University of Toronto – MAT246H1-S – LEC0201/9201

Concepts in Abstract Mathematics

Homework questions – Week 9

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Exercise 1.

1. Prove that $\forall x, y \in \mathbb{R},\ \lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x+y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$.
2. Prove that $\forall n \in \mathbb{N} \setminus \{0\},\ \forall x \in \mathbb{R},\ \left\lfloor \frac{\lfloor nx \rfloor}{n} \right\rfloor = \lfloor x \rfloor$.

Exercise 2.

1. Prove that $\forall n \in \mathbb{N},\ \left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n \in 2\mathbb{N}$.
2. Prove that for every $n \in \mathbb{N}$, $\left\lfloor \left(2+\sqrt{3}\right)^n \right\rfloor$ is odd.

Exercise 3.
Let I and J be two open intervals of $\mathbb{R}$. Prove that $(I \cap \mathbb{Q}) \cap (J \cap \mathbb{Q}) = \varnothing \implies I \cap J = \varnothing$.

Exercise 4.

1. Is the sum of two irrational numbers always an irrational number?
2. Is the product of two irrational numbers always an irrational number?
3. Prove that $\forall x \in \mathbb{R} \setminus \mathbb{Q},\ \forall y \in \mathbb{Q},\ x + y \notin \mathbb{Q}$.
4. Prove that $\forall x \in \mathbb{R} \setminus \mathbb{Q},\ \forall y \in \mathbb{Q} \setminus \{0\},\ xy \notin \mathbb{Q}$.

Exercise 5.
Prove that the following numbers are irrational

1. $\sqrt{3}$
2. $\sqrt{6}$
3. $\sqrt{11}$
4. $\sqrt[3]{3+\sqrt{11}} \notin \mathbb{Q}$
5. $\sqrt{2}+\sqrt{3}$
6. $\left(\sqrt{2}+\sqrt{3}\right)^2$
7. $\sqrt{2}+\sqrt{3}+\sqrt{6}$
8. $\left(3\sqrt{2}+2\sqrt{3}+\sqrt{6}\right)^2$
9. $\sqrt{7}+\sqrt{3}$.

Exercise 6.
Prove that $\forall n \in \mathbb{N},\ \sqrt{n} \in \mathbb{Q} \Leftrightarrow \sqrt{n} \in \mathbb{N} \Leftrightarrow \exists m \in \mathbb{N},\ n = m^2$.

Exercise 7.

Is $\sum_{n=1}^{+\infty} 10^{-\frac{n(n+1)}{2}} = 0.101001000100001000001\ldots$ a rational number?

Exercise 8.

1. We fix $r > 0$ and $n \in \mathbb{N}$. We define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{1}{n!}x^n(1-x)^n$ and we set

$$F(x) = \sum_{k\geq 0}(-1)^k r^{2n-2k} f^{(2k+1)}(x)$$

(*note that the sum is finite since f is a polynomial*).

   (a) Prove that $\forall k \in \mathbb{N},\ f^{(k)}(0) \in \mathbb{Z}$.

   (b) Prove that $\forall k \in \mathbb{N},\ f^{(k)}(1) \in \mathbb{Z}$.

   (c) Prove that $F''(x) = -r^2F(x) + r^{2n+2}f(x)$.

   (d) Compute $\frac{d}{dx}\left(F'(x)\sin(rx) - rF(x)\cos(rx)\right)$.

   (e) Compute $\int_0^1 f(x)\sin(rx)\mathrm{d}x$.

2. Prove that $\forall r \in (0, \pi],\ r \in \mathbb{Q} \implies (\sin(r) \notin \mathbb{Q} \text{ or } \cos(r) \notin \mathbb{Q})$.

3. Prove that $\pi \notin \mathbb{Q}$.

Sample solutions to Exercise 1.

1. Let $x, y \in \mathbb{R}$. By definition of the floor function, we know that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ and $\lfloor y \rfloor \leq y < \lfloor y \rfloor + 1$.
   Therefore $\lfloor x \rfloor + \lfloor y \rfloor \leq x + y$.
   Since $\lfloor x + y \rfloor$ is the greater integer less than or equal to $x + y$, we get that $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$.
   Finally $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq x + y \leq \lfloor x \rfloor + \lfloor y \rfloor + 2$.
   So, either $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ or $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + 1$.
   In both cases we have $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$.

2. Let $n \in \mathbb{N} \setminus \{0\}$ and $x \in \mathbb{R}$. Since $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, we get $n\lfloor x \rfloor \leq nx < n\lfloor x \rfloor + n$.
   Since $\lfloor nx \rfloor$ is the greatest integer less than or equal to nx, we obtain $n\lfloor x \rfloor \leq \lfloor nx \rfloor \leq nx < n\lfloor x \rfloor + n$.
   Thus $\lfloor x \rfloor \leq \frac{\lfloor nx \rfloor}{n} < \lfloor x \rfloor + 1$ and hence $\lfloor x \rfloor = \left\lfloor \frac{\lfloor nx \rfloor}{n} \right\rfloor$.

Sample solutions to Exercise 2.

1. Let $n \in \mathbb{N}$. Then

$$\begin{aligned}\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \sqrt{3}^k + \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-1)^k \sqrt{3}^k \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(1 + (-1)^k\right) 2^{n-k} \sqrt{3}^k\end{aligned}$$

   Note that if k is odd then $\left(1 + (-1)^k\right) = 0$ and that if $k = 2l$ is even then

$$\binom{n}{k} \left(1 + (-1)^k\right) 2^{n-k} \sqrt{3}^k = \binom{n}{2l} \times 2 \times 2^{n-2l} \times 3^l \in 2\mathbb{N}$$

   Therefore $\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n \in 2\mathbb{N}$.

2. Let $n \in \mathbb{N}$. Since $2 - \sqrt{3} \in (0, 1)$, we have that $0 < \left(2-\sqrt{3}\right)^n < 1$.
   Therefore, if we set $S = \left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n$, then $S < \left(2+\sqrt{3}\right)^n < S - 1$, thus

$$S - 1 \leq \left(2+\sqrt{3}\right)^n < S$$

   i.e. $\left\lfloor \left(2+\sqrt{3}\right)^n \right\rfloor = S - 1$ which is odd according to the previous question.

Sample solutions to Exercise 3.

Let's prove the contrapositive, i.e. $I \cap J \neq \varnothing \implies (I \cap \mathbb{Q}) \cap (J \cap \mathbb{Q}) \neq \varnothing$.
Assume that $I \cap J \neq \varnothing$, then there exists $a \in I \cap J$.
Since I is an open interval, there exists $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subset I$.
Similarly, there exists $\eta > 0$ such that $(a - \eta, a + \eta) \subset J$.
Set $\delta = \min(\varepsilon, \eta)$, then $(a - \delta, a + \delta) \subset I \cap J$.
Since between two reals there exists a rational, we know that there exists $q \in \mathbb{Q}$ such that $a - \delta < q < a + \delta$.
Therefore $q \in I \cap \mathbb{Q}$ and $q \in J \cap \mathbb{Q}$, so that $(I \cap \mathbb{Q}) \cap (J \cap \mathbb{Q}) \neq \varnothing$.

Sample solutions to Exercise 4.

1. No: $\sqrt{2}$ and $-\sqrt{2}$ are both irrational but $(\sqrt{2}) + (-\sqrt{2}) = 0 \in \mathbb{Q}$.

2. No: $(\sqrt{2})(\sqrt{2}) = 2 \in \mathbb{Q}$.

3. Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and $y \in \mathbb{Q}$. Assume by contradiction that $x + y \in \mathbb{Q}$ then $x = (x + y) - y \in \mathbb{Q}$.

4. Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and $y \in \mathbb{Q} \setminus \{0\}$. Assume by contradiction that $xy \in \mathbb{Q}$ then $x = \frac{xy}{y} \in \mathbb{Q}$.

Sample solutions to Exercise 5.

1. Assume by contradiction that $\sqrt{3} \in \mathbb{Q}$ then $\sqrt{3} = \frac{a}{b}$ where $a \in \mathbb{N}$ and $b \in \mathbb{N} \setminus \{0\}$. Hence $a^2 = 3b^2$.
   The prime factorization of a^2 contains an even number of primes whereas the prime factorization of $3b^2$ contains an odd number of primes.
   Therefore it contradicts the uniqueness of the prime factorization.

2. Assume by contradiction that $\sqrt{6} \in \mathbb{Q}$ then $\sqrt{6} = \frac{a}{b}$ where $a \in \mathbb{N}$, $b \in \mathbb{N} \setminus \{0\}$ and $\gcd(a, b) = 1$.
   Therefore $6b^2 = a^2$. So $2|a^2$. By Euclid's lemma, $2|a$, so there exists $k \in \mathbb{N}$ such that $a = 2k$.
   Hence we may rewrite $6b^2 = 4k^2$, which implies $3b^2 = 2k^2$. So $2|3b^2$.
   Since $\gcd(2, 3) = 1$, by Gauss' lemma we get $2|b^2$ and then by Euclid's lemma, we get $2|b$.
   Therefore $2|\gcd(a, b) = 1$. Hence a contradiction.

3. Same as for $\sqrt{3}$..

4. Assume by contradiction that $x = \sqrt[3]{3 + \sqrt{11}} \in \mathbb{Q}$. Then $x^3 = 3 + \sqrt{11}$. So $\sqrt{11} = x^3 - 3 \in \mathbb{Q}$.

5. Assume by contradiction that $\sqrt{2} + \sqrt{3} \in \mathbb{Q}$.
   Then $(\sqrt{2} + \sqrt{3})^2 = 2 + 3 + 2\sqrt{6} \in \mathbb{Q}$. Therefore $\sqrt{6} = \frac{(\sqrt{2}+\sqrt{3})^2-5}{2} \in \mathbb{Q}$.

6. Assume by contradiction that $\left(\sqrt{2} + \sqrt{3}\right)^2 \in \mathbb{Q}$.
   Since $\left(\sqrt{2} + \sqrt{3}\right)^2 = 2 + 3 + 2\sqrt{6}$, we get $\sqrt{6} = \frac{\left(\sqrt{2}+\sqrt{3}\right)^2-5}{2} \in \mathbb{Q}$.

7. Assume by contradiction that $x = \sqrt{2} + \sqrt{3} + \sqrt{6} \in \mathbb{Q}$.
   Then $\sqrt{2} + \sqrt{3} = x - \sqrt{6}$. Squaring both sides, we get $5 + 2\sqrt{6} = x^2 + 6 - 2x\sqrt{6}$.
   Therefore $\sqrt{6} = \frac{x^2+1}{2+2x} \in \mathbb{Q}$.

8. Assume by contradiction that $\left(3\sqrt{2} + 2\sqrt{3} + \sqrt{6}\right)^2 \in \mathbb{Q}$.
   Since $\left(3\sqrt{2} + 2\sqrt{3} + \sqrt{6}\right)^2 = 36+12\left(\sqrt{2} + \sqrt{3} + \sqrt{6}\right)$, we get $\sqrt{2}+\sqrt{3}+\sqrt{6} = \frac{\left(3\sqrt{2}+2\sqrt{3}+\sqrt{6}\right)^2-36}{12} \in \mathbb{Q}$.

9. There is an elegant method using the complex conjugate.
   Assume by contradiction that $\sqrt{7} + \sqrt{3} \in \mathbb{Q}$. Then $\left(\sqrt{7} + \sqrt{3}\right)\left(\sqrt{7} - \sqrt{3}\right) = 7 - 3 = 4$. Thus $\sqrt{7} - \sqrt{3} = \frac{4}{\sqrt{7}+\sqrt{3}} \in \mathbb{Q}$.
   Hence $\sqrt{3} = \frac{\left(\sqrt{7}+\sqrt{3}\right)-\left(\sqrt{7}-\sqrt{3}\right)}{2} \in \mathbb{Q}$.

Sample solutions to Exercise 6.

Let $n \in \mathbb{N}$.

- $\sqrt{n} \in \mathbb{Q} \Rightarrow \sqrt{n} \in \mathbb{N}$:
  Assume that $\sqrt{n} \in \mathbb{Q}$, then there exists $(a, b) \in \mathbb{N} \setminus \{0\}$ such that $\sqrt{n} = \frac{a}{b}$ and $\gcd(a, b) = 1$.
  Then $a^2 = nb^2$, thus $b|a^2$. By Gauss' lemma applied twice $b|a$ and then $b|1$. Thus $b = 1$ and $\sqrt{n} = a \in \mathbb{N}$.
- $\sqrt{n} \in \mathbb{N} \Rightarrow \exists m \in \mathbb{N},\ n = m^2$: assume that $\sqrt{n} \in \mathbb{N}$. Then $n = \left(\sqrt{n}\right)^2$. So we can take $m = \sqrt{n}$.
- $\exists m \in \mathbb{N},\ n = m^2 \Rightarrow \sqrt{n} \in \mathbb{Q}$: assume that there exists $m \in \mathbb{N}$ such that $n = m^2$. Then $\sqrt{n} = m \in \mathbb{N} \subset \mathbb{Q}$.

Sample solutions to Exercise 7.

No, $\displaystyle\sum_{n=1}^{+\infty} 10^{-\frac{n(n+1)}{2}} = 0.101001000100001000001\dots$ is not rational since its decimal expansion is not eventually periodic.
We denote the decimals by $(a_k)_{k\geq 1}$: $a_k = 1$ if $\exists n \in \mathbb{N},\ k = \frac{n(n+1)}{2}$ and $a_k = 0$ otherwise.
Let $r \in \mathbb{N}$ and $s \in \mathbb{N} \setminus \{0\}$.
Then there exists $k \in \mathbb{N}$ such that $r + k > \frac{s(s+1)}{2}$ and $a_{r+k} = 1$, so that $0 = a_{r+k+s} \neq a_{r+k} = 1$.

Sample solutions to Exercise 8.

1. (a) Note that $f(x) = \displaystyle\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k x^{n+k} = \frac{1}{n!}\sum_{k=n}^{2n}\binom{n}{k-n}(-1)^{k-n}x^k$.
   Let $k \in \mathbb{N}$. If $k < n$ or $k > 2n$ then $f^{(k)}(0) = 0$.
   Otherwise, if $n \leq k \leq 2n$ then $f^{(k)}(0) = (-1)^{k-n}\displaystyle\frac{k!}{n!}\binom{n}{k-n} \in \mathbb{Z}$.

   (b) Let $k \in \mathbb{N}$. Since $f(x) = f(1-x)$, we get $f^{(k)}(1) = (-1)^k f^{(k)}(0) \in \mathbb{Z}$.

   (c)
$$\begin{aligned}
F''(x) &= \sum_{k\geq 0}(-1)^k r^{2n-2k} f^{(2(k+1)+1)}(x)\\
&= -r^2\sum_{k\geq 0}(-1)^{k+1} r^{2n-2(k+1)} f^{(2(k+1)+1)}(x)\\
&= -r^2\sum_{k\geq 1}(-1)^k r^{2n-2k} f^{(2k+1)}(x)\\
&= -r^2\left(F(x) - r^{2n}f(x)\right)\\
&= -r^2F(x) + r^{2n+2}f(x)
\end{aligned}$$

   (d)
$$\begin{aligned}
\frac{d}{dx}\left(F'(x)\sin(rx) - rF(x)\cos(rx)\right) &= F''(x)\sin(rx) + rF'(x)\cos(rx) - rF'(x)\cos(rx) + rF(x)\sin(rx)\\
&= F''(x)\sin(rx) + rF(x)\sin(rx)\\
&= \left(F''(x) + rF(x)\right)\sin(rx)\\
&= r^{2n+2}f(x)\sin(rx)
\end{aligned}$$

   (e)
$$\begin{aligned}
\int_0^1 f(x)\sin(rx)\mathrm{d}x &= \frac{1}{r^{2n+2}}\int_0^1 r^{2n+2}f(x)\sin(rx)\mathrm{d}x\\
&= \frac{1}{r^{2n+2}}\left[F'(x)\sin(rx) - rF(x)\cos(rx)\right]_0^1\\
&= \frac{1}{r^{2n+2}}\left(F'(1)\sin(r) - rF(1)\cos(r) + rF(0)\right)
\end{aligned}$$

2. Let $r \in (0,\pi]\cap\mathbb{Q}$. Assume by contradiction that $\sin(r), \cos(r) \in \mathbb{Q}$. Then, we may write $\frac{1}{r} = \frac{a}{d}$, $\sin(r) = \frac{b}{d}$ and $\cos(r) = \frac{c}{d}$ where $a, b, c \in \mathbb{Z}$ and $d \in \mathbb{N}\setminus\{0\}$.
   Let $n \in \mathbb{N}$, then using 1.(e), 1.(a) and 1.(b) we get that $I_n = \frac{A_n}{d^{2n+3}}$ for some $A_n \in \mathbb{Z}$.
   Since $I_n > 0$, we get that $A_n \geq 1$, and thus that $I_n \geq \frac{1}{d^{2n+3}}$.
   But we also have that
$$\begin{aligned}
I_n &= \int_0^1 f(x)\sin(rx)\mathrm{d}x\\
&\leq \int_0^1 f(x)\mathrm{d}x \quad \textit{since } \sin > 0 \textit{ on } (0,\pi)\\
&\leq \frac{1}{n!} \quad \textit{since } f(x) \leq \frac{1}{n!} \textit{ on } [0,1]
\end{aligned}$$
   Therefore $\frac{1}{d^{2n+3}} \leq I_n \leq \frac{1}{n!}$ and thus $n! \leq d^{2n+3}$. Which leads to a contradiction for n large enough.

3. We use the contrapositive of the previous question: since $\pi \in (0, \pi]$ and since $\sin(\pi) = 0 \in \mathbb{Q}$ and $\cos(\pi) = -1 \in \mathbb{Q}$, we get that $\pi \notin \mathbb{Q}$.