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Integral Equations and Operator Theory

On the Nature of Ill-Posedness of the Forward-Backward Heat Equation

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Abstract. We study the Cauchy problem with periodic initial data for the forward-backward heat equation defined by a J-self-adjoint linear operator L depending on a small parameter. The problem originates from the lubrication approximation of a viscous fluid film on the inner surface of a rotating cylinder. For a certain range of the parameter we rigorously prove the conjecture, based on numerical evidence, that the complete set of eigenvectors of the operator L does not form a Riesz basis in $\mathcal{L}^2(-\pi,\pi)$. Our method can be applied to a wide range of evolution problems given by PT-symmetric operators.

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1. Introduction

Analysis of the dynamics of a thin film of liquid entrained on the inside of a rotating cylinder is of great importance in many applications. For example, when liquid thermosetting plastic is placed inside a rotating mould, the best quality outcome can be achieved if the distribution of the liquid is as uniform as possible. More details on this application can be found in [35]. The same problem arises in the coating of fluorescent light bulbs in which a suspension consisting of a coating solute and a solvent is placed inside a spinning glass tube. The model for the coating is described in [6].

The lubrication approximation is used extensively to study flows in thin films. Under the assumption that the film is thin enough for viscous entrainment to

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compete with gravity, the time evolution of a thin film of liquid on the inner surface of a cylinder rotating in a gravitational field can be described by the forward-backward heat equation:

$$h_t + Lh = 0, \quad \theta \in (-\pi, \pi), \quad t \in (0, T), \tag{1.1}$$

where

$$Lh = \varepsilon\, \partial_\theta(\sin(\theta)\, h_\theta) + h_\theta, \quad h(-\pi) = h(\pi), \quad \varepsilon > 0. \tag{1.2}$$

The effect of the surface tension is neglected in this linearized model derived by Benilov, O'Brien and Sazonov in [3].

We prove that the Cauchy problem related to this equation

$$h|_{t=0} = h_0(\theta), \quad h(-\pi, t) = h(\pi, t) \tag{1.3}$$

does not have a weak solution in the Sobolev sense $h(\theta, t)$, even locally in time, if $h_0(\theta)$ belongs to the class of finitely smooth functions with $\operatorname{supp} h_0 \cap (\delta, \pi - \delta) \neq \varnothing$ (see Section 3).

The statement above can be roughly understood from the classic theory of parabolic equations that states that regularity of a generalized solution depends on the regularity of the equation coefficients (in our case all coefficients are in $C^\infty(-\pi, \pi)$) and on the time-reversibility of the equation, i.e., simultaneous changes of the time variable t to $-t$ and the space variable x to $-x$ lead to the same partial differential equation. Time-reversibility and infinite regularity generally imply ill-posedness.

The physical explanation of this explosive blow-up of solutions is related to a drop of fluid that in the absence of surface tension will be detached from the film in the upper part of cylinder, where the effect of the gravity is the strongest [3, p. 217].

Benilov, O'Brien and Sazonov [3] studied the eigenvalues of the operator L asymptotically, with an application of the modified WKB approximation and numerically, with an application of the analytic continuation method. They came to a very interesting set of hypotheses: all eigenvalues of the operator L are located on the imaginary axis; they are all simple; and the set of eigenfunctions is complete in $\mathcal{L}^2(-\pi, \pi)$ which is not typical for an ill-posed time-evolution problem.

The analysis of the spectral properties of this operator was continued by Chugunova, Pelinovsky [10] and by Davies [12]. In particular, it was shown that if the parameter ε is in the interval $0 < \varepsilon < 2$, then the operator is well defined in the sense that it admits closure in $\mathcal{L}^2(-\pi, \pi)$ with non-empty resolvent set. Analyzing a tridiagonal matrix representation of the operator L with respect to the Fourier basis, Davies [12] showed that L admits an orthogonal decomposition with respect to three invariant subspaces $\mathcal{H}^{2,0}(\mathbb{D})$, $\mathcal{H}^{2,0}(\overline{\mathbb{C}} \setminus \mathbb{D})$, and $\operatorname{Ker}(L) = \{c\mathbb{1},\ c \in \mathbb{C}\}$ (see Section 4 below) and used this fact to prove that the nontrivial part $\widetilde{L} := L \restriction \mathcal{H}^{2,0}(\mathbb{D}) \oplus \mathcal{H}^{2,0}(\overline{\mathbb{C}} \setminus \mathbb{D})$ of L has a compact inverse of the Hilbert-Schmidt type. We prove that, actually, the inverse operator is nuclear, i.e., belongs to the trace class $\mathfrak{S}_1$ (see Proposition 4.7).

It was proved by Weir [39] that if there exists an eigenvalue λ of the operator L, then $\mu = i\frac{2\lambda}{\varepsilon}$ is an eigenvalue of some symmetric operator; hence λ can be only purely imaginary. This elegant proof is based on the continuation of the eigenfunctions into the Hardy space $\mathcal{H}^2(\mathbb{D})$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In the preprint [5], it was proved that a wider class of operators possess only purely imaginary eigenvalues.

It was shown numerically in [10, 12] that the angle between the subspace spanned by the first N eigenfunctions and the $(N+1)$th eigenfunction of the operator L tends to 0 as N tends to infinity. This provides numerical evidence that the eigenfunctions do not form a Riesz basis in $\mathcal{L}^2(-\pi,\pi)$ because the related projectors are not uniformly bounded. The main result of this paper is an analytical proof of this numerical conjecture. Namely, we show that the operator iL is not similar to a self-adjoint operator and, consequently, its eigenfunctions do not form a Riesz basis (see Section 5).

In Section 4, we rigorously prove the completeness of eigenfunctions and the existence of an infinite number of eigenvalues of the operator L. Combining this result with [12, 39], it is easy to see that the sequence of eigenvalues is purely imaginary and accumulates at $\pm i\infty$. As a consequence of the linearity, the original Cauchy problem has infinitely many global in time solutions which are linear combinations of harmonics $e^{\lambda_n t}u_{\lambda_n}(x)$ where λ_n is a purely imaginary eigenvalue of the operator L and $u_{\lambda_n}(x)$ is the related eigenfunction. We wish to thank Weir for informing us about [40], in which the existence of the infinite number of eigenvalues is obtained by a different method.

The operator L is J-self-adjoint in the Krein space with $J(f(\theta)) = f(\pi-\theta)$ and therefore it belongs to the class of PT-symmetric operators. Interesting development of the spectral theory of PT-symmetric operators which are not similar to self-adjoint ones can be found in [26, 33, 34, 5].

Notation. In the sequel, $C_1, C_2, \dots$ denote constants that may change from line to line but remain independent of the appropriate quantities. We use $h(\theta)$ when h is considered as a function of one variable and $h(\theta,\, t)$ when h is considered as a function of two variables. The symbol $\mathbb{1}$ denotes the function that identically equals 1 for $\theta \in (-\pi,\pi)$. Let T be a linear operator in a Hilbert space H. The following classic notations are used: $\mathrm{Dom}(T)$, $\mathrm{Ker}(T)$, $\mathrm{Ran}(T)$ are the domain, the kernel, and the range of T, respectively; $\sigma(T)$ and $\rho(T)$ denote the spectrum and the resolvent set of T; $\sigma_p(T)$ stands for the set of eigenvalues of T. We write $f(x) \asymp g(x) \quad (x \to x_0)$ if both f/g and g/f are bounded functions in a certain neighborhood of x_0. By $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ we denote the open unit disc in $\mathbb{C}$. In Sections 2 and 3, we use the Sobolev spaces $W_2^k(-\pi,\pi)$ and $W_2^{k_1,k_2}(Q)$ defined as in [25, Section I.1.2].

2. Analysis of the differential equation

The main aim of this section is to show that some of the spectral properties of the differential operator explicitly defined by (2.15) and of the operator considered in [12, 39] coincide. We use this fact in Proposition 4.2 to prove that these operators coincide. The latter is crucial for Section 5, where results obtained in Section 3 for the parabolic equation (1.1) are combined with results of [12, 39] to prove Theorem 5.6.

One of the most general linear second-order differential equations with periodic coefficients that can be solved by a trigonometric series with a three-term recursion relation between the coefficients was studied by Magnus and Winkler [1] and has the form

$$(A + B\cos(2\theta))\, y'' + C\sin(2\theta)\, y' + (D + E\cos(2\theta))y = 0, \tag{2.1}$$

where A, B, C, D, E are constants. Under the additional condition that the coefficient $A + B\cos(2\theta)$ does not have zeros located on the real axis they studied existence of the periodic solutions to (2.1).

In this section we study the basic properties of the differential equation $\ell[h](x) = f(x)$, where the differential expression ℓ is given by

$$\ell[h] := \varepsilon\frac{d}{d\theta}\left(\sin(\theta)\frac{dh}{d\theta}\right) + \frac{dh}{d\theta}, \quad \theta \in (-\pi, \pi), \tag{2.2}$$

and use these properties to define the maximal periodic differential operator associated with ℓ. This equation can be transformed to the form (2.1), but all singularities are located on the real axis, so the additional condition of Magnus and Winkler is not satisfied.

Let $f \in \mathcal{L}^2(-\pi, \pi)$ and $\varepsilon > 0$. Denote $\mathcal{I}_+ = (0, \pi)$, $\mathcal{I}_- = (-\pi, 0)$. Consider the differential equation

$$\ell[h](x) = f(x) \quad \text{a.e. on} \quad (-\pi, \pi) \tag{2.3}$$

assuming that the function h is such that

$$h,\, \varepsilon\sin(\theta)h' + h \in AC_{\text{loc}}(\mathcal{I}_- \cup \mathcal{I}_+), \tag{2.4}$$

i.e., h and $\varepsilon\sin(\theta)h' + h$ are absolutely continuous on each closed subinterval of $\mathcal{I}_- \cup \mathcal{I}_+$.

Lemma 2.1. *Let h satisfy* (2.4)*. Then h is a solution of the equation $\ell[h](x) = f(x)$ if and only if h has the form*

$$h(\theta) = |\cot(\theta/2)|^{1/\varepsilon}\left(k_2^{\pm} - \int_{\pm\pi/2}^{\theta} f(t)|\tan(t/2)|^{1/\varepsilon}dt\right) + \int_0^{\theta} f(t)dt + k_1^{\pm}, \tag{2.5}$$

where $\theta \in \mathcal{I}_\pm$, and $k_1^{\pm}$, $k_2^{\pm}$ are arbitrary constants.

The proof is based on direct calculations.

Proposition 2.2. *Assume that* $\varepsilon \in (0,2)$*. A function* $h \in \mathcal{L}^2(-\pi,\pi)$ *satisfies* (2.4) *and is a solution of the equation* $\ell[h] = f$ *with* $f \in \mathcal{L}^2(-\pi,\pi)$ *if and only if* h *has the form*

$$h(\theta) = -\left|\cot\frac{\theta}{2}\right|^{1/\varepsilon} \int_0^\theta f(t)\left|\tan\frac{t}{2}\right|^{1/\varepsilon} dt + \int_0^\theta f(t)dt + k_1^\pm, \quad \theta \in \mathcal{I}_\pm, \tag{2.6}$$

where $k_1^\pm$ *are arbitrary constants.*

Proof. Assume that h is an $\mathcal{L}^2(-\pi,\pi)$-solution of $\ell[h] = f$. Then it has the form (2.5). Note that

$$f(t)\left|\tan\frac{t}{2}\right|^{1/\varepsilon} \in \mathcal{L}^1(0,\delta) \quad \text{for any} \quad \delta \in (0,\pi).$$

Therefore there exist the finite limit

$$C_1 := \lim_{\theta\to+0}\left(k_2^\pm - \int_{\pm\pi/2}^\theta f(t)\left|\tan\frac{t}{2}\right|^{1/\varepsilon} dt\right).$$

If $C_1 \neq 0$, then (2.5) implies that $|h(\theta)| \geq |C_2|\ \theta^{-1/\varepsilon}$ for $\theta > 0$ small enough, where $C_2 > 0$. Since $\varepsilon < 2$, we see that $h \notin \mathcal{L}^2(0,\pi)$. This shows that $C_1 = 0$, and therefore h has the form (2.6) on $(0,\pi)$. Similarly, one can show that h has the form (2.6) on $(-\pi,0)$.

Let us prove that any function h of the form (2.6) belongs to $\mathcal{L}^2(0,\pi)$ (the proof for $\mathcal{L}^2(-\pi,0)$ is the same). It is enough to check that $h \in \mathcal{L}^2(0,\delta)$ and $h \in \mathcal{L}^2(\pi-\delta,\pi)$ for sufficiently small $\delta > 0$.

For $\theta \in (0,\delta)$, we have

$$\int_0^\theta |f(t)|\ \left|\tan\frac{t}{2}\right|^{1/\varepsilon} dt \leq C_3\|f\|_{\mathcal{L}^2}\left(\int_0^\theta t^{2/\varepsilon}dt\right)^{1/2} = C_3\|f\|_{\mathcal{L}^2}\ \theta^{1/2+1/\varepsilon}.$$

Hence,

$$\left|\cot\frac{\theta}{2}\right|^{1/\varepsilon}\int_0^\theta |f(t)|\ \left|\tan\frac{t}{2}\right|^{1/\varepsilon} dt \leq 2^{-1/\varepsilon}C_3\ \|f\|_{\mathcal{L}^2}\ \theta^{1/2}, \tag{2.7}$$

and we finally see that $h \in \mathcal{L}^2(0,\delta)$.

For $\theta \in (\pi-\delta,\pi)$, we have

$$\begin{aligned}\int_0^\theta |f(t)|\ \left|\tan\frac{t}{2}\right|^{1/\varepsilon} dt &\leq C_4 + 2^{1/\varepsilon}\ \|f\|_{\mathcal{L}^2}\left(\int_{\pi-\delta}^\theta (\pi-t)^{-2/\varepsilon}dt\right)^{1/2}\\ &\leq C_5 + 2^{1/\varepsilon}\ \|f\|_{\mathcal{L}^2}\,(\pi-\theta)^{1/2-1/\varepsilon}.\end{aligned}$$

Hence,

$$\left|\cot\frac{\theta}{2}\right|^{1/\varepsilon}\int_0^\theta |f(t)|\ \left|\tan\frac{t}{2}\right|^{1/\varepsilon} dt \leq C_6(\pi-\theta)^{1/\varepsilon} + C_7\ \|f\|_{\mathcal{L}^2}(\pi-\theta)^{1/2}.$$

So $h \in \mathcal{L}^2(\pi-\delta,\pi)$. $\square$

In particular, we have proved that

$$\lim_{\theta\to\pm 0} h(\theta) = k_1^{\pm} \quad \text{and} \quad \lim_{\theta\to\pm\pi\mp 0} h(\theta) = k_1^{\pm} + \int_0^{\pm\pi} f(t)dt$$

hold for any $\mathcal{L}^2$-solution h. This implies that the condition

$$h \text{ is continuous on } [-\pi,\pi] \text{ and periodic} \tag{2.8}$$

is fulfilled exactly when

$$k_1^+ = k_1^- \quad \text{and} \quad f \perp \mathbb{1}.$$

Let the symbol $\mathrm{W}^k_{2p}(-\pi,\pi)$ stand for the subspace of the space $\mathrm{W}^k_2(-\pi,\pi)$ consisting of periodic functions, i.e., functions satisfying the conditions

$$u^{(i)}(\pi) = u^{(i)}(-\pi), \qquad i = 0,1,\dots,k-1.$$

The norm in this space coincides with that of the Sobolev space $\mathrm{W}^k_2(-\pi,\pi)$.

Proposition 2.3. *Let $\varepsilon \in (0,2)$, $f \in \mathcal{L}^2(-\pi,\pi)$, and $\int_{-\pi}^{\pi} f(\theta)d\theta = 0$. Then an $\mathcal{L}^2$-solution of $\ell[h] = f$ satisfies* (2.8) *if and only if*

$$h(\theta) = -\left|\cot\frac{\theta}{2}\right|^{1/\varepsilon} \int_0^\theta f(t)\left|\tan\frac{t}{2}\right|^{1/\varepsilon} dt + \int_0^\theta f(t)dt + k_1 \tag{2.9}$$

for a.a. $\theta \in (-\pi,\pi)$, where k_1 is an arbitrary constant. Moreover, any function h of the form (2.9) *possesses the following properties:*

- (i) *$h \in AC[-\pi,\pi]$, $h' \in \mathcal{L}^2(-\pi,\pi)$, and*

$$\|h'\|_{\mathcal{L}^2} \le K(|k_1| + \|f\|_{\mathcal{L}^2}), \tag{2.10}$$

  where K is a constant independent of f.
- (ii) *$\sin(\theta)h' \in AC[-\pi,\pi]$ and $(\sin(\theta)h')' \in \mathcal{L}^2(-\pi,\pi)$.*

Proof. To show that $h' \in \mathcal{L}^2(-\pi,\pi)$, it is enough to prove that $h' \in \mathcal{L}^2(0,\delta)$ for any $\delta > 0$ small enough. Since

$$h'(\theta) = -\frac{1}{\varepsilon\sin(\theta)} \cot^{1/\varepsilon}\left(\frac{\theta}{2}\right) \int_0^\theta f(t)\tan^{1/\varepsilon}\left(\frac{t}{2}\right) dt, \quad \theta \in (0,\delta),$$

it is sufficient to show that

$$\int_0^\delta \theta^{-2-2/\varepsilon}\left(\int_0^\theta f(t)t^{1/\varepsilon}dt\right)^2 d\theta \le C_1 \int_0^\delta |f(\theta)|^2 d\theta \tag{2.11}$$

for any $f \in \mathcal{L}^2(0,\delta)$.

Denote $g(t) := f(t)t^{1/\varepsilon}$. Then (2.11) takes the form

$$\int_0^\delta \theta^{-2-2/\varepsilon}\left(\int_0^\theta g(t)dt\right)^2 d\theta \le C_1 \int_0^\delta |g(\theta)|^2\theta^{-2/\varepsilon}d\theta \ . \tag{2.12}$$

This is a weighted norm inequality for the Hardy operator. Applying [28] (see also [32] and references therein), we see that

$$\sup_{\theta\in[0,\delta)}\left(\int_\theta^\delta t^{-2-2/\varepsilon}dt\right)^{1/2}\left(\int_0^\theta t^{2/\varepsilon}\,dt\right)^{1/2}<\infty,\tag{2.13}$$

and therefore (2.12) holds true. It is easy to see that the latter implies (2.10) and statement (i) of the theorem.

If $f\perp\mathbb{1}$, then $\sin(\theta)\,h'+h\in \mathrm{W}^1_{2p}(-\pi,\pi)$ and, by claim (i), so is $\sin(\theta)\,h'$. $\square$

Introduce the space $X_2=\{h\in \mathrm{W}^1_{2p}(-\pi,\pi)\,:\,\sin(\theta)h'\in \mathrm{W}^1_2(-\pi,\pi)\}$ endowed with the quasi-norm

$$\|h\|^2_{X_2}=\|h'\|^2_{\mathcal{L}^2(-\pi,\pi)}+\|\sin(\theta)h'\|^2_{\mathrm{W}^1_2(-\pi,\pi)}.$$

Denote by X_2^0 the subspace of X_2 comprising the functions h with the property $\int_{-\pi}^{\pi}h(\theta)\,d\theta=0$. In turn, one can get the following statement.

Proposition 2.4. *The set X_2^0 equipped with the norm $\|\cdot\|_{X_2}$ is a Hilbert space.*

As a consequence of the definitions, we obtain that if $h\in X_2$, then the function $\sin(\theta)\,h'$ is absolutely continuous (may be after a change on a set of zero measure) and

$$\sin(\theta)\,h'|_{\theta=0}=\sin(\theta)\,h'|_{\theta=\pi}=\sin(\theta)\,h'|_{\theta=-\pi}=0.\tag{2.14}$$

Let the symbol $\mathcal{L}^2_p(-\pi,\pi)$ stand for the subspace of $\mathcal{L}^2(-\pi,\pi)$ comprising the functions f with the property $\int_{-\pi}^{\pi}f(\theta)\,d\theta=0$.

We present below a set of corollaries of Proposition 2.3. We also give the alternative proof of this result using the Galerkin method in Appendix A.

Assume $0<\varepsilon<2$ and denote by L the operator acting in $\mathcal{L}^2(-\pi,\pi)$ and defined by

$$Lf=\ell[f]\quad\text{for}\quad f\in\mathrm{Dom}(L):=X_2.\tag{2.15}$$

Clearly, $\mathrm{Ker}(L)=\{c\mathbb{1},\ c\in\mathbb{C}\}$.

Let us denote by $\widetilde{L}$ the restriction of L to $\mathcal{L}^2_p(-\pi,\pi)$ $\big(=(\mathrm{Ker}(L))^\perp\big)$,

$$\widetilde{L}:=L\restriction\mathcal{L}^2_p(-\pi,\pi),\quad \mathrm{Dom}(\widetilde{L}):=\mathrm{Dom}(L)\cap\mathcal{L}^2_p(-\pi,\pi).\tag{2.16}$$

It follows from the remark after (2.8) that $\mathrm{Ran}(L)\subset\mathcal{L}^2_p(-\pi,\pi)$. So $\widetilde{L}$ is an operator in the Hilbert space $\mathcal{L}^2_p(-\pi,\pi)$.

To find the inverse operator $\widetilde{L}^{-1}$, let us symmetrize (2.9) as

$$h(\theta)=-\left|\cot\frac{\theta}{2}\right|^{1/\varepsilon}\int_0^\theta f(t)\left(\left|\tan\frac{t}{2}\right|^{1/\varepsilon}-\left|\tan\frac{\theta}{2}\right|^{1/\varepsilon}\right)dt+k_1,\tag{2.17}$$

where $\theta\in(-\pi,\pi)$. Solving the equation $(h,\mathbb{1})=0$, we get

$$k_1=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left[\left|\cot\frac{\theta}{2}\right|^{1/\varepsilon}\int_0^\theta f(t)\left(\left|\tan\frac{t}{2}\right|^{1/\varepsilon}-\left|\tan\frac{\theta}{2}\right|^{1/\varepsilon}\right)dt\right]d\theta.\tag{2.18}$$

So for $f \in \mathrm{Ran}(L)$, we have $\widetilde{L}^{-1} f = h$ with h defined by (2.17)-(2.18).

Corollary 2.5. *Assume* $0 < \varepsilon < 2$.

(i) *The operator* L *defined by* (2.15) *is a closed operator in* $\mathcal{L}^2(-\pi,\pi)$ (*with the dense domain* X_2).

(ii) *Its kernel* $\mathrm{Ker}(L)$ *is the one-dimensional subspace of constants* $\{c\mathbb{1},\ c \in \mathbb{C}\}$.

(iii) *The range* $\mathrm{Ran}(L)$ *of* L *is the orthogonal complement to* $\mathrm{Ker}(L)$, *i.e.,* $\mathrm{Ran}(L) = \mathcal{L}^2_p(-\pi,\pi)$.

(iv) *The operator* $\widetilde{L}$ *defined by* (2.16) *has a compact inverse* $\widetilde{L}^{-1}$.

(v) *The operator* $\widetilde{L} : X_2^0 \to \mathcal{L}^2_p(-\pi,\pi)$ *is an isomorphism of* X_2^0 *onto* $\mathcal{L}^2_p(-\pi,\pi)$.

Proof. (i)–(iii). Let $k_1 = k_1(f)$ be the linear functional defined by (2.18). It is easy to see that

$$k_1 \quad \text{is bounded on} \quad \mathcal{L}^2(-\pi,\pi). \tag{2.19}$$

This and Proposition 2.3 imply immediately that $\mathrm{Ran}(L) = \mathcal{L}^2_p(-\pi,\pi)$. Therefore, we have $\mathrm{Dom}(\widetilde{L}^{-1}) = \mathcal{L}^2_p(-\pi,\pi)$. It follows from (2.10) and

$$\|h'\|_{\mathcal{L}^2(-\pi,\pi)} \geq \|h\|_{\mathcal{L}^2(-\pi,\pi)}, \quad h \in \mathcal{L}^2_p(-\pi,\pi),$$

that $\widetilde{L}^{-1}$ is a bounded linear operator on $\mathcal{L}^2_p(-\pi,\pi)$. Therefore $\widetilde{L}$ is closed and so is L.

(iv)–(v). It follows from (2.10) and (2.19) that $\widetilde{L}^{-1}$ is bounded as an operator from $\mathcal{L}^2_p(-\pi,\pi)$ onto X_2^0. This proves statement (v). Note also that

$$\|h\|_{W_2^1} \leq 2\|h\|_{X_2^0} \qquad \text{for any } h \in X_2^0.$$

This implies that $\widetilde{L}^{-1}$ is a compact operator in $\mathcal{L}^2_p(-\pi,\pi)$. □

The adjoint differential operation ℓ^* is given by

$$\ell^*[h] := \varepsilon \frac{d}{d\theta}\left(\sin(\theta)\frac{dh}{d\theta}\right) - \frac{dh}{d\theta}, \quad \theta \in (-\pi,\pi).$$

It is easy to see that $\ell^*[h] = J\ell[Jh]$, where $(Jh)(\theta) = h(\pi - \theta)$.

Proposition 2.6. *Let* $0 < \varepsilon < 2$. *The adjoint operator* L^* *has the form*

$$L^* h = \ell^*[h], \qquad \mathrm{Dom}(L^*) = X_2.$$

Moreover, $L = JL^*J$ *and so* L^* *is unitary equivalent to* L.

Proof. Integrating by parts and taking into account (2.14), one can see that $X_2 \subset \mathrm{Dom}(L^*)$.

Let us show that $L = JL^*J$ and therefore $X_2 = \mathrm{Dom}(L^*)$. Consider the operator F defined by $F = JLJ$. Then $F \subset L^*$. Since $L = 0 \oplus \tilde{L}$ with respect to the decomposition

$$\mathcal{L}^2(-\pi,\pi) = \{c\mathbb{1},\ c \in \mathbb{C}\} \oplus \mathcal{L}^2_p(-\pi,\pi),$$

we see that $L^* = 0 \oplus \tilde{L}^*$ and $F = 0 \oplus \tilde{F}$ with respect to the same decomposition. Here $\tilde{L}^* = (\tilde{L})^*$ and $\tilde{F} := \tilde{J}\tilde{L}\tilde{J}$, where $\tilde{J} := J \restriction \mathcal{L}^2_p(-\pi,\pi)$ is a unitary operator in $\mathcal{L}_p^2(-\pi,\pi)$. By Corollary 2.5 (iii), $\mathrm{Ran}(\tilde{L}) = \mathrm{Ran}(\tilde{F}) = \mathcal{L}_p^2(-\pi,\pi)$ and $\tilde{F}^{-1}$ is a compact operator on $\mathcal{L}_p^2(-\pi,\pi)$. On the other hand, $(\tilde{L}^*)^{-1} = (\tilde{L}^{-1})^*$ is also a compact operator on $\mathcal{L}_p^2(-\pi,\pi)$ and $\tilde{F}^{-1} \subset (\tilde{L}^*)^{-1}$. We see that $\tilde{F}^{-1} = (\tilde{L}^*)^{-1}$ and therefore $L^* = F$. ☐

The last proposition implies that the statements analogous to that of Corollary 2.5 are valid for L^*.

3. The ill-posedness of the Cauchy problem for the forward-backward heat equation

The linearized model of the thin film dynamic (1.1) was derived without taking into account the smoothing effect of the surface tension. It is very natural to expect that a drop of fluid will be detached from the “ceiling” of the rotating cylinder. That can be described as a blow-up and it perfectly fits into the ill-posed nature of the Cauchy problem for the forward-backward heat equation (1.1). The intuition based on the classic theory of backward heat equation, says that global in time classic solutions can exist only for some class of functions analytic in vertical strip $(-\pi,\pi)$ with exponentially fast decaying Fourier coefficients.

In this section we assume that $\varepsilon > 0$. Let us consider the parabolic problem

$$h_t + Lh = 0, \quad \theta \in (-\pi,\pi), \quad t \in (0,T), \tag{3.1}$$

$$h|_{t=0} = h_0(\theta), \quad h(-\pi,t) = h(\pi,t), \tag{3.2}$$

where L is understood in the sense of the differential expression (1.2). Note that, after the change of variables $\theta \to -\theta$, equation (3.1) can be replaced with the equation

$$u_t - Lu = 0, \qquad \text{where } u(\theta) = h(-\theta).$$

Let $Q = (-\pi,\pi) \times (0,T)$. We prove in this section that the problem (3.1)–(3.2) is ill-posed in the classes of finite smoothness. In what follows, the symbol $(\cdot,\cdot)$ stands for the inner product in the space $\mathcal{L}^2$ in the corresponding domain $(Q, Q_1, \ldots)$.

Definition 3.1. A function $h \in \mathcal{L}^2\left(0,T; \mathrm{W}_{2p}^1(-\pi,\pi)\right) \cap C\left([0,T]; \mathcal{L}^2(-\pi,\pi)\right)$ such that

$$-(h, v_t) - \varepsilon\,(\,\sin(\theta)\,h_\theta\,,\,v_\theta\,) + (h_\theta, v) = \int_{-\pi}^{\pi} h_0(\theta) v(\theta,0)\,d\theta$$

for all $v \in \mathcal{L}^2(0,T; \mathrm{W}_{2p}^1(-\pi,\pi))$, $v_t \in \mathcal{L}^2(Q)$, $v(\theta,T) = 0$, will be called a generalized solution to the problem (3.1)–(3.2) from the space $\mathrm{W}_{2p}^{1,0}(Q)$.

Similar definitions can be found in [25, sect. 1 of chap. 3, sect. 3 of chap. 1].

Theorem 3.2 (Nonexistence). *Let* $h_0 \in \mathcal{L}^2(-\pi,\pi)$. *If there exist* $\delta \in (0,\pi/2)$ *and* $k \in \mathbb{N}$ *such that* $h_0 \notin \mathrm{W}_2^k(\delta, \pi-\delta)$ *then the problem* (3.1)–(3.2) *does not have a generalized solution in* $\mathrm{W}_{2p}^{1,0}(Q)$.

Proof. Assume that a solution h of the problem exists. Take an arbitrary positive number $\delta \in (0, \pi/2)$ and put $Q_0 = (\delta, \pi - \delta) \times (0, T/2)$. We have

$$-(h, v_t) - \varepsilon\,(\,\sin(\theta)\,h_\theta\;,\;v_\theta) + (h_\theta, v) = \int_{-\pi}^{\pi} h_0(\theta) v(\theta, 0)\,d\theta \tag{3.3}$$

for all functions v as in Definition 3.1 such that

$$\operatorname{supp} v \;\subset\; Q_0 \cup \{\,(\theta, 0) : \theta \in (\delta, \pi - \delta)\,\}.$$

Make the change of variables $\tau = T/2 - t$. Then the function $\tilde{h}(\theta, \tau) = h(\theta, T/2 - \tau)$ is a generalized solution of the equation $\tilde{h}_\tau - L\tilde{h} = 0$ in Q_0 and

$$\tilde{h}|_{\tau = T/2} = h_0(\theta). \tag{3.4}$$

More exactly, we have $\tilde{h} \in \mathcal{L}^2\left(0, T/2;\, \mathrm{W}_2^1(\delta, \pi - \delta)\right) \cap C\left([0, T/2]; \mathcal{L}^2(\delta, \pi - \delta)\right)$ and

$$-(\tilde{h}, v_\tau) + \varepsilon\,(\,\sin(\theta)\,\tilde{h}_\theta\;,\;v_\theta\,) - (\tilde{h}_\theta, v) = -\int_{\delta}^{\pi - \delta} h_0(\theta)\,v(\theta, T/2)\,d\theta \tag{3.5}$$

for all

$$v \in \mathcal{L}^2\left(0, T/2;\, \mathrm{W}_2^1(\delta, \pi - \delta)\right) \;:\; v_t \in \mathcal{L}^2(Q_0),\; v(\theta, 0) = 0,\; v|_{\theta = \delta} = v|_{\theta = \pi - \delta} = 0.$$

Now construct a function h_1 being a generalized solution to the problem

$$h_\tau - Lh = 0, \;\; h|_{\theta = \delta} = h|_{\theta = \pi - \delta} = 0, \;\; h|_{\tau = T/2} = h_0(\theta),$$

in the domain $Q_1 = (\delta, \pi - \delta) \times (T/2, T)$. Using the conventional parabolic theory (see, for instance, [25, Theorem III.4.2], or [14, Theorem 3 of Sect. 7.1.2]), we obtain that a solution of this problem exists and

$$h_1 \in \mathcal{L}^2\left(0, T/2; \mathrm{W}_2^1(\delta, \pi - \delta)\right) \;\cap\; C\left([T/2, T]; \mathcal{L}^2(\delta, \pi - \delta)\right).$$

This solution satisfies the integral identity (see the definitions of a generalized solution in [25, Section III.1]).

$$-(h_1, v_\tau) + \varepsilon(\sin(\theta)\,h_{1\theta}, v_\theta) - (h_{1\theta}, v) = \int_{\delta}^{\pi - \delta} h_0(\theta) v(\theta, T/2)\,d\theta \tag{3.6}$$

for all $v \in \mathcal{L}^2(T/2, T; \mathrm{W}_2^1(\delta, \pi - \delta)) : v_\tau \in \mathcal{L}^2(Q_1), v(\theta, T) = 0, v|_{\theta = \delta} = v|_{\theta = \pi - \delta} = 0$. Put $Q_2 = (\delta, \pi - \delta) \times (0, T)$. By adding (3.5) and (3.6) we conclude that

$$-(h_2, v_\tau) + \varepsilon(\sin(\theta)\,h_{2\theta}, v_\theta) - (h_{2\theta}, v) = 0 \tag{3.7}$$

for all

$$v \in \mathcal{L}^2\left(0, T;\, \mathrm{W}_2^1(\delta, \pi - \delta)\right) \;:\; v_\tau \in \mathcal{L}^2(Q_2),\, v(\theta, T) = 0,\, v(\theta, 0) = 0,$$
$$v|_{\theta = \delta} = v|_{\theta = \pi - \delta} = 0.$$

Here the function h_2 coincides with $\tilde{h}$ for $t \in (0, T/2)$ and with h_1 for $t \in (T/2, T)$. So the function h_2 is a generalized solution of the equation

$$h_\tau - Lh = 0$$

in Q_2 in the sense of the integral identity (3.7). Since the coefficients of the operator L are infinitely differentiable, we can apply [25, Theorem III.12.1] in which it is demonstrated that interior smoothness of a generalized solution of the parabolic equation is determined by smoothness of the coefficients of L. By this theorem, we have $h_2 \in C^\infty(Q_2)$. Therefore,

$$h_2(\theta, T/2) = h_1(\theta, T/2) = h_0(\theta) \in C^\infty(\delta, \pi - \delta).$$

Since the quantity δ is arbitrary, we can conclude that $h_0(\theta) \in C^\infty(0, \pi)$. It contradicts to the condition of the theorem that $h_0 \notin \mathrm{W}_2^k(\delta, \pi - \delta)$ for some $\delta > 0$. □

Remark 3.3. In the proof of the theorem, we actually have established the following statement: if the problem (3.1)–(3.2) has a generalized solution in $\mathrm{W}_{2p}^{1,0}(Q)$ for some initial data $h_0 \in \mathcal{L}^2(-\pi, \pi)$, then $h_0 \in C^\infty(0, \pi)$.

Definition 3.4. The problem (3.1)–(3.2) is said to be densely solvable in $\mathrm{W}_{2p}^k(-\pi, \pi)$ ($k \geq 0$ is an integer number) if there exists a dense subset K of this space such that, for any $h_0 \in K$, the problem (3.1)–(3.2) has a generalized solution h in the sense of the Definition 3.1.

Theorem 3.5 (Instability). *Let the problem* (3.1)–(3.2) *be densely solvable in the space* $\mathrm{W}_{2p}^k(-\pi, \pi)$ *for some nonnegative integer* k*, and let* K *be the corresponding dense subset. Then there is no a constant* $c = c(k) > 0$ *such that, for every generalized solution* h *to the problem* (3.1)–(3.2) *with an initial value* $h_0 \in K$*, the estimate*

$$\|h\|_{\mathcal{L}^2(Q)} \leq c \|h_0\|_{\mathrm{W}_{2p}^k(-\pi,\pi)} \tag{3.8}$$

holds.

Proof. We use the arguments of Theorem 3.2. Assume to the contrary that (3.8) holds for some constant $c > 0$ and all $h_0 \in K$. We have that $K \subset \mathrm{W}_{2p}^k(-\pi, \pi)$ and $\overline{K} = \mathrm{W}_{2p}^k(-\pi, \pi)$. Given $\delta \in (0, \pi/2)$, find a function $h_0 \in \mathrm{W}_{2p}^k(-\pi, \pi)$ such that $h_0 \notin \mathrm{W}_2^{k+1}(\delta, \pi - \delta)$. Construct a sequence $h_{0n} \in K : \|h_{0n} - h_0\|_{\mathrm{W}_2^k(-\pi,\pi)} \to 0$ as $n \to \infty$. In accord with the conditions of the theorem, there exists a generalized solution h_n to the problem (3.1)–(3.2). By Remark 3.3, $h_{0n} \in C^\infty(0, \pi)$. As it is easily seen,

$$\|h_{0n}\|_{\mathrm{W}_2^{k+1}(\delta,\pi-\delta)} \to \infty \quad \text{as} \quad n \to \infty. \tag{3.9}$$

Otherwise, there exists a subsequence h_{0n_k} which is bounded in $\mathrm{W}_2^{k+1}(\delta, \pi - \delta)$ and, as a consequence, we can assume that $h_{0n_k} \to \tilde{h}_0$ as $k \to \infty$ in $\mathrm{W}_2^{k+1}(\delta, \pi - \delta)$ and $\mathrm{W}_2^k(\delta, \pi - \delta)$ weakly for some $\tilde{h}_0 \in \mathrm{W}_2^{k+1}(\delta, \pi - \delta)$. Due to the uniqueness of the limit, $\tilde{h}_0 = h_0$ on $(\delta, \pi - \delta)$. This contradicts to the fact that $h_0 \notin \mathrm{W}_2^{k+1}(\delta, \pi - \delta)$. Next, we repeat the arguments of the proof of Theorem 3.2. After the change of variables $\tau = T/2 - t$ we find that the functions $\tilde{h}_n = h_n(\theta, T/2 - \tau)$ satisfy the integral identity

$$-(\tilde{h}_n, v_\tau) + \varepsilon\,(\sin(\theta)\,\tilde{h}_{n\theta}\,, v_\theta\,) - (\tilde{h}_{n\theta}, v) = -\int_{\delta/2}^{\pi-\delta/2} h_{0n}(\theta)\, v(\theta, T/2)\, d\theta \tag{3.10}$$

for all $v \in \mathcal{L}^2\left(0, T/2; \mathrm{W}_2^1(\delta/2, \pi-\delta/2)\right)$ such that

$$v_t \in \mathcal{L}^2(Q_0),\ v(\theta,0)=0,\ v|_{\theta=\delta/2} = v|_{\theta=\pi-\delta/2}=0.$$

This means that the functions $\tilde{h}_n$ are generalized solutions to a parabolic equation in Q_0 in the sense of an integral identity (see [25, Section III.1]). Note that coefficients of our parabolic equation (3.1) are infinitely differentiable, and, by [25, Theorem III.12.1], this solution belongs to any Hölder space $H^{2+\alpha,1+\alpha/2}(Q_0)$ (see the definition of this space in [25]). We put

$$Q_1 = (\delta', \pi-\delta') \times (\varepsilon_0, T/2), \quad \text{where} \quad \delta/2 < \delta' < \pi/2, \quad 0 < \varepsilon_0 < T/2.$$

Since our solution can be extended to a generalized solution in the domain $Q_2 = (\delta/2, \pi-\delta/2) \times (0,T)$ (see the proof of Theorem 3.2), we can say that $\tilde{h}_n \in H^{2+\alpha,1+\alpha/2}(\overline{Q_1})$ for all $\delta' > \delta/2$, $\varepsilon_0 > 0$. By [25, Theorem III.8.1], for every $\delta' > \delta/2$ and $\varepsilon_0 > 0$ the norm of $\tilde{h}_n$ in $C(\overline{Q_1})$ is estimated by some constant depending on δ', $\varepsilon_0 > 0$, and the norm of $\tilde{h}_n$ in $L_2(Q_0)$. Using this fact and applying [25, Theorem IV.10.1], we obtain that the norm of $\tilde{h}_n$ in any Hölder space $H^{2+\alpha,1+\alpha/2}(\overline{Q_1})$ is estimated by some constant depending on $\delta', \varepsilon_0, \alpha$, and the norm $\|\tilde{h}_n\|_{\mathcal{L}^2(Q_0)}$. Assume that $\delta' < \delta$ and $\alpha \geq k-1$. In view of (3.8) with h_n, h_{0n} substituted for h, h_0 and the fact that the norms $\|h_{0n}\|_{\mathrm{W}_2^k(-\pi,\pi)}$ are bounded, we can assume that this constant is independent of n. As a consequence, we have the estimate

$$\|\tilde{h}_n(\theta, T/2)\|_{\mathrm{W}_2^{k+1}(\delta,\pi-\delta)} = \|h_{0n}(\theta)\|_{\mathrm{W}_2^{k+1}(\delta,\pi-\delta)} \leq c(\delta,k), \tag{3.11}$$

where the constant $c(\delta, k)$ is independent of n. Comparing (3.11) and (3.9), we arrive at a contradiction. ☐

4. The completeness property for the operator L

In this section we restrict the parameter to the interval $0 < \varepsilon < 2$. In Proposition 4.2, we show that the operator L defined by (2.15) coincides with the closed operator introduced in [12]. The main aim of this section is to prove that the system of all eigenvectors and generalized eigenvectors of the operator L is complete in $\mathcal{L}^2(-\pi,\pi)$. In particular, this implies that L has an infinite number of eigenvalues.

Denote by $\mathcal{H}^{2,0}(\mathbb{D})$ and $\mathcal{H}^{2,0}(\overline{\mathbb{C}} \setminus \mathbb{D})$ the subspaces of the Hardy spaces $\mathcal{H}^2(\mathbb{D})$ and $\mathcal{H}^2(\overline{\mathbb{C}} \setminus \mathbb{D})$, respectively, that are orthogonal to the function $\mathbb{1}$ (for basic facts on $\mathcal{H}^2(\mathbb{D})$ and $\mathcal{H}^2(\overline{\mathbb{C}} \setminus \mathbb{D})$ see e.g. [15, Section 2.1]). In the sequel, we use the standard identification of the function $u(z) \in \mathcal{H}^2(\mathbb{D})$ with the function $u(e^{i\theta}) := \lim_{r \nearrow 1} u(re^{i\theta})$, which belongs to $\mathcal{L}^2(-\pi,\pi)$, and also use the similar agreement for $u(z) \in \mathcal{H}^2(\overline{\mathbb{C}} \setminus \mathbb{D})$. Then $\mathcal{H}^2(\mathbb{D})$ and $\mathcal{H}^2(\overline{\mathbb{C}} \setminus \mathbb{D})$ are the subspaces of $\mathcal{L}^2(-\pi,\pi)$ and $\mathcal{H}^2(\mathbb{D}) \cap \mathcal{H}^2(\overline{\mathbb{C}} \setminus \mathbb{D}) = \{c\mathbb{1},\ c \in \mathbb{C}\}$. In these terms, the space $\mathcal{L}^2(-\pi,\pi)$ admits the orthogonal decomposition

$$\mathcal{L}^2(-\pi,\pi) = \mathcal{H}^{2,0}(\mathbb{D}) \oplus \{c\mathbb{1},\ c \in \mathbb{C}\} \oplus \mathcal{H}^{2,0}(\overline{\mathbb{C}} \setminus \mathbb{D}). \tag{4.1}$$

Define the operator L_{fin} in the Hilbert space $\mathcal{L}^2(-\pi,\pi)$ by $L_{\text{fin}} h := \ell[h]$, $\operatorname{Dom}(L_{\text{fin}}) = P_{\text{fin}}$, where ℓ is the differential expression defined in Section 2 and P_{fin} is the set of finite trigonometric polynomials

$$h(\theta) = (2\pi)^{-1/2} \sum_{n=-N}^{N} v_n e^{in\theta}, \quad N < \infty, \quad v_n \in \mathbb{C}.$$

It is easy to see that L_{fin}^* is densely defined, and hence the closure

$$L_{\min} := \overline{L_{\text{fin}}}$$

exists as an operator in $\mathcal{L}^2(-\pi,\pi)$.

Let L be the operator defined by (2.15) and let $\widetilde{L}$ be its restriction defined by (2.16). Below we give a proof for $L_{\min} = L$ using the results of [12]. This proof allows us to use the orthogonal decomposition of L obtained in [12] (see also [10]).

Proposition 4.1 (Theorems 11 and 13 in [12]**).** (i) *The operator $L_{\min}$ admits the orthogonal decomposition $L_{\min} = L_- \oplus \mathbf{0} \oplus L_+$ with respect to* (4.1).

(ii) *The operators $L_\pm$ are invertible, and their inverses $L_\pm^{-1}$ are Hilbert-Schmidt operators.*

(iii) *L_+^{-1} and $(-L_-)^{-1}$ are unitary equivalent.*

Proposition 4.2. *Let L and $\widetilde{L}$ be the operators defined by* (2.15) *and* (2.16)*, respectively. Then $L = L_{min}$ and $\widetilde{L} = L_- \oplus L_+$.*

Proof. By Corollary 2.5 (i), L is a closed extension of L_{fin}. Hence $L_{\min} \subset L$. Let us show that $L = L_{\min}$.

By Proposition 4.1, the operator $L_+^{-1} \oplus L_-^{-1}$ is compact and is acting on $\mathcal{L}_p^2(-\pi,\pi)$. So $\sigma(L_- \oplus L_+)$ is at most countable. Then

$$\operatorname{Ran}(L_- \oplus L_+ - \lambda I) = \mathcal{L}_p^2(-\pi,\pi) \quad \text{for any} \quad \lambda \in \rho(L_- \oplus L_+). \tag{4.2}$$

By Corollary 2.5 (iv), the operator $\widetilde{L}^{-1}$ acting on $\mathcal{L}_p^2(-\pi,\pi)$ is compact. So $\widetilde{L}$ possesses the same properties, that is, $\sigma(\widetilde{L})$ is at most countable and (4.2) holds for $\widetilde{L}$.

Assume that $L_+ \oplus L_- \subsetneqq \widetilde{L}$ (which is equivalent to $L_{\min} \subsetneqq L$). Then (4.2) and the analogous equation for $\widetilde{L}$ imply that $\lambda \in \sigma_p(\widetilde{L})$ whenever $\lambda \in \rho(\widetilde{L}) \cap \rho(L_+ \oplus L_-)$, a contradiction. □

Definition 4.3 (see e.g. [16]**).** By $\mathfrak{S}_p$, $0 < p < \infty$, we denote the class of all bounded linear operators A acting on a Hilbert space H for which

$$|A|_p := \left(\sum_{j=1}^{\infty} s_j^p(A) \right)^{1/p} < \infty$$

where $s_j(A)$ are singular numbers of A, i.e, eigenvalues of the self-adjoint operator $(A^*A)^{1/2}$ that are enumerated in decreasing order, counted with multiplicities.

Two $\mathfrak{S}$-classes were given special names: $\mathfrak{S}_2$ is the class of Hilbert-Schmidt operators and $\mathfrak{S}_1$ is the class of nuclear operators. It was proved by Davies [12] that the operators $A_{\pm}^{-1}$, where

$$A_{\pm} := -iL_{\pm}, \tag{4.3}$$

belong to the class $\mathfrak{S}_2$ and so does $\widetilde{L}^{-1}$. We prove below the stronger statement that the operator $\widetilde{L}^{-1}$ is nuclear.

We need the following result of Gohberg and Markus (see [17], a weaker version can be found e.g. in [16, Section III.7.8]).

Theorem 4.4 **([17]).** *If $0 < p \leq 2$, then a linear operator A acting on a Hilbert space H belongs to $\mathfrak{S}_p$ if and only if for at least one orthonormal basis $\{e_j\}$ of H the inequality*

$$\sum_{j=1}^{\infty} \|Ae_j\|^p < \infty \tag{4.4}$$

holds. In addition,

$$|A|_p^p \leq \sum_{j=1}^{\infty} \|Ae_j\|^p \leq \sum_{j,k=1}^{\infty} |(Ae_j, e_k)|^p.$$

It follows from [12, Theorem 11 and Eq. (15)] that the operator $iL_+^{-1} (= A_+^{-1})$ in the Fourier basis $\{e_n\}_1^{\infty}$, $e_n(\theta) = e^{in\theta}$, is represented by the matrix $(\rho_{m,n})$ which has the following properties:

$$\begin{array}{ll} |\rho_{m,n}| \leq C_1\, m^{-1+1/\varepsilon}\; n^{-1-1/\varepsilon}, & m \leq n, \\ |\rho_{m,n}| \leq C_1\, m^{-1-1/\varepsilon}\; n^{-1+1/\varepsilon}, & n < m. \end{array} \tag{4.5}$$

Lemma 4.5 **([12]).** *Then $\sum\limits_{m=1}^{\infty} |\rho_{m,n}|^2 \leq K_1 n^{-3}$, where $K_1 > 0$ is a certain constant.*

Remark 4.6. This fact was obtained in [12, the proof of Theorem 11]. The authors thank E.B. Davies for communicating the proof of Lemma 4.5 to us.

Proposition 4.7. *The operator $\widetilde{L}^{-1}$ is nuclear. More precisely, $\widetilde{L}^{-1} \in \mathfrak{S}_p$ for any $p > 2/3$.*

Proof. Since $\|L_+^{-1}e_n\|_{\mathcal{L}^2}^2 = \sum\limits_{m=1}^{\infty} |\rho_{m,n}|^2$, Lemma 4.5 shows that

$$\sum_{n=1}^{\infty} \|L_+^{-1}e_n\|_{\mathcal{L}^2}^p \leq C_3 \sum_{n=1}^{\infty} n^{-3p/2}.$$

So the Gohberg-Markus criterion (Theorem 4.4) implies $L_+^{-1} \in \mathfrak{S}_p$ for any $2/3 < p \leq 2$. Thus L_+^{-1} belongs to $\mathfrak{S}_p$ for any $p > 2/3$ and so does $\widetilde{L}^{-1}$ due to Propositions 4.1 (ii) and 4.2. □

Although the weaker result that the operator $\widetilde{L}^{-1} \in \mathfrak{S}_p$ for $p > 1$ can be obtained directly from the factorization of the operator L found by Chugunova and Strauss in [11], the fact that the operator $\widetilde{L}^{-1}$ actually belongs to the class of nuclear operators $\mathfrak{S}_1$ is crucial for the proof of the completeness property (see below).

Following [16, Section IV.4], we call an operator T acting in a Hilbert space H *dissipative* if

$$\operatorname{Im}(Tf, f) \geq 0 \quad \text{for all} \quad f \in \operatorname{Dom}(T). \tag{4.6}$$

Proposition 4.8. *The operators L_+, $(-L_+)^{-1}$, $-L_-$ and L_-^{-1} are dissipative.*

Proof. Using the tridiagonal matrix representation of $A_+(= -iL+)$ with respect to the Fourier basis $\{e^{in\theta}\}_1^\infty$ (see [10, 12]), we get

$$A_+ = (a_{n,m})_1^\infty = \begin{bmatrix} 1 & -\varepsilon & 0 & 0 & \cdots \\ \varepsilon & 2 & -3\varepsilon & 0 & \cdots \\ 0 & 3\varepsilon & 3 & -6\varepsilon & \cdots \\ 0 & 0 & 6\varepsilon & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{4.7}$$

$$a_{n,n} = n, \quad a_{n-1,n} = \frac{\varepsilon}{2} n(n-1), \quad a_{n,n+1} = -\frac{\varepsilon}{2} n(n+1), \quad n = 1, 2, \dots .$$

This representation implies that

$$\operatorname{Im}(L_+h, h) = \operatorname{Re}(A_+h, h) \geq 0 \tag{4.8}$$

for all $h \in \mathcal{H}^{2,0}(\mathbb{D}) \cap P_{\text{fin}}$. Since $L = \overline{L_{\text{fin}}}$, one gets (4.8) for all $h \in \operatorname{Dom}(L^+)$, i.e., L_+ is dissipative. Substituting $h = (L_+)^{-1}f$ into (4.8), we see that so is $(-L_+)^{-1}$. Proposition 4.1 (iii) completes the proof. □

Theorem 4.9 (Lidskii, see e.g. [16, Theorem V.2.3]**).** *If the dissipative operator A acting on a Hilbert space H belongs to the class $\mathfrak{S}_1$, then its system of all eigenvectors and generalized eigenvectors is complete in H.*

Now the main result of this section can be obtained using Propositions 4.1, 4.7, 4.8, and Lidskii's theorem.

Theorem 4.10. *The operator L has infinitely many eigenvalues. The system of its eigenvectors and generalized eigenvectors is complete in $\mathcal{L}^2(-\pi, \pi)$.*

Proof. By Propositions 4.7, 4.8, and Lidskii's theorem, the systems of all eigenvectors and generalized eigenvectors of the operators L_+^{-1}, L_-^{-1}, and $\widetilde{L}^{-1}$ are complete in $\mathcal{H}^{2,0}(\mathbb{D})$, $\mathcal{H}^{2,0}(\overline{\mathbb{C}} \setminus \mathbb{D})$, and $\mathcal{L}_p^2(-\pi, \pi)$, respectively. Since $\operatorname{Ker}(\widetilde{L}^{-1}) = \{0\}$, all generalized eigenspaces of $\widetilde{L}^{-1}$ corresponding to its eigenvalues α_n are generalized eigenspaces of $\widetilde{L}$ corresponding to eigenvalues $\lambda_n = 1/\alpha_n$. Since all eigenvalues of the compact operator $\widetilde{L}^{-1}$ have finite algebraic multiplicities, we see that $\widetilde{L}^{-1}$ and $\widetilde{L}$ have infinitely many eigenvalues and that standard arguments imply completeness property for eigenvectors of $\widetilde{L}$ (and, consequently, for eigenvectors of L). (In

Section 5 (see Proposition 5.5), it is shown that all eigenvalues of $\widetilde{L}$ are simple, so, actually, $\widetilde{L}^{-1}$ has no generalized eigenvectors.) $\square$

Remark 4.11. Theorem 4.3 was announced in [10] with arguments partially based on numerical results. The rigorous proof given above uses essentially Proposition 4.7, and so uses the results of [12]. The fact that the operator L has an infinite number of eigenvalues was obtained independently and by a different method in [40].

5. Purely imaginary eigenvalues and the Riesz basis property

In this section we prove that if $0 < \varepsilon < 2$, then the set of eigenvectors $\{u_n\}_1^\infty$ of the operator L normalized by $\|u_n\|_{\mathcal{L}^2} = 1$ does not form an unconditional basis in $\mathcal{L}^2(-\pi, \pi)$. Recall that a basis is called *unconditional* if it remains a basis for any ordering of the elements. By the well-known theorem of Lorch (see e.g. [16, Theorem VI.2.2]), $\{u_n\}_1^\infty$ is an unconditional basis if and only if there exists a bounded and boundedly invertible operator T such that $\{Tu_n\}_1^\infty$ is a complete orthonormal set in $\mathcal{L}^2(-\pi, \pi)$ or, in other words, $\{u_n\}_1^\infty$ is *a Riesz basis.*

Recall that L_+ is an operator in the Hilbert space

$$\mathcal{H}^{2,0}(\mathbb{D}) = \mathcal{H}^2(\mathbb{D}) \ominus \{c\mathbb{1},\, c \in \mathbb{C}\}.$$

We identify the function $u(z) \in \mathcal{H}^2(\mathbb{D})$ with $u(e^{i\theta}) \in \mathcal{L}^2(-\pi, \pi)$. Note that

$$u \perp \mathbb{1} \quad \text{is equivalent to} \quad u(0) = 0.$$

So the last equality holds for all $u \in \mathcal{H}^{2,0}(\mathbb{D})$.

Let $u, f \in \mathcal{H}^2(\mathbb{D})$. Consider the restrictions $\mathfrak{u}$ and $\mathfrak{f}$ of the functions u and f on the interval $[0, 1) \subset \mathbb{D}$,

$$\mathfrak{u}(x) = u(x), \quad \mathfrak{f}(x) = f(x) \quad \text{for} \quad x \in [0, 1). \tag{5.1}$$

The following proposition was essentially obtained in [39] (formally for $f = \lambda u$, see [39, Lemma 2.1 and the proof of Theorem 2.3]). For the sake of completeness, we give another proof that avoids the use of the Fourier representation of L and so is closer to the main approach of the present paper.

Proposition 5.1 ([39]). *Let $0 < \varepsilon < 2$. Assume that $u \in \mathcal{H}^{2,0}(\mathbb{D})$, $u \in \mathrm{Dom}(L_+)$, and $Lu = f$. Then the restrictions $\mathfrak{u}, \mathfrak{f}$ defined by* (5.1) *satisfy the equation*

$$\mathfrak{b}[\mathfrak{u}](x) = -\frac{2i}{\varepsilon}\mathfrak{f}(x) \quad \textit{for all} \quad x \in (0, 1)\ ,$$

where $\mathfrak{b}[\cdot]$ is the Sturm-Liouville differential expression defined by

$$\mathfrak{b}[u] = -\frac{1}{w}(pu')',$$

$$p(x) = (1-x)^{1+1/\varepsilon}(x+1)^{1-1/\varepsilon}, \quad w(x) = x^{-1}(1-x)^{1/\varepsilon}(x+1)^{-1/\varepsilon}.$$

Proof. First, assume additionally that u belongs to $P_{\text{fin}} \cap \mathcal{H}^{2,0}(\mathbb{D})$, i.e., $u(z)$ is a polynomial such that $u(0)=0$. Then so does f. Since $\frac{du(e^{i\theta})}{d\theta} = ie^{i\theta}\frac{du}{dz}(e^{i\theta})$, we see that

$$\begin{aligned} &f(e^{i\theta}) = \ell[u](e^{i\theta}) \\ &= \varepsilon\sin(\theta)\left(-e^{i\theta}\frac{du}{dz}(e^{i\theta}) - e^{2i\theta}\frac{d^2u}{dz^2}(e^{i\theta})\right) + \varepsilon\cos(\theta)ie^{i\theta}\frac{du}{dz}(e^{i\theta}) + ie^{i\theta}\frac{du}{dz}(e^{i\theta}). \end{aligned}$$

Since both sides are polynomials in $z = e^{i\theta}$, we get

$$f(z) = \frac{i\varepsilon}{2}(z^2-1)z\frac{d^2u(z)}{dz^2} + i(\varepsilon z^2 + z)\frac{du(z)}{dz} \tag{5.2}$$

for all $z \in \mathbb{C}$. In particular, for $z = x \in (0,1)$, the last equality takes the form $\mathfrak{f}(x) = \frac{i\varepsilon}{2}\mathfrak{b}[\mathfrak{u}]$.

Now, for arbitrary $u, f \in \mathcal{H}^{2,0}(\mathbb{D})$ such that $u \in \operatorname{Dom}(L_+)$ and $Lu = f$, consider sequences $\{f_n\}_1^\infty$ and $\{u_n\}_1^\infty$ in $P_{\text{fin}} \cap \mathcal{H}^{2,0}(\mathbb{D})$ such that

$$\lim_{n\to\infty}\|f - f_n\|_{\mathcal{L}^2} = 0, \quad \lim_{n\to\infty}\|u - u_n\|_{\mathcal{L}^2} \quad \text{and} \quad L_+u_n := f_n, \quad n \in \mathbb{N}.$$

Note that such sequences exist since $L = \overline{L_{\text{fin}}}$. We complete the proof substituting the polynomials f_n and u_n into (5.2) and passing to the limit. ☐

It was noticed in [39], that Proposition 5.1 implies that if $u(e^{i\theta})$ is an eigenfunction of the operator L_+, $L_+u = \lambda u$, then the restriction $\mathfrak{u}$ is a solution of a Sturm-Liouville eigenvalue problem with real-values coefficients. Indeed,

$$\mathfrak{b}[\mathfrak{u}](x) = \mu\mathfrak{u}(x), \quad \text{where} \quad \mu = -2i\lambda/\varepsilon. \tag{5.3}$$

Consider the weighted space $\mathcal{L}^2((0,1);w)$, where the weight function w is defined as in Proposition 5.1. Let $B_{\max}$ be an operator in $\mathcal{L}^2((0,1);w)$ associated with the differential expression $\mathfrak{b}[\cdot]$ and defined on its maximal domain, $B_{\max}\mathfrak{u} = \mathfrak{b}[\mathfrak{u}]$ and

$$\operatorname{Dom}(B_{\max}) = \{\mathfrak{u} \in \mathcal{L}^2((0,1);w) : \mathfrak{u}, \mathfrak{u}' \in AC_{\text{loc}}(0,1),\ \mathfrak{b}[\mathfrak{u}] \in \mathcal{L}^2((0,1);w)\}.$$

Note that all points of the interval $(0,1)$ are regular for the differential expression $\mathfrak{b}$, however the endpoints 0 and 1 are singular (1 is singular since $p^{-1} \notin \mathcal{L}^1(1/2,1)$).

Proposition 5.2. *Let $\varepsilon > 0$.*

(i) $\mathfrak{b}$ *is in the limit-point case at* 0*,*
(ii) $\mathfrak{b}$ *is in the limit-point case at* 1 *exactly when* $\varepsilon \le 1$*.*
(iii) $B_{\max}$ *is self-adjoint in* $\mathcal{L}^2((0,1);w)$ *exactly when* $0 < \varepsilon \le 1$*.*

Proof. (i). Clearly, $\mathbb{1}$ is a solution of $\mathfrak{b}[u] = 0$ and $\mathbb{1} \notin \mathcal{L}^2((0,1/2);w)$. Weyl's alternative (see e.g. [38, Theorem 5.6]) completes the proof.

(ii). The general solution of $\mathfrak{b}[u] = 0$ on $(0,1)$ is

$$u(x) = k_1\int_{1/2}^x \frac{1}{p(s)}ds + k_2, \quad k_1, k_2 \in \mathbb{C}.$$

If $k_1 \neq 0$,

$$u(x) \asymp k_1 \int_{1/2}^{x} (1-s)^{-1-1/\varepsilon} ds \asymp k_1 (1-x)^{-1/\varepsilon}, \quad x \to 1-0.$$

Hence all solutions of $\mathfrak{b}[u] = 0$ belong to $\mathcal{L}^2((1/2,1);w)$ if and only if $\varepsilon > 1$.

(iii) follows from (i) and (ii). □

Proposition 5.3. *If $u(e^{i\theta}) \in \mathcal{H}^{2,0}(\mathbb{D})$, then the function $\mathfrak{u}$ defined by* (5.1) *belongs to $\mathcal{L}^2((0,1);w)$.*

Proof. By (5), we have $\mathfrak{u}(0) = 0$. Since $\mathfrak{u}$ is analytic at 0, we see that $\mathfrak{u} \in \mathcal{L}^2((0,1/2);w)$. The measure $w(x)dx$ on $[1/2,1)$ induces a measure $M(S) := \int_{S \cap [\frac{1}{2},1)} w(x)dx$ on $\mathbb{D}$. For any sector

$$S = \{re^{i\theta} : 1-l \leq r < 1, |\theta - \theta_0| < l\}, \quad l \in (0,1),$$

we have

$$M(S) \leq 2 \int_{\max\{1/2,1-l\}}^{1} dx \leq 2l$$

since $\max_{x \in [1/2,1)} w(x) \leq 2$. So $M(\cdot)$ is a Carleson measure (see e.g. [15, Sec. 4.3]). Therefore,

$$\int_{1/2}^{1} |\mathfrak{u}(x)|^2 w(x) dx \leq C_1 \|u\|^2_{\mathcal{H}^{2,0}} ,$$

where C_1 is a constant independent of u. This completes the proof. □

The following result was essentially obtained by John Weir in [39, Theorem 2.3] (see also [5] for a more general result). The proof of [39, Theorem 2.3] was given under additional condition $1/\varepsilon \notin \mathbb{Z}$, but John Weir informed us that the case $1/\varepsilon \in \mathbb{Z}$ can also be treated by similar arguments. For the sake of completeness of the paper, we give below a proof for all $\varepsilon \in (0,2)$, which is different, but is based on the same idea of Weir (Proposition 5.1). We use the theory of Hardy spaces and this makes our proof simpler than that of [39, Theorem 2.3], where Frobenius theory and certain delicate estimates obtained in [12] were used.

Proposition 5.4 ([39], see also [5]). *Let $\varepsilon \in (0,2)$. Then all eigenvalues of the operator $A_+ (= -iL_+)$ are real and positive.*

Proof. Let $u(e^{i\theta}) \in \mathcal{H}^{2,0}(\mathbb{D})$ and $L_+ u = \lambda u$. By Proposition 5.3, $\mathfrak{u} \in \mathcal{L}^2((0,1);w)$. Proposition 5.1 implies that $\mathfrak{b}[\mathfrak{u}] = \mu \mathfrak{u}$ with $\mu = -2i\lambda/\varepsilon$, and therefore $\mathfrak{u} \in \mathrm{Dom}(B_{\max})$.

Let us consider two cases: $\varepsilon \in (0,1]$ and $\varepsilon \in (1,2)$. In the case $\varepsilon \leq 1$, Proposition 5.2 (iii) makes the proof simple. Since $\mathfrak{u} \in \mathrm{Dom}(B_{\max})$, μ is an eigenvalue of the nonnegative self-adjoint operator $B_{\max}$. Thus, $\mu \geq 0$.

If $1 < \varepsilon < 2$, then the proof requires additional analysis. By Proposition 2.3, $g(e^{i\theta}) := \frac{du}{d\theta}(e^{i\theta}) \in \mathcal{L}^2(-\pi,\pi)$. It is easy to see from the representation $u(e^{i\theta}) = (2\pi)^{-1/2} \sum_{n=1}^{\infty} v_n e^{in\theta}$ that $g(e^{i\theta}) \in \mathcal{H}^2(\mathbb{D})$ (on the other hand, the latter follows from [12, Theorem 16]) and $g(e^{i\theta}) = \lim\limits_{r\to 1-0} g(re^{i\theta})$, where $g(z) = z\frac{du(z)}{dz}$, $z \in \mathbb{D}$. By [15, Problem II.5 (a)], $|g(x)| \le \|g\|_{\mathcal{L}^2}(1-|x|^2)^{-1/2}$ for $x \in (0,1)$ and therefore, for $x \in (\,1/2\,,\,1\,)$,

$$\left|\frac{d\mathfrak{u}(x)}{dx}\right| \;\le\; \|g\|_{\mathcal{L}^2}\,|x|^{-1}(1-|x|^2)^{-1/2} \;\le\; C_1(1-x)^{-1/2}. \tag{5.4}$$

By [38, Theorem 5.8 (ii)], the operator $B_{\mathbb{1}}$ defined by $B_{\mathbb{1}}\mathfrak{u} := \mathfrak{b}[\mathfrak{u}]$ on the domain

$$\mathrm{Dom}(B_{\mathbb{1}}) := \{\mathfrak{u} \in \mathrm{Dom}(B_{\max}) : [\mathbb{1},\mathfrak{u}]_1 = 0\}, \quad [\mathbb{1},\mathfrak{u}]_1 := \lim_{x\to 1-0} p(x)\,\mathfrak{u}'(x),$$

is self-adjoint. Note that the limit $[\mathbb{1},\mathfrak{u}]_1$ exists for any $\mathfrak{u} \in \mathrm{Dom}(B_{\max})$ due to [38, Theorem 3.10]).

It follows from (5.4) that, for any eigenvector $u(e^{i\theta})$ of L_+, its restriction $\mathfrak{u}$ belongs to $\mathrm{Dom}(B_{\mathbb{1}})$. Indeed, it was shown above that $\mathfrak{u} \in \mathrm{Dom}(B_{\max})$. On the other hand, it follows from (5.4) that

$$[\mathbb{1},\mathfrak{u}]_1 = \lim_{x\to 1-0}(1-x)^{1+1/\varepsilon}(x+1)^{1-1/\varepsilon}u'(x) = 0.$$

So μ is an eigenvalue of the operator $B_{\mathbb{1}} = B_{\mathbb{1}}^*$.

It follows from (5.4) that $\mathfrak{u}(x) = \mathfrak{u}(1/2) + \int_{1/2}^{x}\mathfrak{u}'(t)dt$ has a finite limit as $x \to 1-0$ (this fact also follows from [12, Theorem 16]). Therefore,

$$\begin{aligned}(B_{\mathbb{1}}\mathfrak{u},\mathfrak{u})_{\mathcal{L}^2((0,1);w)} &= -\int_0^1 (p(x)\mathfrak{u}'(x))'\overline{\mathfrak{u}(x)}dx \\ &= \int_0^1 p(x)|\mathfrak{u}'(x)|^2dx - \lim_{x\to 1-0} p(x)\mathfrak{u}'(x)\overline{\mathfrak{u}(x)} = \int_0^1 p(x)|\mathfrak{u}'(x)|^2dx \ge 0.\end{aligned}$$

Thus, $B_{\mathbb{1}} \ge 0$ and therefore $\mu \ge 0$.

Finally, note that $\mathrm{Ker}(L_+) = \{0\}$ and therefore $\mu \ne 0$. ☐

Proposition 5.5. *Let* $\varepsilon \in (0,2)$. *Then all eigenvalues of the operator* L_+ *are simple.*

Proof. Let $u(e^{i\theta}) \in \mathcal{H}^{2,0}(\mathbb{D})$ be an eigenvector of the operator L_+, $L_+u = \lambda u$. Assume that there exists a non-zero vector $u_1(e^{i\theta}) \in \mathcal{H}^{2,0}(\mathbb{D})$ such that

$$(L_+ - \lambda I)u_1 = u.$$

Consider the restrictions $\mathfrak{u}$, $\mathfrak{u}_1$ of the functions u and u_1 on the interval $[0,1) \subset \mathbb{D}$. By Proposition 5.3, $\mathfrak{u}$, $\mathfrak{u}_1 \in \mathcal{L}^2((0,1);w)$. Proposition 5.1 shows that

$$\mathfrak{b}[\mathfrak{u}_1] - \mu\,\mathfrak{u}_1 = -\frac{2i}{\varepsilon}\mathfrak{u} \qquad \text{with} \quad \mu = -\frac{2i\,\lambda}{\varepsilon}. \tag{5.5}$$

In the case $\varepsilon \in (0,1]$, (5.5) immediately implies that $\mathfrak{u}_1$ is a generalized eigenvector of the self-adjoint operator $B_{\max}$. This is not possible.

Consider the case $\varepsilon \in (1,2)$. Since $u_1 \in \mathrm{Dom}(L)$, arguing as in the proof of Proposition 5.4, we see that $\mathfrak{u}_1 \in \mathrm{Dom}(B_{\mathbb{1}})$ and therefore $\mathfrak{u}_1$ is a generalized eigenvector of the operator $B_{\mathbb{1}} = B_{\mathbb{1}}^*$. This is also a contradiction. ☐

Theorem 5.6. *Let $\varepsilon \in (0,2)$ and let $\{u_n\}_1^\infty$ be a maximal system of linearly independent eigenvectors of L. We assume that u_n are normalized by $\|u_n\|_{\mathcal{L}^2} = 1$. Then $\{u_n\}_1^\infty$ is complete in $\mathcal{L}^2(-\pi,\pi)$, but does not form a Riesz basis in $\mathcal{L}^2(-\pi,\pi)$.*

Proof. The first statement follows directly from Proposition 5.5 and Theorem 4.10. We should prove the second one. Assume that the set $\{u_n\}_1^\infty$ forms a Riesz basis in $\mathcal{L}^2(-\pi,\pi)$. Then Proposition 5.4 implies that iL is similar to a certain self-adjoint operator Q. That is, there exists a bounded and boundedly invertible operator S such that $S\,\mathrm{Dom}(Q) = \mathrm{Dom}(L)$ and $iL = SQS^{-1}$.

The spectral theorem for a self-adjoint operator implies that, for arbitrary $u_0 \in \mathrm{Dom}(L)(= X_2)$, the problem

$$u_t + Lu = 0, \quad u\,|_{t=0} = u_0, \quad t \in \mathbb{R},$$

has a unique solution $u(\cdot,t)$ in the sense of [22, Definition I.1.1 and Eq. (I.1.2)] (such solutions are sometimes called *strong solutions*). Moreover, this solution has the form $u(\cdot,t) = Se^{-itQ}S^{-1}u_0(\cdot)$. Therefore, for any $T > 0$,

$$u \in C([0,T]; X_2) \subset C([0,T]; \mathrm{W}^1_{2p}(-\pi,\pi)),$$

$$u_t(\cdot,t) \in C([0,T]; \mathcal{L}^2(-\pi,\pi)), \quad \text{and} \quad Lu(\cdot,t) \in C([0,T]; \mathcal{L}^2(-\pi,\pi)).$$

It is easy to see that u is a generalized solution of (3.1)–(3.2) in the sense of Definition 3.1. Since e^{-itQ} is a unitary operator,

$$\|u(\cdot,t)\|_{\mathcal{L}^2(-\pi,\pi)} \leq \|S\|\,\|S^{-1}\|\,\|u_0\|_{\mathcal{L}^2(-\pi,\pi)}, \quad t \in \mathbb{R}.$$

Hence, for any $T > 0$, we have

$$\int_0^T \|u(\cdot,t)\|^2_{\mathcal{L}^2(-\pi,\pi)} dt \leq CT\|u_0\|^2_{\mathcal{L}^2(-\pi,\pi)},$$

where $C = \|S\|^2\,\|S^{-1}\|^2 < \infty$. The latter contradicts Theorem 3.5 since $\mathrm{Dom}(L)$ is dense in $\mathrm{W}^1_{2p}(-\pi,\pi)$ and $\|u_0\|_{\mathcal{L}^2(-\pi,\pi)} \leq \|u_0\|_{\mathrm{W}^k_{2p}(-\pi,\pi)}$ for any $k \geq 0$. ☐

We would like to note that the linear partial differential equation (1.1) is an interesting example when the nature of explosive blow-up and instability of solutions has its roots not in the location of the eigenvalues but in geometric properties of the eigenfunctions.

6. Further discussion

When the eigenfunctions related to neutrally stable eigenvalues of some linearized problem form a complete set, the representation of a solution of the nonlinear problem as a series of these eigenfunctions is one of the general approaches to the nonlinear stability problem. The lack of a basis property of the eigenfunction set is an obstacle for the applicability of this particular method.

Due to the ill-posed nature of the forward-backward heat equation all eigenmodes are linearly unstable [3] and it is common to use the smoothing effect of the surface tension to stabilize them. The lubrication approximation that takes into account the influence of the capillarity effects and/or surface tension leads to the initial value problem for the fourth order nonlinear partial differential equation described in [8]. Some stability properties of its linearization were studied in [2, 4, 7]. The authors came to the conclusion that almost all modes, except some first ones, become stable even if the surface tension is relatively weak.

We would also like to mention that the main assumption of the parameter range $|\varepsilon| < 2$ comes naturally from the theory of mixed type equations. For the case when $|\varepsilon| > 2$ the domain of the operator changes completely, so one needs to introduce a different kind of boundary conditions.

Let us consider the equation

$$k(x,t)\,u_{tt} + \alpha(t,x)\,u_t + \Delta u = 0, \;\; x \in \Omega, \;\; t > 0 \tag{6.1}$$

where the coefficient $k(x,t)$ can change sign in the domain where the operator is considered. So equation (6.1) is an equation of the mixed type, i.e. it is of the same type as the well-known Tricomi equation. On the lateral boundary of the cylinder $\Omega \times (0,T)$ we pose the Dirichlet boundary condition and there are two additional boundary conditions on the lower and upper base of the cylinder:

$$u|_{t=T} = 0, \quad u_t|_{S^+} = 0, \quad u_t|_{S^-} = 0,$$
$$S^+ = \{(0,x): \; k(x,0) > 0\}, \quad S^- = \{(T,x): \; k(x,T) < 0\}.$$

This boundary value problem and closely related problems were studied by many authors (see, for instance, [36, 37]). It was demonstrated that the condition

$$\alpha - \frac{k_t}{2} \geq \delta_0 > 0 \;\; \forall (x,t),$$

where δ_0 is a positive constant, ensures the existence of generalized solutions to the above-described boundary value problem. Stronger conditions of the type

$$\alpha - \frac{|k_t|(2k-1)}{2} \geq \delta_0 > 0 \;\; \forall (x,t)$$

ensure existence of smooth solutions and uniqueness of generalized solutions. The existence of solutions of non-linear forward-backward heat equations was studied by Hollig [19] and by Pyatkov [30]. Amongst the most recent results devoted to the nonlinear forward-backward parabolic problems we would like to mention the Kuznecov papers [23, 24].

Note also that Eq. (1.1) can be written as a second order equation with a nonnegative characteristic form (see e.g. [21, 29]) or as an abstract kinetic equation (see e.g. [13, 18, 20, 31] and references therein). However boundary value problems arising in these theories are different. One more difficulty in application of the general theory appears because of the strong degeneracy of the coefficient $\sin(\theta)$ at points 0 and π. In particular, the theory of abstract kinetic equations requires writing the differential expression L in the $\mathcal{J}$-self-adjoint form with respect to $\mathcal{J}: f(x) \to \operatorname{sgn}(x) f(x)$. The latter leads to the operator

$$\frac{\operatorname{sgn}(\theta)}{|\tan\theta/2|^{1/\varepsilon}} \frac{d}{d\theta}\, |\tan\theta/2|^{1/\varepsilon}\, |\sin(\theta)|\, \frac{d}{d\theta}$$

in the weighted Hilbert space $\mathcal{L}^2\left((-\pi,\pi);|\tan\theta/2|^{1/\varepsilon}\right)$ (see [10, Appendix A]). The boundary conditions associated with this operator and hence its spectral properties are completely different from that of the periodic operator L acting on $\mathcal{L}^2(-\pi,\pi)$.

Appendix A. Proof of Corollary 2.5 using the Galerkin method

The second proof of Corollary 2.5. Let $\{\omega_j\}_{j=1}^{\infty}$ be a basis for the Hilbert space $H=\{h\in \mathrm{W}_{2p}^2(-\pi,\pi): \int_{-\pi}^{\pi}h(\theta)\,d\theta=0\}$. Find functions φ_j such that $\varphi_j'=\omega_j$, $\int_{-\pi}^{\pi}\varphi_j(\theta)\,d\theta=0$. We look for an approximate solution to equation (2.3) in the form

$$h_n=\sum_{j=1}^{n}c_{jn}\varphi_j,$$

where the constants c_{jn} are determined from the system of algebraic equations

$$(Lh_n,\omega_j)=(f,\omega_j),\quad j=1,2,\dots,n, \tag{A.1}$$

(the brackets denote the inner product in $\mathcal{L}^2(-\pi,\pi)$, i.e., $(h,v)=\int_{-\pi}^{\pi}h(\theta)v(\theta)\,d\theta$).

Multiplying (A.1) by c_{jn} and summing the equalities obtained, we arrive at the relation

$$(Lh_n,h_n')=(f,h_n').$$

Integrating by parts we derive the estimate

$$\|h_n'\|_{\mathcal{L}^2(-\pi,\pi)}\le c\|f\|_{\mathcal{L}^2(-\pi,\pi)}, \tag{A.2}$$

where c is a constant independent of n. This estimate implies that the system (A.1) is solvable (we can refer, for instance, to [27, lemma 4.3 of ch. 1]). Note that there exists a constant c_1 independent of n such that

$$\|h_n\|_{\mathcal{L}^2(-\pi,\pi)}\le c_1\|h_n'\|_{\mathcal{L}^2(-\pi,\pi)} \tag{A.3}$$

From (A.2), (A.3) we conclude that there exists a subsequence h_{n_k} and a function $h\in \mathrm{W}_2^1(-\pi,\pi)$, $h(-\pi)=h(\pi)$ and $\int_{-\pi}^{\pi}h(\theta)\,d\theta=0$, such that

$$h_{n_k}\to h \ \text{ in } \mathcal{L}^2(-\pi,\pi),\quad h_{n_k}'\to h' \ \text{ weakly in } \mathcal{L}^2(-\pi,\pi). \tag{A.4}$$

Let us multiply (A.1) with $n=n_k$ by constants α_j $(1\le j\le m\le n_k)$ and sum the results. Fix m assuming that $n_k\ge m$. We infer

$$-\varepsilon(\sin(\theta)\,h_{n_k}',\omega')+(h_{n_k}',\omega)=(f(\theta),\omega),\quad \omega=\sum_{j=1}^{m}\alpha_j\omega_j.$$

Passing to the limit as $k\to\infty$ we arrive at the equality

$$-\varepsilon(\sin(\theta)\,h',\omega')+(h',\omega)=(f(\theta),\omega),\quad \omega=\sum_{j=1}^{m}\alpha_j\omega_j. \tag{A.5}$$

The functions ω of the form $\omega=\sum_{j=1}^{m}\alpha_j\omega_j$ are dense in H and thus (A.5) holds for all functions in H. Since $\int_{-\pi}^{\pi}f(\theta)\,d\theta=0$, we can see that (A.5) also holds for all functions of the form $\omega + c$ (c is an arbitrary constant) and therefore for all functions in $\mathrm{W}^2_{2p}(-\pi,\pi)$. In particular, it holds for $\omega \in C^\infty_0(-\pi,\pi)$. From the definition of the generalized derivative (in the Sobolev sense) we have that there exist the generalized derivative $(\sin(\theta)\, h')'$ and

$$\varepsilon(\sin(\theta)\, h')' = (f - h') \in \mathcal{L}^2(-\pi,\pi).$$

Thereby, $\sin(\theta)\, h' \in \mathrm{W}^1_2(-\pi,\pi)$. Integrating by parts in (A.5) we obtain that the equation (2.3) is satisfied almost everywhere on $(-\pi,\pi)$. We have proved that $\tilde{L}$ is an isomorphism of X^0_2 onto $\mathcal{L}^2_p(-\pi,\pi)$. Since $X^0_2 \subset \mathrm{W}^1_2(-\pi,\pi)$ and the embedding $\mathrm{W}^1_2(-\pi,\pi) \subset \mathcal{L}_2(-\pi,\pi)$ is compact, the operator $\tilde{L}^{-1} : \mathcal{L}^2_p(-\pi,\pi) \to \mathcal{L}^2_p(-\pi,\pi)$ is compact. The remaining assertions are more or less obvious. ☐

Remark A.1. We can take the set $\{\sin(j\theta), \cos(j\theta)\}^\infty_{j=1}$ rather than an abstract basis $\{\omega_j\}$.

Remark A.2. The compactness of the operator $\tilde{L}^{-1} : \mathcal{L}^2_p(-\pi,\pi) \to \mathcal{L}^2_p(-\pi,\pi)$ implies that the spectrum of $\tilde{L}$ and the operator L itself is discrete with the only accumulation point ∞. The fact that the operator L has no real spectrum can be easily proved by integrating by parts. Indeed, assume the contrary that there exist $\lambda \in \mathbb{R}$ such that

$$Lh = \lambda h, \quad h \in X^0_2.$$

Let us multiply this equation by h and integrate the result over $(-\pi,\pi)$. Integrating by parts and taking the real part, we arrive at the inequality $\|h_\theta\|^2_{\mathcal{L}^2(-\pi,\pi)} \le 0$ which yields $h \equiv 0$.

Remark A.3. It is possible to prove the following statement. Let $1-\varepsilon(k+1/2) > 0$. Then for every $f \in \mathrm{W}^k_{2p}(-\pi,\pi)$ with $\int_{-\pi}^{\pi} f(\theta)\, d\theta = 0$ there exists a unique solution of the equation (2.3) such that $h \in \mathrm{W}^{k+1}_{2p}(-\pi,\pi)$, $\sin(\theta)\, h^{(k+1)} \in \mathrm{W}^1_2(-\pi,\pi)$. We do not need this result, so the proof is omitted.

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