University of Toronto – MAT246H1-S – LEC0201/9201

Concepts in Abstract Mathematics

Homework questions – Week 8

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Exercise 1.

Prove that $\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}} \in \mathbb{N}$.

Exercise 2.

1. Prove that $\forall a, b \in \mathbb{R},\ ab \leq \frac{a^2+b^2}{2}$.
2. Prove that $\forall a, b, c \in \mathbb{R},\ ab+bc+ac \leq a^2+b^2+c^2$.
3. Prove that $\forall a, b, c \in \mathbb{R},\ 3ab+3bc+3ac \leq (a+b+c)^2$.

Exercise 3.

Prove that $\forall x \in \mathbb{R},\ |x-1| \leq x^2-x+1$.

Exercise 4.

1. Prove that $\forall x, y \in \mathbb{R},\ |x|+|y| \leq |x+y|+|x-y|$.
2. Prove that $\forall x, y \in \mathbb{R},\ \frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|}+\frac{|y|}{1+|y|}$.

Exercise 5.

Let $A \subset \mathbb{R}$ be non-empty and bounded. We set $B = \{|x-y| \ :\ x, y \in A\}$.

1. Prove that B admits a supremum.
2. Prove that $\sup B = \sup A - \inf A$.

Exercise 6.

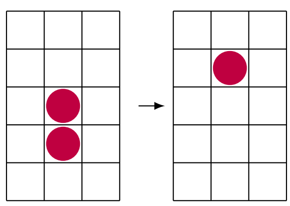
Prove that if $f : [0,1] \to [0,1]$ is non-decreasing then f admits a fixed point, i.e. $\exists a \in [0,1],\ f(a) = a$.
Hint: study $\{x \in [0,1] \ :\ f(x) \geq x\}$.

Exercise 7.

Let $A \subset \mathbb{R}$ be non-empty and bounded from above. Set $M = \sup(A)$.
Prove that if $M \notin A$ then for all $\varepsilon > 0$ the set $(M-\varepsilon, M) \cap A$ contains infinitely many elements.

Exercise 8. *Conway's Soldiers, or why geometric series are useful*

We consider an infinite checkerboard represented by $\mathbb{Z}\times\mathbb{Z}$ with pieces on it. The pieces are allowed to move using the peg solitaire rules: a move consists of one piece jumping over another piece into an empty cell (either horizontally or vertically), the piece which was jumped over is then removed.



The goal of this exercise is to show that there is no initial configuration with finitely many pieces located on $\mathbb{Z} \times \mathbb{Z}_{\leq 0}$ allowing to reach cells with y-coordinate 5.

1. Prove that there exists initial configurations allowing to reach cells with y-coordinate $1, 2, 3$ and 4.

2. We denote by σ the positive root of $x^2 + x - 1 = 0$ and we fix a target cell on $\mathbb{Z} \times \mathbb{Z}$.
   We label each cell of $\mathbb{Z} \times \mathbb{Z}$ with σ^n where n is the Manhattan distance from the target to the cell.

σ^4	σ^3	σ^2	σ^3
σ^3	σ^2	σ^1	σ^2
σ^2	σ^1	$\cancel{\sigma^0}$	σ^1
σ^3	σ^2	σ^1	σ^2
σ^4	σ^3	σ^2	σ^3

   Given a finite configuration C (i.e. finitely pieces on the checkerboard), we define $F(C) = \sum_{i \in C} \sigma^{n_i}$ where n_i is the Manhattan from the target cell to the cell i.
   Prove that if C' is a configuration obtained after one move then $F(C') - F(C) \leq 0$.

3. Compute $\displaystyle\sum_{n=2}^{+\infty} \sigma^n$.

4. Assume that the target cell is $(0, 5)$. Compute $F(C)$ where C contains all the cells with non-positive y-coordinates (hence C contains infinitely many cells).

5. Conclude that there is no finite initial configuration in $\mathbb{Z} \times \mathbb{Z}_{\leq 0}$ allowing to reach $(0, 5)$.

Sample solutions to Exercise 1.

Set $\alpha = \sqrt{7+4\sqrt{3}} + \sqrt{7-4\sqrt{3}}$ then

$$\alpha^2 = 14 + 2\sqrt{\left(7+4\sqrt{3}\right)\left(7-4\sqrt{3}\right)} = 14 + 2\sqrt{7^2 - 4^2 \times 3} = 14 + 2\sqrt{1} = 16$$

So $\alpha = \pm 4$, but since $\alpha > 0$, we get $\alpha = 4$.

Sample solutions to Exercise 2.

1. Let $a, b \in \mathbb{R}$, then $0 \leq (a-b)^2 = a^2 + b^2 - 2ab$, so that $ab \leq \frac{a^2+b^2}{2}$.

2. Let $a, b, c \in \mathbb{R}$. We know from the previous question that $ab \leq \frac{a^2+b^2}{2}$, $bc \leq \frac{b^2+c^2}{2}$ and $ac \leq \frac{a^2+c^2}{2}$.
By summing these three inequalities, we get $ab + bc + ac \leq \frac{a^2+b^2+b^2+c^2+a^2+c^2}{2} = a^2 + b^2 + c^2$.

3. Let $a, b, c \in \mathbb{R}$. Then

$$\begin{aligned}(a+b+c)^2 &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ac \\ &\geq ab + bc + ac + 2ab + 2bc + 2ac \qquad \text{from the previous question.} \\ &= 3ac + 3bc + 3ac\end{aligned}$$

Sample solutions to Exercise 3.

Let $x \in \mathbb{R}$.

- First case: $x \leq 1$ then $x^2 - x + 1 - |x-1| = x^2 - x + 1 + (x-1) = x^2 \geq 0$, therefore $|x-1| \leq x^2 - x + 1$.
- Second case: $x > 1$ then $x^2 - x + 1 - |x-1| = x^2 - x + 1 - (x-1) = x^2 - 2x + 2 = (x-1)^2 + 1 > 0$, therefore $|x-1| \leq x^2 - x + 1$.

Sample solutions to Exercise 4.

1. Let $x, y \in \mathbb{R}$. Then $2|x| = |2x| = |(x+y)+(x-y)| \leq |x+y| + |x-y|$ and similarly $2|y| \leq |x+y| + |x-y|$.
Summing these two inequalities, we obtain $2(|x|+|y|) \leq 2(|x+y|+|x-y|)$.

2. Define $f : [0, +\infty) \to \mathbb{R}$ by $f(u) = \frac{u}{1+u}$ then f is differentiable and $f'(u) = \frac{1}{(1+u)^2} > 0$.
Therefore f is increasing.
Let $x, y \in \mathbb{R}$. Since $|x+y| \leq |x| + |y|$, we obtain

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|+|y|}{1+|x|+|y|} = \frac{|x|}{1+|x|+|y|} + \frac{|y|}{1+|x|+|y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$

Sample solutions to Exercise 5.

1. Since A is non-empty, there exists $x \in A$ and then $0 = |x-x| \in B$. Therefore B is non-empty.
Since A is bounded, there exists $M \in \mathbb{R}$ such that $\forall x \in A$, $|x| \leq M$.
Therefore, if $x, y \in A$, then $|x-y| \leq |x| + |y| \leq 2M$. Thus $2M$ is an upper bound of B.
Since B is a non-empty subset of $\mathbb{R}$ which is bounded from above, it admits a supremum.

2. Since A is non-empty and bounded from below, there exists $m = \inf(A)$.
Similarly, since A is non-empty and bounded from above, there exists $M = \sup(A)$.
Let $x, y \in A$, then $m \leq x \leq M$ and $-M \leq -y \leq -m$, thus $-(M-m) \leq x - y \leq M - m$, i.e. $|x-y| \leq M - m$.
Thus $M - m$ is an upper bound of B. Let's check it is the least one.
Let $\varepsilon > 0$. Since $m = \inf(A)$, there exists $y \in A$ such that $y \leq m + \frac{\varepsilon}{2}$. Since $M = \sup(A)$, there exists $x \in A$ such that $M - \frac{\varepsilon}{2} \leq x$. Therefore $|x-y| \geq x - y \geq M - m + \varepsilon$.
We proved that for every $\varepsilon > 0$, there exists $|x-y| \in B$ such that $|x-y| \geq M - m + \varepsilon$.
Therefore $\sup(B) = M - m$.

Sample solutions to Exercise 6.

Set $E = \{x \in [0,1] \; : \; f(x) \geq x\}$. Since $f(0) \in [0,1]$, we have that $f(0) \geq 0$, so $0 \in E$.
Besides E is bounded from above by 1.
Thence, by the least upper bound principle, E admits a supremum $a = \sup(E)$.
Assume by contradiction that $f(a) \neq a$, then

- Either $f(a) < a$. Since a is the least upper bound of E, $f(a)$ is not an upper bound, so there exists $b \in E$ such that $f(a) < b \leq a$.
  But then $b \leq a$ and $f(a) < b \leq f(b)$ (since $b \in E$), which is impossible since f is non-decreasing.
- Or $f(a) > a$. Then, since f is non-decreasing, we get $f(f(a)) \geq f(a)$. So $f(a) \in E$. Which is impossible since for every $x \in [0,1]$, $x \leq a < f(a)$ (since a is an upper bound).

Sample solutions to Exercise 7.

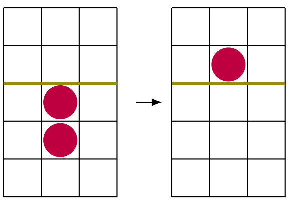
Let $\varepsilon > 0$. Assume by contradiction that $(M-\varepsilon, M) \cap A$ contains finitely many elements, i.e.

$$(M-\varepsilon, M) \cap A = \{a_1, a_2, \dots, a_p\}.$$

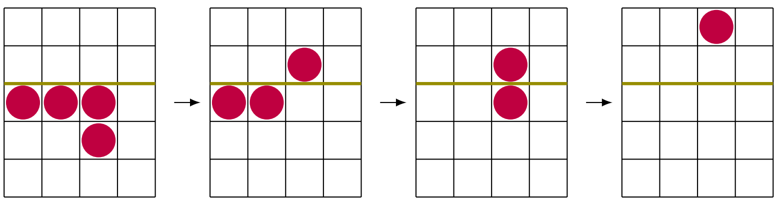
Note that $a := \max(a_1, \dots, a_p) < M$.
Set $\delta = M - a$. Since $\delta > 0$, there exists $b \in A$ such that $M - \delta < b \leq M$.
Since $M \notin A$, we have $b < M$. Besides $b > M - \delta = a \geq M - \varepsilon$.
Therefore $b \in (M-\varepsilon, M) \cap A$.
But, $\forall i = 1, \dots, p$, $b > a > a_i$.
Hence a contradiction

Sample solutions to Exercise 8.

1. To reach the first row:



   To reach the second row:



   I let you continue for the third and fourth rows!

2. There are three cases to handle:
   - The piece moves towards the target cell: then if the piece is initially located at a cell labeled σ^n, then it jumps over piece in a cell labeled σ^{n-1} to reach a cell labeled σ^{n-2}.
     Therefore $F(C') - F(C) = -\sigma^n - \sigma^{n-1} + \sigma^{n-2} = \sigma^{n-2}(-\sigma^2 - \sigma + 1) = 0$.
   - The piece remains at the same distance to the target cell: then if the piece is initially located at a cell labeled σ^n, then it jumps over piece in a cell labeled σ^{n-1} to reach a cell labeled σ^n.
     Therefore $F(C') - F(C) = -\sigma^n - \sigma^{n-1} + \sigma^n = -\sigma^{n-1}$.

- The piece moves away from the target cell: then if the piece is initially located at a cell labeled σ^n, then it jumps over piece in a cell labeled σ^{n+1} to reach a cell labeled σ^{n+2}.
  Therefore $F(C') - F(C) = -\sigma^n - \sigma^{n+1} + \sigma^{n+2} = \sigma^n(-1-\sigma+\sigma^2) = -2\sigma^{n+1}$.

3. For those who like geometric series, like Cherge: since $0 < \sigma < 1$, we have

$$\sum_{n=2}^{+\infty} \sigma^n = \frac{\sigma^2}{1-\sigma} = 1$$

Otherwise, if you don't like geometric series: since $\forall n \in \mathbb{N}$, $\sigma^{n+2} = \sigma^n - \sigma^{n+1}$, we have a telescoping series:

$$\sum_{n=2}^{K} \sigma^n = \sum_{n=0}^{K-2} \sigma^{n+2} = \sum_{n=0}^{K-2} \left(\sigma^n - \sigma^{n+1}\right) = \sigma^0 - \sigma^{K-1} \xrightarrow[K\to+\infty]{} 1 - 0 = 1$$

4.

$\cdots$	σ^{10}	σ^9	σ^8	σ^7	σ^6	σ^5	σ^6	σ^7	σ^8	σ^9	σ^{10}	$\cdots$
$\cdots$	σ^{11}	σ^{10}	σ^9	σ^8	σ^7	σ^6	σ^7	σ^8	σ^9	σ^{10}	σ^{11}	$\cdots$
$\cdots$	σ^{12}	σ^{11}	σ^{10}	σ^9	σ^8	σ^7	σ^8	σ^9	σ^{10}	σ^{11}	σ^{12}	$\cdots$

The cells on $y = 0$ give

$$\sigma^5 + 2\sigma^6 \sum_{k=0}^{+\infty} \sigma^k = \sigma^5 + \frac{2\sigma^6}{1-\sigma} = \sigma^5 + 2\sigma^4 = \sigma^3\left(\sigma^2 + 2\sigma\right) = \sigma^3(1+\sigma) = \sigma^2(\sigma+\sigma^2) = \sigma^2$$

Therefore, the cells on $y = n$ give σ^{2+n} and

$$F(C) = \sum_{n=2}^{+\infty} \sigma^n = 1$$

5. Assume that we have a finite initial configuration C_0, then $F(C_0) < F(C) = 1$ from the previous question.
If C_n is a configuration obtained after n moves then $(F(C_n))_n$ is decreasing by Question 2.
Assume that we reach 5 after n moves then $F(C_n) \geq \sigma^0 = 1$ (since it contains at least a piece located at $(5,0)$). But $F(C_n) \leq F(C_0) < 1$. Hence a contradiction.