.$$

**Theorem 2.** *Let two spectra $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ be given so that $\lambda_0 < \mu_0 < \lambda_1 < \mu_1 < \cdots$, equalities (7), (8) take place, moreover $a_0 \neq a_0'$. Then there exist an absolutely continuous function $q(x)$, real numbers $h_1$, $h_2$ and a rational function $\Theta(\lambda)$ for which $\operatorname{Im}\Theta(\lambda)\operatorname{Im}\lambda \leq 0$, such that $\lambda_n$ is the spectrum of the problem (2), $\mu_n$ is the spectrum of the problem (3).*

**II.** Let us consider the regular differential equation

$$-y'' + q(x)y = \lambda y, \tag{9}$$

where $q(x)$ $(0 \leq x \leq l)$ is a real-valued continuous function and two boundary-value problems:

$$y(0) = \Theta(\lambda)y'(0), \quad y'(l) + H\,y(l) = 0, \tag{10}$$

$$y(0) = (\Theta(\lambda) + h)y'(0), \quad y'(l) + H\,y(l) = 0, \tag{11}$$

where $h$ and $H$ are finite real numbers, $h \neq 0$ and $\Theta(\lambda)$ is a rational function for which $\operatorname{Im}\Theta(\lambda)\operatorname{Im}\lambda \geq 0$. Denote by $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$; $\mu_0 < \mu_1 < \mu_2 < \cdots$, the eigenvalues of boundary-value problems (9), (10) and (9), (11), respectively. Let $u_1(x,\lambda)$ and $u_2(x,\lambda)$ be the solutions of (9) satisfying the following initial conditions:

$$u_1(0,\lambda) = 1, \quad u_1'(0,\lambda) = 0,$$
$$u_2(0,\lambda) = 0, \quad u_2'(0,\lambda) = -1.$$

Then the solutions of the differential equation (9)

$$\varphi(x,\lambda) = \Theta_1(\lambda)u_1(x,\lambda) - \Theta_2(\lambda)u_1'(x,\lambda),$$
$$\psi(x,\lambda) = (\Theta_1(\lambda) + h\,\Theta_2(\lambda))u_1(x,\lambda) - \Theta_2(\lambda)u_1'(x,\lambda)$$

satisfy right boundary condition (10) and right boundary condition (11), respectively.

Then the spectra of the boundary-value problems are the zero-sequences of the entire functions [5]:

$$\Phi_1(\lambda) = \varphi'(l,\lambda) + H\varphi(l,\lambda),$$
$$\Phi_2(\lambda) = \psi'(l,\lambda) + H\psi(l,\lambda).$$

Next we will show how to use $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ to obtain $\alpha_n$ the normalizing numbers of eigenfunctions of the first boundary-value problem. Following the work [3], we shall determine

$$\alpha_n = \alpha_{n1} + \alpha_{n2}, \quad \text{where} \quad \alpha_{n1} = \int_0^l |\varphi(x,\lambda_n)|^2 dx, \quad \alpha_{n2} = \Theta'(\lambda_n)|\Theta_2(\lambda_n)|^2.$$

Let us define

$$f(x,\lambda) = \psi(x,\lambda) + m(\lambda)\varphi(x,\lambda).$$

The Weyl function $m(\lambda)$ is defined by the condition $f'(l,\lambda) + H\,f(l,\lambda) = 0$. Hence we obtain $m(\lambda) = -\frac{\Phi_1(\lambda)}{\Phi_2(\lambda)}$.

The formula shows that $m(\lambda)$ is a meromorphic function, whose poles and zero-sequences are spectra of appropriate boundary-value problems (9), (10) and (9),

(11) respectively. In addition, by Green's formula, we have

$$(\lambda-\overline{\mu})\int_0^l f(x,\lambda)\overline{f(x,\mu)}dx = f'(0,\lambda)\overline{f(0,\mu)} - f(0,\lambda)\overline{f'(0,\mu)}$$

$$= (\Theta_2(\lambda)+m(\lambda)\Theta_2(\lambda))(\overline{\Theta_1(\mu)}+h\overline{\Theta_2(\mu)}+\overline{m(\mu)}\,\overline{\Theta_1(\mu)})$$
$$-(\Theta_1(\lambda)+h\Theta_2(\lambda)+m(\lambda)\Theta_1(\lambda))(\overline{\Theta_2(\mu)}+\overline{m(\mu)}\,\overline{\Theta_2(\mu)})$$
$$= (\overline{\Theta(\mu)}-\Theta(\lambda))\Theta_2(\lambda)\,\overline{\Theta_2(\mu)}+h\Theta_2(\lambda)\overline{\Theta_2(\mu)}(m(\lambda)-\overline{m(\mu)})$$
$$+m(\lambda)\overline{m(\mu)}(\overline{\Theta(\mu)}-\Theta(\lambda))\Theta_2(\lambda)\overline{\Theta_2(\mu)}$$
$$+(\overline{m(\mu)}+m(\lambda))(\Theta_2(\lambda)\overline{\Theta_1(\mu)}-\Theta_1(\lambda)\overline{\Theta_2(\mu)}).$$

On the other hand, if $\mu=\lambda$, then

$$(12)\quad \int_0^l |f(x,\lambda)|^2dx = -\frac{\operatorname{Im}\Theta(\lambda)}{\operatorname{Im}\lambda}|\Theta_2(\lambda)|^2 + h\,\frac{\operatorname{Im}\, m(\lambda)}{\operatorname{Im}\lambda}|\Theta_2(\lambda)|^2$$
$$-\frac{\operatorname{Im}\Theta(\lambda)}{\operatorname{Im}\lambda}|\Theta_2(\lambda)|^2|m(\lambda)|^2 - (\overline{m(\lambda)}+m(\lambda))\frac{\operatorname{Im}\Theta(\lambda)}{\operatorname{Im}\lambda}|\Theta_2(\lambda)|^2.$$

It is obvious that $1+|m(\lambda)|^2+2\operatorname{Re} m(\lambda)\geq 0$. Hence we obtain

$$\int_0^l |f(x,\lambda)|^2dx+\frac{\operatorname{Im}\Theta(\lambda)}{\operatorname{Im}\lambda}|\Theta_2(\lambda)|^2(\quad 1+|m(\lambda)|^2+2\operatorname{Re} m(\lambda)) = h\,\frac{\operatorname{Im}\, m(\lambda)}{\operatorname{Im}\lambda}|\Theta_2(\lambda)|^2$$

in the case $h>0$, $m(\lambda)$ is a Nevanlinna function (in the case $h<0$, $-m(\lambda)$ is a Nevanlinna function). We have thus proved

**Theorem 3.** *The spectra of the boundary-value problems (10) and (11) are alternating.*

Let us use Green's formula once more

$$(13)\quad (\lambda-\lambda_n)\int_0^l f(x,\lambda)\varphi(x,\lambda_n)dx = f'(0,\lambda)\varphi(0,\lambda_n)-f(0,\lambda)\varphi'(0,\lambda_n)$$
$$= (\Theta_2(\lambda)+m(\lambda)\Theta_2(\lambda))(\Theta_1(\lambda_n))$$
$$-(\Theta_1(\lambda)+h\,\Theta_2(\lambda)+m(\lambda)\Theta_1(\lambda))(\Theta_2(\lambda_n)),$$

where $m(\lambda)=-\frac{\Phi_2(\lambda)}{\Phi_1(\lambda)}$. On the other hand,

$$(14)\quad (\lambda-\lambda_n)\int_0^l f(x,\lambda)\varphi(x,\lambda_n)dx$$
$$= (\lambda-\lambda_n)\int_0^l \psi(x,\lambda)\varphi(x,\lambda_n)dx - (\lambda-\lambda_n)\frac{\Phi_2(\lambda)}{\Phi_1(\lambda)}\int_0^l \varphi(x,\lambda)\varphi(x,\lambda_n)dx.$$

But now as $\lambda \to \lambda_n$ using the statement (13) we see that

$$\alpha_{n1} = -\Theta'(\lambda_n)|\Theta_2(\lambda_n)|^2 - h\,|\Theta_2(\lambda_n)|^2 \frac{\Phi_1'(\lambda_n)}{\Phi_2(\lambda_n)}$$

From this it follows that

$$\alpha_n = \frac{h|\Theta_2(\lambda_n)|^2}{\mu_n - \lambda_n} {\prod_{k=0}^{\infty}}' \frac{\lambda_k - \lambda_n}{\mu_k - \lambda_n}.$$

**Theorem 4.** *Let two spectra $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$ and $\Theta_2(\lambda)$ be given so that $\lambda_0 < \mu_0 < \lambda_1 < \mu_1 < \cdots$, equalities (7), (8) take place, moreover $a_0 \neq a_0'$. Then there exist an absolutely continuous function $q(x)$, a real number $h$ and a rational function $\Theta(\lambda)$ for which $\operatorname{Im}\Theta(\lambda)\operatorname{Im}\lambda \geq 0$, such that $\lambda_n$ is the spectrum of the problem (10), $\mu_n$ is the spectrum of the problem (11).*

## References


[1] Krein M.G. On some cases of effective determination of the density of a nonhomogeneous string by its spectral function. Dokl.Akad.Nauk SSSR, (1953), **93**, N4, 617–620.

[2] M.G. Krein, On inverse problems for inhomogeneous strings, Dokl. Akad. Nauk SSSR, 82 (1952), 669–672.

[3] I.M. Gel'fand and B.M. Levitan, On a simple identity for the characteristic values of a differential operator of the second order, Dokl. Akad. Nauk SSSR, 88 (1953), 593–596.

[4] Strauss A.V. On selfadjoint operators acting in the orthogonal sum of Hilbert spaces. Dokl. Akad. Nauk SSSR, (1962), **144**, N3, 512–515.

[5] A.V.Strauss, On the spectral functions of differential operators, Izv. Akad. Nauk SSSR, Ser. Mat., 19 (1955), 201–220.

[6] M.V. Chugunova, On the inverse spectral problem for the finite interval, Funct. Analysis, Ulyanovsk, 35 (1994), 113–123.

[7] M.V. Chugunova, On the effective methods for the solution of some inverse problems, Funct. Analysis, Ulyanovsk, 36 (1997), 66–75.

[8] N. Aronszajn, W.F. Donoghue, On the exponential representations of analytic functions in the upper half-plane with positive imaginary part, J. d'Analyse Math., 5 (1956), 321–388.



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