University of Toronto – MAT246H1-S – LEC0201/9201

Concepts in Abstract Mathematics

Homework questions – Week 11

Jean-Baptiste Campesato

April 5^{th}, 2021 to April 9^{th}, 2021

Exercise 1.
Given three sets E, F, G, prove that if $E \subset F \subset G$ and $|E| = |G|$ then $|E| = |F|$.

Exercise 2.
Given a set S, prove that $|\mathcal{P}(S)| = \left|\{0,1\}^S\right|$ where $\{0,1\}^S$ denotes the set of functions $S \to \{0,1\}$.
Remark: this formula generalizes the fact that if S is a finite set with $n = |S|$ then $|\mathcal{P}(S)| = 2^n$.
Therefore it is common to denote the powerset of a set S by $2^S \coloneqq \mathcal{P}(S)$.

Exercise 3.

1. What is $\left|\{0,1\}^{\mathbb{N}}\right|$? i.e. what is the cardinality of the set of functions $\mathbb{N} \to \{0,1\}$?
2. What is $\left|\mathbb{N}^{\{0,1\}}\right|$? i.e. what is the cardinality of the set of functions $\{0,1\} \to \mathbb{N}$?

Exercise 4.

1. What is the cardinality of $S = \{A \in \mathcal{P}(\mathbb{N}) \ : \ A \text{ is finite}\}$.
2. Is $T = \{A \in \mathcal{P}(\mathbb{N}) \ : \ A \text{ is infinite}\}$ countable?

Exercise 5.
Prove that any set X of pairwise disjoint intervals which are non-empty and not reduced to a singleton is countable,
i.e. if $X \subset \mathcal{P}(\mathbb{R})$ satisfies
(i) $\forall I \in X,\ I$ is an interval which is non-empty and not reduced to a singleton
(ii) $\forall I, J \in X,\ I \neq J \implies I \cap J = \varnothing$
then X is countable.

Exercise 6.
Prove that a set is infinite if and only if it admits a proper subset of same cardinality.

Exercise 7.

1. Prove that $\mathbb{R} \setminus \mathbb{Q}$ is not countable.
2. Prove that $|\mathbb{R} \setminus \mathbb{Q}| = |\mathbb{R}|$.

Exercise 8.
Prove that $|(0,1)| = |\mathbb{R}|$.

Exercise 9.

1. Prove that $|\mathbb{R}^2| = |\mathbb{R}|$.
2. Prove that $\forall n \in \mathbb{N} \setminus \{0\}$, $|\mathbb{R}^n| = |\mathbb{R}|$.
3. Prove that $\left|\mathbb{R}^{\mathbb{N}}\right| = |\mathbb{R}|$ where $\mathbb{R}^{\mathbb{N}}$ is the set of sequences/functions $\mathbb{N} \to \mathbb{R}$.

Exercise 10.
Set $S^2 \coloneqq \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\right\}$. Prove that $|S^2| = |\mathbb{R}|$.

Exercise 11.
What is the cardinality of the set S of all circles in the plane?

Cheatsheet

Recollection of some results about cardinality

Definition. We say that two sets E and F have same *cardinality*, denoted by $|E| = |F|$, if there exists a bijection $f : E \to F$.

Proposition.
1. *If E is a set then $|E| = |E|$.*
2. *Given two sets E and F, if $|E| = |F|$ then $|F| = |E|$.*
3. *Given three sets E, F and G, if $|E| = |F|$ and $|F| = |G|$ then $|E| = |G|$.*

Theorem. *A set E is infinite if and only if for every $n \in \mathbb{N}$ there exists $S \subset E$ such that $|S| = n$.*

Definition. Given two sets E and F, we write $|E| \le |F|$ if there exists an injective function $f : E \to F$.

Proposition.
1. *If E is a set then $|E| \le |E|$.*
2. *Given two sets E and F, if $|E| \le |F|$ and $|F| \le |E|$ then $|E| = |F|$* Cantor–Schröder–Bernstein theorem.
3. *Given three sets E, F and G, if $|E| \le |F|$ and $|F| \le |G|$ then $|E| \le |G|$.*

Proposition. *If $E \subset F$ then $|E| \le |F|$.*

Proposition. *If $|E_1| = |E_2|$ and $|F_1| = |F_2|$ then $|E_1 \times F_1| = |E_2 \times F_2|$.*

Theorem. *Given two sets E and F, $|E| \le |F|$ if and only if there exists a surjective function $g : F \to E$.*

Theorem. *Given two sets E and F, if $|E| = |F|$ then $|\mathcal{P}(E)| = |\mathcal{P}(F)|$.*

Notation. We set $\aleph_0 := |\mathbb{N}|$ (pronounced *aleph nought*).

Definition. A set E is countable if either E is finite or $|E| = \aleph_0$.

Proposition. *If $S \subset \mathbb{N}$ is infinite then $|S| = \aleph_0$.*

Proposition. *A set E is countable if and only if $|E| \le \aleph_0$ (i.e. there exists an injection $f : E \to \mathbb{N}$), otherwise stated E is countable if and only if there exists a bijection between E and a subset of $\mathbb{N}$.*

Proposition. $|\mathbb{N} \times \mathbb{N}| = \aleph_0$

Theorem. *A countable union of countable sets is countable, i.e. if I is countable and if for every $i \in I$, E_i is countable then $\bigcup_{i\in I} E_i$ is countable.*

Theorem. *If E is an infinite set then there exists $T \subset E$ such that $|T| = \aleph_0$, i.e. $\aleph_0$ is the least infinite cardinal.*

Theorem. $|\mathbb{Z}| = \aleph_0$

Theorem. $|\mathbb{Q}| = \aleph_0$

Theorem. $\aleph_0 < |\mathbb{R}|$

Theorem (Cantor's theorem)**.** *Given a set E, $|E| < |\mathcal{P}(E)|$.*

Theorem. $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$

Sample solutions to Exercise 1.
Since $E \subset F$, we know that $|E| \leq |F|$.
Besides, since $F \subset G$, we have $|F| \leq |G| = |E|$.
By Cantor–Schröder–Bernstein theorem, we have $|E| = |F|$.

Sample solutions to Exercise 2.
We define $\psi : \mathcal{P}(S) \to \{0,1\}^S$ by $\psi(A)(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$.
Let's prove that ψ is a bijection:

- ψ is injective.
  Let $A, B \subset S$ be such that $A \neq B$.
  WLOG we may assume that there exists $x \in S$ such that $x \in A$ and $x \notin B$.
  Therefore $\psi(A)(x) = 1$ and $\psi(B)(x) = 0$. Thus $\psi(A) \neq \psi(B)$.

- ψ is surjective.
  Let $f : S \to \{0,1\}$ be a function. Define $A = \{x \in S : f(x) = 1\}$. Then $f = \psi(A)$.

Therefore $|\mathcal{P}(S)| = \left|\{0,1\}^S\right|$.

Sample solutions to Exercise 3.
1. $\left|\{0,1\}^{\mathbb{N}}\right| = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$

2. The idea here is that a function $\{0,1\} \to \mathbb{N}$ is characterized by the values of 0 and 1.
   Define $\psi : \mathbb{N}^{\{0,1\}} \to \mathbb{N} \times \mathbb{N}$ by $\psi(f) = (f(0), f(1))$.

   - ψ is injective: let $f, g : \mathbb{N} \to \{0,1\}$ be such that $\psi(f) = \psi(g)$. Then $(f(0), f(1)) = (g(0), g(1))$ so that $f(0) = g(0)$ and $f(1) = g(1)$. Therefore $f = g$.
   - ψ is surjective: let $(a,b) \in \mathbb{N} \times \mathbb{N}$. Define $f : \{0,1\} \to \mathbb{N}$ by $f(0) = 1$ and $f(1) = b$. Then $\psi(f) = (a,b)$.

   Therefore $\left|\mathbb{N}^{\{0,1\}}\right| = |\mathbb{N} \times \mathbb{N}| = \aleph_0$.

Sample solutions to Exercise 4.
1. Define $f : S \to \mathbb{N}$ by $f(A) = \sum_{k \in A} 2^k$.
   Then f is bijective by existence and uniqueness of the binary positional representation of a natural number. Therefore $|S| = |\mathbb{N}| = \aleph_0$.

2. Assume that T is countable then $\mathcal{P}(\mathbb{N}) = S \sqcup T$ is countable as the union of countable sets.
   Which is a contradiction since $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}| > \aleph_0$.

Sample solutions to Exercise 5.
Let X be as in the statement.
For $I \in X$, we can find $q_I \in I \cap \mathbb{Q}$ since I is an interval which is non-empty and not reduced to a singleton.
Define $f : X \to \mathbb{Q}$ by $f(I) = q_I$. Let's prove that f is injective.
Let $I, J \in X$ such that $q \coloneqq f(I) = f(J)$. Then $q = q_I \in I$ and $q = q_J \in J$. Therefore $q \in I \cap J \neq \varnothing$. Thus $I = J$ (use the contrapositive of (ii)).
Hence $|X| \leq |\mathbb{Q}| = \aleph_0$. So X is countable.

Sample solutions to Exercise 6.

$\Rightarrow$ Let's prove that any infinite set admits a proper subset of same cardinality.
Let X be an infinite set. We want to construct $S \subsetneq X$ satisfying $|S| = |X|$.
Since X is infinite, $\aleph_0 \leq |X|$, i.e. there exists an injective function $f : \mathbb{N} \to X$.

We define $g : \begin{cases} X & \to & X \\ x & \mapsto & f(n+1) & \text{if } \exists n \in \mathbb{N},\ x = f(n) \\ x & \mapsto & x & \text{if } x \notin \operatorname{Im}(f) \end{cases}$.

- g is well-defined: given $x \in X$, if $\exists n, m \in \mathbb{N}$, $x = f(n) = f(m)$ then $n = m$ since f is injective.
- g is injective: let $x, y \in X$ be such that $g(x) = g(y)$.
  - First case: $g(x) = g(y) \in \operatorname{Im}(f)$ then there exists $n, m \in \mathbb{N}$ such that $x = f(n)$ and $y = f(m)$. Since $f(n+1) = g(x) = g(y) = f(m+1)$, we get that $n = m$ by injectiveness of f. Therefore $x = f(n) = f(m) = y$.
  - Second case: $g(x) = g(y) \notin \operatorname{Im}(f)$ then $x = g(x) = g(y) = y$.

  Note that $f(0) \notin \operatorname{Im}(g)$, thus $f(0) \in X \setminus \operatorname{Im}(g)$. Besides $g : X \to \operatorname{Im}(g)$ is a bijection. Hence $S = \operatorname{Im}(g)$ satisfies $S \subsetneq X$ and $|X| = |S|$.

$\Leftarrow$ We are going to prove the contrapositive: if a set is finite then it doesn't admit a proper subset of same cardinality.
Let X be a finite set. Let $S \subsetneq X$ be a proper subset.
Then there exists $x_0 \in X \setminus S$ so that $S \sqcup \{x_0\} \subset X$ and hence $|S \sqcup \{x_0\}| = |S| + 1 \leq |E|$, i.e. $|S| < |E|$.

Sample solutions to Exercise 7.

1. Assume by contradiction that $\mathbb{R} \setminus \mathbb{Q}$ is countable. Then $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}$ is countable as the union of two countable sets. Hence a contradiction.

2. One way to solve this question is to take an injective function $\mathbb{R} \to \mathbb{R}$ whose range is a proper interval of $\mathbb{R}$ and then to move the rational values in the complement of the range after making them irrational.
   For instance:
   Define $f : \mathbb{R} \to \mathbb{R} \setminus \mathbb{Q}$ by
   $$f(x) = \begin{cases} e^x & \text{if } e^x \notin \mathbb{Q} \\ -e^x - e & \text{otherwise} \end{cases}$$
   - f is well-defined: if $e^x \in \mathbb{Q}$ then $-e^x - e \in \mathbb{R} \setminus \mathbb{Q}$ (since $-e^x \in \mathbb{Q}$ and $-e \in \mathbb{R} \setminus \mathbb{Q}$).
   - f is injective: let $x, y \in \mathbb{R}$ be such that $f(x) = f(y)$.
     - First case: $f(x) = f(y) > 0$ then $f(x) = e^x$ and $f(y) = e^y$ thus $e^x = f(x) = f(y) = e^y$ and then $x = y$ since exp is injective.
     - Second case: $f(x) = f(y) < 0$ then $f(x) = -e^x - e$ and $f(y) = -e^y - e$ thus $-e^x - e = f(x) = f(y) = -e^y - e$, so that $e^x = e^y$ and hence $x = y$ since exp is injective.

     Note that $f(x) = f(y) \neq 0$ since $0 \in \mathbb{Q}$.

   Thus $|\mathbb{R}| \leq |\mathbb{R} \setminus \mathbb{Q}|$. Besides $|\mathbb{R} \setminus \mathbb{Q}| \leq |\mathbb{R}|$ since $\mathbb{R} \setminus \mathbb{Q} \subset \mathbb{R}$.
   Hence $|\mathbb{R}| = |\mathbb{R} \setminus \mathbb{Q}|$ by Cantor–Schröder–Bernstein theorem.

   Comment: (using the axiom of choice) it is true that if A and B are infinite sets then $|A \cup B| = \max(|A|, |B|)$ (but this statement was not proved in class, so you can't use it).
   Therefore, since $|\mathbb{Q}| < |\mathbb{R} \setminus \mathbb{Q}|$, $|\mathbb{R}| = |(\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}| = \max(|\mathbb{R} \setminus \mathbb{Q}|, |\mathbb{Q}|) = |\mathbb{R} \setminus \mathbb{Q}|$.

Sample solutions to Exercise 8.

Define $f : \mathbb{R} \to (0,1)$ by $f(x) = \frac{\arctan(x)+\frac{\pi}{2}}{\pi}$. Then

- f is well-defined:
  for $x \in \mathbb{R}$, $-\frac{\pi}{2} < \arctan(x) < \frac{\pi}{2}$ thus $0 < \arctan(x)+\frac{\pi}{2} < \pi$ and hence $0 < \frac{\arctan(x)+\frac{\pi}{2}}{\pi} < 1$, i.e. $f(x) \in (0,1)$.
- f is bijective: *prove it using that* $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *is bijective.*

Therefore $|(0,1)| = |\mathbb{R}|$.

There are lots of such bijections, for instance:

$$\begin{array}{ccc} (0,1) & \to & \mathbb{R} \\ x & \mapsto & \frac{1}{1+e^x} \end{array} \qquad \begin{array}{ccc} (0,1) & \to & \mathbb{R} \\ x & \mapsto & \frac{2x-1}{x-x^2} \end{array} \qquad \begin{array}{ccc} \mathbb{R} & \to & (0,1) \\ x & \mapsto & e^{-e^x} \end{array}$$

Sample solutions to Exercise 9.

1. **First method:**
   We define $f : (0,1) \times (0,1) \to (0,1)$ as follows. Let $(x,y) \in (0,1) \times (0,1)$.
   Denote the proper decimal expansions of x and y by $x = \sum_{k=1}^{+\infty} a_k 10^{-k} = 0.a_1a_2\dots$ where $a_k \in \{0,1,\dots,9\}$ are not all equal to 0 and $y = \sum_{k=1}^{+\infty} b_k 10^{-k} = 0.b_1b_2\dots$ similarly.
   Then we set $f(x,y) = \sum_{k=0}^{+\infty} a_k 10^{-(2k+1)} + \sum_{k=1}^{+\infty} b_k 10^{-2k} = 0.a_1b_1a_2b_2\dots = \sum_{k=1}^{+\infty} c_k 10^{-k}$ where
   $$c_k = \begin{cases} a_n & \text{if } \exists n \in \mathbb{N}\setminus\{0\},\ k = 2n \\ b_n & \text{if } \exists n \in \mathbb{N},\ k = 2n+1 \end{cases}$$
   Then f a bijection by existence and uniqueness of the proper decimal expansion.
   Hence $|(0,1) \times (0,1)| = |(0,1)|$.
   Since $|(0,1)| = |\mathbb{R}$, we get $|\mathbb{R} \times \mathbb{R}| = |(0,1) \times (0,1)| = |(0,1)| = |\mathbb{R}|$.

   Second method:
   Define $f : \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by $f(A,B) = \{2k \ :\ k \in A\} \cup \{2l+1 \ :\ l \in B\}$.
   Then f is bijective (*prove it*).
   Thus $|\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{N})|$.
   Since $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$, we get $|\mathbb{R} \times \mathbb{R}| = |\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

2. Let's prove by induction on $n \in \mathbb{N}\setminus\{0\}$ that $|\mathbb{R}^n| = |\mathbb{R}|$.
   - *Base case at $n = 1$:* then $\mathbb{R}^1 = \mathbb{R}$ thus $|\mathbb{R}^1| = |\mathbb{R}|$.
   - *Inductive step:* assume that $|\mathbb{R}^n| = |\mathbb{R}|$ for some $n \in \mathbb{N}\setminus\{0\}$. Then
   $$\begin{aligned} |\mathbb{R}^{n+1}| &= |\mathbb{R}^n \times \mathbb{R}| \\ &= |\mathbb{R} \times \mathbb{R}| \quad \text{since } |\mathbb{R}^n| = |\mathbb{R}| \text{ and } |\mathbb{R}| = |\mathbb{R}| \\ &= |\mathbb{R}| \quad \text{by the previous question} \end{aligned}$$

3. One idea here is to notice that $\left|\mathbb{R}^{\mathbb{N}}\right| = \left|\left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}}\right| = \left|\{0,1\}^{\mathbb{N}\times\mathbb{N}}\right| = \left|\{0,1\}^{\mathbb{N}}\right| = |\mathbb{R}|$.
   Since we have not covered arithmetic of cardinals, we need to prove each equality.
   - Since $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = \left|\{0,1\}^{\mathbb{N}}\right|$, there exists a bijection $\psi : \mathbb{R} \to \{0,1\}^{\mathbb{N}}$.
     We define $\varphi : \mathbb{R}^{\mathbb{N}} \to \left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}}$ by $\varphi(f) = \psi \circ f : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$.
     Then φ is a bijection (*check it*), and thus $\left|\mathbb{R}^{\mathbb{N}}\right| = \left|\left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}}\right|$.

- We define $\xi : \{0,1\}^{\mathbb{N}\times\mathbb{N}} \to \left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}}$ by $\xi(f) : \left\{ \begin{array}{ccc} \mathbb{N} & \to & \{0,1\}^{\mathbb{N}} \\ n & \mapsto & (m \mapsto f(n,m)) \end{array} \right.$.

  Check that ξ is a bijection. Therefore $\left|\left(\{0,1\}^{\mathbb{N}}\right)^{\mathbb{N}}\right| = \left|\{0,1\}^{\mathbb{N}\times\mathbb{N}}\right|$.

- Since $|\mathbb{N}\times\mathbb{N}| = |\mathbb{N}|$, there exists a bijection $\zeta : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$.
  We define $\gamma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ by $\gamma(f) = f \circ \zeta$.
  Check that γ is a bijection. Therefore $\left|\{0,1\}^{\mathbb{N}\times\mathbb{N}}\right| = \left|\{0,1\}^{\mathbb{N}}\right|$.

Sample solutions to Exercise 10.
Define $f : (0,1) \to S^2$ by $f(t) = (\cos t, \sin t, 0)$. Then f is well-defined and injective.
Thus $|\mathbb{R}| = |(0,1)| \leq |S^2|$.
Besides, since $S^2 \subset \mathbb{R}^3$, we have that $|S^2| \leq |\mathbb{R}^3| = |\mathbb{R}|$.
By Cantor–Schröder–Bernstein theorem, we get that $|S^2| = |\mathbb{R}|$.

Sample solutions to Exercise 11.
A circle is characterized by its center and its radius. Therefore there is a bijection $\mathbb{R}^2 \times (0,+\infty) \to S$ mapping (x,y,r) to the circle centered at (x,y) of radius r.
Thus $|S| = |\mathbb{R}^2 \times (0,+\infty)|$.
Since $\exp : \mathbb{R} \to (0,+\infty)$ is a bijection, we have $|(0,+\infty)| = |\mathbb{R}|$. Hence $|\mathbb{R}^2 \times (0,+\infty)| = |\mathbb{R}^3| = |\mathbb{R}|$.
Therefore $|S| = |\mathbb{R}|$.