University of Toronto – MAT246H1-S – LEC0201/9201

Concepts in Abstract Mathematics

Homework questions – Week 10

Jean-Baptiste Campesato

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Exercise 1.

Let E be a set.

1. Prove that $\forall A, B, C \in \mathcal{P}(E),\ A \cup B = B \cap C \implies A \subset B \subset C$.
2. Prove that $\forall A, B \in \mathcal{P}(E),\ A \cap B = A \cup B \implies A = B$.

Exercise 2.

Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be three functions.
Prove that $g \circ f$ and $h \circ g$ are bijective if and only if f, g and h are bijective.

Exercise 3.

Let $f : E \to F$.

1. Prove that $\forall A \in \mathcal{P}(E),\ A \subset f^{-1}(f(A))$.
2. Prove that $\forall B \in \mathcal{P}(F),\ f(f^{-1}(B)) \subset B$.
3. Can these inclusions be strict?

Exercise 4.

Let $f : E \to F$.

1. Prove that $\forall A, B \in \mathcal{P}(F),\ A \subset B \implies f^{-1}(A) \subset f^{-1}(B)$.
   Does the converse hold?
2. Prove that $\forall A, B \in \mathcal{P}(F),\ f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.
3. Prove that $\forall A, B \in \mathcal{P}(F),\ f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

Exercise 5.

Let $f : E \to F$.

1. Prove that $\forall A, B \in \mathcal{P}(E),\ A \subset B \implies f(A) \subset f(B)$.
   Does the converse hold?
2. Prove that $\forall A, B \in \mathcal{P}(E),\ f(A \cap B) \subset f(A) \cap f(B)$.
   Can the inclusion be strict?
3. Prove that $\forall A, B \in \mathcal{P}(E),\ f(A \cup B) = f(A) \cup f(B)$.

Exercise 6.

Let $f : E \to F$. Prove that f is injective if and only if $\forall A, B \in \mathcal{P}(E),\ f(A \cap B) = f(A) \cap f(B)$.

Exercise 7.

Let E be a finite set. For $A, B \in \mathcal{P}(E)$ we define the symmetric difference of A and B by $A \Delta B = (A \cup B) \setminus (A \cap B)$.
Prove that $\forall A, B \in \mathcal{P}(E),\ |A \Delta B| = |A| + |B| - 2|A \cap B|$.

Exercise 8.

Let E and F be two finite sets.

1. Prove that F^E (the set of functions $E \to F$) is finite and express $\left|F^E\right|$ in terms of $|E|$ and $|F|$.
2. Prove that the set $\{f \in E^F : f \text{ is injective}\}$ is finite and express its cardinal in terms of $|E|$ and $|F|$.
3. Prove that the set $\{f \in E^E : f \text{ is bijective}\}$ is finite and express its cardinal in terms of $|E|$.

The case of surjective functions is more tricky.

Exercise 9.
Let E be a finite set and $k \in \{0, 1, \dots, |E|\}$. What is the cardinal of $\{A \in \mathcal{P}(E) \ : \ |A| = k\}$?

Exercise 10.
Prove that a set E is finite if and only if $\mathcal{P}(E)$ is finite.
In this case, give an expression of $|\mathcal{P}(E)|$ in terms of $|E|$.

Exercise 11. *The pigeonhole principle or Dirichlet's drawer principle*
I had no enough time to cover this topic in lectures, so here it is :-).

1. Let E and F be two finite sets. Prove that $|E| \leq |F|$ if and only if there exists an injection $f : E \to F$.
2. Let E and F be two finite sets. Prove that if $|E| > |F|$ then there is no injective function $E \to F$.
   This statement is pigeonhole principle or Dirichlet's drawer principle: if you have n elements put in $k < n$ boxes, then at least one box contains two elements.
3. During a post-covid party with $n > 1$ participants, we may always find two people who shook hands to the same number of people.
4. Let $n \in \mathbb{N} \setminus \{0\}$. Let $a_1, a_2, \dots, a_n \in \mathbb{Z}$. Prove that there exists distinct $i_1, \dots, i_r \in \{1, \dots, n\}$, $r \geq 1$, so that $n | \sum_{k=1}^{r} a_{i_k}$.
5. Prove that among 13 distinct real numbers, there always exist two x, y satisfying $0 < \dfrac{x-y}{1+xy} < 2 - \sqrt{3}$.
   Hint: it looks like a trigonometric formula you know!

Sample solutions to Exercise 1.

1. Let $A, B, C \in \mathcal{P}(E)$. Assume that $A \cup B = B \cap C$.
   Let $x \in A$ then $x \in A \cup B = B \cap C$. Therefore $x \in B$. So $A \subset B$.
   Let $x \in B$ then $x \in A \cup B = B \cap C$. Therefore $x \in C$. So $B \subset C$.

2. *Using the previous question.*
   Let $A, B \in \mathcal{P}(E)$. Assume that $A \cap B = A \cup B$.
   From the previous question we get that $A \subset B \subset A$. Hence $A = B$.

   Direct proof.
   Let $A, B \in \mathcal{P}(E)$. Assume that $A \cap B = A \cup B$.
   Let $x \in A$ then $x \in A \cup B = A \cap B$. Thus $x \in B$. Therefore $A \subset B$.
   Let $x \in B$ then $x \in A \cup B = A \cap B$. Thus $x \in A$. Therefore $B \subset A$.
   Hence $A = B$.

   Proof by contrapositive.
   Let $A, B \in \mathcal{P}(E)$. Assume that $A \neq B$. Then

   - either $A \setminus B \neq \varnothing$ and then there exists $x \in E$ such that $x \in A$ and $x \notin B$. Thus $x \in A \cup B$ but $x \notin A \cap B$. Therefore $A \cap B \neq A \cup B$.
   - or $B \setminus A \neq \varnothing$ and then there exists $x \in E$ such that $x \in B$ and $x \notin A$. Thus $x \in A \cup B$ but $x \notin A \cap B$. Therefore $A \cap B \neq A \cup B$.

Sample solutions to Exercise 2.

$\Leftarrow$ Assume that f, g and h are bijective then $g \circ f$ and $h \circ g$ are too.
$\Rightarrow$ Assume that $g \circ f$ and $h \circ g$ are bijective.
Since $g \circ f$ is surjective, g is too. Since $h \circ g$ is injective, g is too.
Hence g is bijective, so it admits an inverse $g^{-1} : C \to B$.
Then $f = g^{-1} \circ (g \circ f)$ and $h = (h \circ g) \circ g^{-1}$ are bijective as composition of bijective functions.

Sample solutions to Exercise 3.

1. Let $A \in \mathcal{P}(E)$. Let $x \in A$. Then $f(x) \in f(A)$. Therefore $x \in f^{-1}(f(A))$.
   We proved that $A \subset f^{-1}(f(A))$.

2. Let $B \subset \mathcal{P}(F)$. Let $y \in f(f^{-1}(B))$. Then there exists $x \in f^{-1}(B)$ such that $y = f(x)$. But since $x \in f^{-1}(B)$, $y = f(x) \in B$.
   We proved that $f(f^{-1}(B)) \subset B$.

3. Define
   $$f : \left\{ \begin{array}{ccc} \{1,2\} & \to & \{1,2\} \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 1 \end{array} \right.$$
   Then $f(f^{-1}(\{1,2\})) = f(\{1,2\}) = \{1\} \subsetneq \{1,2\}$.
   And $f^{-1}(f(\{1\})) = f^{-1}(\{1\}) = \{1,2\} \supsetneq \{1\}$.

Sample solutions to Exercise 4.

1. Let $A, B \in \mathcal{P}(F)$ be such that $A \subset B$. Take $x \in f^{-1}(A)$. Then $f(x) \in A \subset B$. Thus $x \in f^{-1}(B)$.
   The converse doesn't hold. Indeed, define
   $$f : \left\{ \begin{array}{ccc} \{1\} & \to & \{1,2\} \\ 1 & \mapsto & 1 \end{array} \right.$$
   then $f^{-1}(\{2\}) = \varnothing \subset f^{-1}(\{1\})$. But $\{2\} \not\subset \{1\}$.

2. Let $A, B \in \mathcal{P}(F)$.
   Since $A \cap B \subset A$ and $A \cap B \subset B$, we get that $f^{-1}(A \cap B) \subset f^{-1}(A)$ and $f^{-1}(A \cap B) \subset f^{-1}(B)$. Thus $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$.
   For the other inclusion, let $x \in f^{-1}(A) \cap f^{-1}(B)$. Then $f(x) \in A$ and $f(x) \in B$. Thus $f(x) \in A \cap B$ so that $x \in f^{-1}(A \cap B)$. Thus $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$.

3. Let $A, B \in \mathcal{P}(F)$.
   Since $A, B \subset A \cup B$, we get $f^{-1}(A), f^{-1}(B) \subset f^{-1}(A \cup B)$. Thus $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$.
   For the other inclusion, let $x \in f^{-1}(A \cup B)$. Then $f(x) \in A \cup B$. Either $f(x) \in A$ and then $x \in f^{-1}(A) \subset f^{-1}(A) \cup f^{-1}(B)$ or $f(x) \in B$ and then $x \in f^{-1}(B) \subset f^{-1}(A) \cup f^{-1}(B)$. Thus $x \in f^{-1}(A) \cup f^{-1}(B)$. We proved that $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$.

Sample solutions to Exercise 5.

1. Let $A, B \in \mathcal{P}(E)$ be such that $A \subset B$. Let $y \in f(A)$. Then $y = f(x)$ for some $x \in A$. But $x \in A \subset B$. Thus $y = f(x) \in f(B)$. We proved that $f(A) \subset f(B)$.
   The converse doesn't hold. Indeed define
   $$f : \begin{cases} \{1,2\} & \to & \{1\} \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 1 \end{cases}$$
   then $f(\{1\}) = f(\{2\}) = \{1\}$ but $\{1\} \not\subset \{2\}$.

2. Let $A, B \in \mathcal{P}(E)$. Since $A \cap B \subset A, B$ we get $f(A \cap B) \subset f(A), f(B)$. Thus $f(A \cap B) \subset f(A) \cap f(B)$.
   The inclusion can be strict using the same example as above.

3. Let $A, B \in \mathcal{P}(E)$. Since $A, B \subset A \cup B$, we get $f(A), f(A) \subset f(A \cup B)$. Thus $f(A) \cup f(B) \subset f(A \cup B)$.
   For the other inclusion, let $y \in f(A \cup B)$. Then $y = f(x)$ for some $x \in A \cup B$. So either $x \in A$ and then $y = f(x) \in f(A) \subset f(A) \cup f(B)$, or $x \in B$ and then $y = f(x) \in f(B) \subset f(A) \cup f(B)$. In both cases $y \in f(A) \cup f(B)$. So we proved that $f(A \cup B) \subset f(A) \cup f(B)$.

Sample solutions to Exercise 6.

$\Rightarrow$ Assume that f is injective. Let $A, B \in \mathcal{P}(E)$.
We already know that $f(A \cap B) \subset f(A) \cap f(B)$ holds (see the previous exercise).
Let's prove that $f(A) \cap f(B) \subset f(A \cap B)$.
Let $y \in f(A) \cap f(B)$. Then $y = f(x_1)$ for some $x_1 \in A$ and $y = f(x_2)$ for some $x_2 \in B$.
Since $f(x_1) = f(x_2)$ and f is injective, we obtain that $x_1 = x_2 \in A \cap B$. Therefore $y = f(x_1) \in f(A \cap B)$.
We proved that $f(A) \cap f(B) \subset f(A \cap B)$. Thus $f(A) \cap f(B) = f(A \cap B)$.

$\Leftarrow$ Assume that $\forall A, B \in \mathcal{P}(E),\ f(A \cap B) = f(A) \cap f(B)$.
Let $x_1, x_2 \in E$ be such that $f(x_1) = f(x_2)$. Set $y \coloneqq f(x_1) = f(x_2)$.
Then $f(\{x_1\} \cap \{x_2\}) = f(\{x_1\}) \cap f(\{x_2\}) = \{y\} \cap \{y\} = \{y\}$.
Particularly $\{x_1\} \cap \{x_2\} \neq \varnothing$, thus $x_1 = x_2$.

Sample solutions to Exercise 7.

$$\begin{aligned} |A \Delta B| &= |(A \cup B) \setminus (A \cap B)| \\ &= |A \cup B| - |A \cap B| \\ &= |A| + |B| - |A \cap B| - |A \cap B| \\ &= |A| + |B| - 2|A \cap B| \end{aligned}$$

Sample solutions to Exercise 8.

1. Let $\varphi : \{k \in \mathbb{N} \; : \; k < |E|\} \to E$.
   Then $\psi \; : \; F^E \to F^{|E|}$ defined by $\psi(f) = (f(\varphi(0)), \dots, f(\varphi(|E|-1)))$ is a bijection (prove it).
   Therefore $\left|F^E\right| = \left|F^{|E|}\right| = |F|^{|E|}$.

2. According to the last exercise, there exists an injective function $E \to F$ if and only if $|E| \leq |F|$.
   Next, since E is finite, there exists a bijection $\varphi : \{k \in \mathbb{N} \; : \; k < |E|\} \to E$.
   For $f(\varphi(0))$ we have $|F|$ possible choices. For $f(\varphi(1))$ we have $|F \setminus \{f(\varphi(0))\}| = |F| - 1$ choices. For $f(\varphi(2))$ we have $|F \setminus \{f(\varphi(0)), f(\varphi(1))\}| = |F| - 2$ choices. And so on.
   Therefore, $|\{f \in E^F \; : \; f \text{ is injective}\}| = |F|(|F|-1)\cdots(|F|-|E|+1) = \frac{|F|!}{(|F|-|E|)!}$.
   Thus $|\{f \in E^F \; : \; f \text{ is injective}\}| = \begin{cases} 0 & \text{if } |E| > |F| \\ \frac{|F|!}{(|F|-|E|)!} & \text{if } |E| \leq |F| \end{cases}$.

3. It a special case of the above question when $|E| = |F|$: $|\{f \in E^E \; : \; f \text{ is bijective}\}| = \frac{|E|!}{(|E|-|E|)!} = |E|!$.

Sample solutions to Exercise 9.

The number of subsets with cardinality k included in a set of cardinality n is denoted $\binom{n}{k}$ read "n choose k".
We are going to prove that $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
Let E a finite set. Set $n = |E|$. Fix $k \in \{0, 1, \dots, n\}$.
An ordered list of k distinct elements is the same as fixing an injection $\{0, 1, \dots, k-1\} \to E$. So, using the previous question there are $\frac{n!}{(n-k)!}$ such ordered lists.
Two ordered lists of k elements give the same subset if and only if one is obtained from the other one permuting its elements, which is the same as constructing a bijection $\{0, 1, \dots, k-1\} \to \{0, 1, \dots, k-1\}$. From the previous question there are $k!$ such bijections.
Therefore $\binom{n}{k} = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)!k!}$.

Sample solutions to Exercise 10.

$\Rightarrow$
Method 1 (by induction):
Let's prove by induction on $n = |E|$ that $\mathcal{P}(E)$ is finite and that $|\mathcal{P}(E)| = 2^{|E|}$.

- *Base case at $n = 0$:* if $E = \varnothing$ then $\mathcal{P}(E) = \{\varnothing\}$ is finite.

- *Induction step:* assume that the statement holds for some $n \in \mathbb{N}$, i.e. if E is a set with $|E| = n$ then $\mathcal{P}(E)$ is finite and $|\mathcal{P}(E)| = 2^n$.
  Let E be a set such that $|E| = n + 1$. Since $|E| > 0$, there exists $x \in E$.
  By the induction hypothesis, since $|E \setminus \{x\}| = n$, we get that $\mathcal{P}(E \setminus \{x\})$ is finite and $|\mathcal{P}(E \setminus \{x\})| = 2^n$.
  Note that $\mathcal{P}(E \setminus \{x\}) = \{A \in \mathcal{P}(E) \; : \; x \notin A\}$ and that
  $$\begin{array}{ccc} \{A \in \mathcal{P}(E) \; : \; x \notin A\} & \to & \{A \in \mathcal{P}(E) \; : \; x \in A\} \\ A & \mapsto & A \cup \{x\} \end{array}$$
  is a bijection.
  Therefore $\mathcal{P}(E) = \{A \in \mathcal{P}(E) \; : \; x \notin A\} \sqcup \{A \in \mathcal{P}(E) \; : \; x \in A\}$ is finite and
  $|\mathcal{P}(E)| = |\{A \in \mathcal{P}(E) \; : \; x \notin A\}| + |\{A \in \mathcal{P}(E) \; : \; x \in A\}| = 2^n + 2^n = 2^{n+1}$.

Method 2 (*using the previous exercise*):
Let E be a finite set. We know that for $k = 0, \dots, |E|$, the number of subsets with k elements is $\binom{n}{k}$.
Therefore the number of subsets included in E is

$$|\mathcal{P}(E)| = \sum_{k=0}^{|E|} \binom{n}{k} = \sum_{k=0}^{|E|} \binom{n}{k} 1^k 1^{|E|-k} = (1+1)^{|E|} = 2^{|E|}$$

Method 3 (*which generalizes to infinite sets*):
Let E be a finite set. We define $\psi : \mathcal{P}(E) \to \{0,1\}^E$ by $\psi(A)(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$.
Then ψ is a bijection thus $\mathcal{P}(E)$ is finite since $\{0,1\}^E$ is and moreover $|\mathcal{P}(E)| = \left|\{0,1\}^E\right| = 2^{|E|}$.

$\Leftarrow$ Let E be a set. Assume that $\mathcal{P}(E)$ is finite.
Note that $\Phi : E \to \mathcal{P}(E)$ defined by $\Phi(x) = \{x\}$ is injective. Therefore E is finite too.

Sample solutions to Exercise 11.

1. $\Rightarrow$ Assume that $|E| \le |F|$.
   There exist bijections $\varphi : \{k \in \mathbb{N} : k < |E|\} \to E$ and $\psi : \{k \in \mathbb{N} : k < |F|\} \to F$.
   Since $|E| \le |F|$, $f = \psi \circ \varphi^{-1} : E \to F$ is well-defined and injective.
   $\Rightarrow$ Assume that there exists an injection $f : E \to F$.
   Then f induces a bijection $f : E \to f(E)$, so that $|E| = |f(E)|$.
   And since $f(E) \subset F$, we have $|f(E)| \le |F|$.

2. It is a consequence of the previous question.

3. A participant shook either 0, 1,... or $n-1$ hands. So we have n "boxes". Not that it is not possible to have at the same time the boxes 0 and $n-1$ non-empty. Therefore we have only $n-1$ boxes for n participants, so two participants must have shaken the same number of boxes.
   Formally:

   - First case: there is at least one participant who didn't shake any hand. Then $f : \{\text{participants}\} \to \{0, 1, \dots, n-2\}$ mapping each participant to the number of hands he shook is well-defined. Since $|\{\text{participants}\}| = n > n-1 = |\{0, 1, \dots, n-2\}|$, f can't be injective. Therefore at least two participants shooke the same number of hands.
   - Second case: all participants shooke at least one hand. Then $f : \{\text{participants}\} \to \{1, \dots, n-1\}$ mapping each participant to the number of hands he shook is well-defined. Since $|\{\text{participants}\}| = n > n-1 = |\{1, 2, \dots, n-1\}|$, f can't be injective. Therefore at least two participants shooke the same number of hands.

4. For $r = 1, 2, \dots, n$, set $s_r = \sum_{k=1}^{r} a_k$.

   - First case: there exists r such that $n | s_r$. Then we are done.
   - Second case: otherwise, we have n numbers $s_1, \dots, s_n$ whose remainders for the Euclidean division by n are among $1, \dots, n-1$ (i.e. $n-1$ possible remainders). Hence at least two have the same remainders, let's say s_p and s_q with $q > p$. Then $n | s_q - s_p = \sum_{k=p+1}^{q} a_k$.

5. First, note that $\frac{\tan a - \tan b}{1 + \tan a \tan b} = \tan(a-b)$ and that

$$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{4}} = \frac{\sqrt{3}-1}{1+\sqrt{3}} = \frac{\left(\sqrt{3}-1\right)^2}{2} = 2 - \sqrt{3}$$

Let $x_1, \dots, x_{13}$ be 13 distinct real numbers. We set $\alpha_k = \arctan x_k \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Note that the α_k are distinct since arctan is injective.
Note that $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = I_1 \sqcup I_2 \sqcup I_3 \sqcup \cdots \sqcup I_{12}$ where

$$I_1 = \left(-\frac{\pi}{2}, -\frac{\pi}{2} + \frac{\pi}{12}\right], \quad I_2 = \left(-\frac{\pi}{2} + \frac{\pi}{12}, -\frac{\pi}{2} + 2\frac{\pi}{12}\right], \quad \dots, \quad I_{11} = \left(-\frac{\pi}{2} + 10\frac{\pi}{12}, \frac{\pi}{2} + 11\frac{\pi}{12}\right], \quad I_{12} = \left(-\frac{\pi}{2} + 11\frac{\pi}{12}, \frac{\pi}{2}\right)$$

We define $f : \{\alpha_1, \dots, \alpha_{13}\} \to \{1, \dots, 12\}$ by $f(\alpha_k) = r$ where $\alpha_k \in I_r$.
Since $|\{\alpha_1, \dots, \alpha_{13}\}| = 13 > 12 = |\{1, \dots, 12\}|$, f is not injective. So there exists $alpha_k < \alpha_l$ and $r = 1, \dots, 12$ such that $\alpha_k, \alpha_l \in I_r$. Then $0 < \alpha_l - \alpha_k < \frac{\pi}{12}$.
Since tan is increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we get that $\tan 0 < \tan(\alpha_l - \alpha_k) < \tan\frac{\pi}{12} = 2 - \sqrt{3}$.
Note that $\tan(\alpha_l - \alpha_k) = \frac{\tan\alpha_l - \tan\alpha_k}{1 + \tan\alpha_l \tan\alpha_k} = \frac{x_l - x_k}{1 + x_l x_k}$. Thus $0 < \frac{x_l - x_k}{1 + x_l x_k} <= 2 - \sqrt{3}$ as requested.