# Homework questions - Week 5

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### Exercise 1.

- 1. Prove that  $\forall n \in \mathbb{N}, 5 | 2^{2n+1} + 3^{2n+1}$
- 2. Prove that  $\forall n \in \mathbb{N}, |17|2^{7n+1} + 3^{2n+1} + 5^{10n+1} + 7^{6n+1}$

# Exercise 2.

Find all the  $x \in \mathbb{Z}$  such that  $x^2 + 3 \equiv 0 \pmod{7}$ .

# Exercise 3.

- 1. Determine the remainder of the Euclidean division of  $2^n$  by 5 for  $n \in \mathbb{N}$ .
- 2. Determine the remainder of  $1357^{2021}$  by 5.

# Exercise 4.

- 1. Find a criterion for divisibility by 5.
- 2. Find a criterion for divisibility by 8.
  - Use it on 958547 and on 123456789336.
- 3. Find a criterion for divisibility by 11. Use it on 123456789 and 715.

# Exercise 5.

- 1. Find the integer solutions of  $x^2 5y^2 = 3$ .
- 2. Find the integer solutions of  $15x^2 7y^2 = 9$ .
- 3. Find the integer solutions of  $x^2 + y^2 = 4003$  (*Hint: work modulo 4*).

# Exercise 6.

Prove that  $13|3^{126} + 5^{126}$ .

# Exercise 7.

- For which  $n \in \mathbb{N}$ , is it true that  $8|3^n + 4n + 1$ ?
- For which  $n \in \mathbb{N}$ , is it true that  $21|2^{2^n} + 2^n + 1$ ?

# Exercise 8.

- 1. Prove that  $\forall a, b \in \mathbb{Z}$ ,  $(3|a \text{ and } 3|b) \Leftrightarrow 3|(a^2 + b^2)$ .
- 2. Prove that  $\forall a, b \in \mathbb{Z}$ ,  $(7|a \text{ and } 7|b) \Leftrightarrow 7|(a^2 + b^2)$ .
- 3. Prove that  $\forall a, b \in \mathbb{Z}, 21 | (a^2 + b^2) \implies 441 | (a^2 + b^2).$

# Exercise 9.

Compute gcd  $(2^{445} + 7, 15)$ .

# Exercise 10.

Find all the prime numbers *p* such that  $2^p + p^2$  is also prime.

# Exercise 11.

What is the last digit in the decimal expansion of  $7^{3^{8^4}}$ ?

# Exercise 12.

- 2. Convert the following number from decimal to the Babylonian cuneiform numeral system: 42137.
- 3. Convert the following number from hexadecimal (with digits 0, 1, 2, ..., 9, A, B, ..., F) to base 10:  $\overline{F420C}^{16}$ .
- 4. Convert the following number from decimal to hexadecimal: 11211.
- 5. Compute in hexadecimal (without converting to decimal):  $\overline{9AB7}^{16} + \overline{3C0D}^{16}$ .
- 6. Perform the above computation using decimals. Is it easier?

# Exercise 13.

The scene takes place on an island inhabited by chameleons which are either blue, green, or red. When two chameleons of different colors meet, they both change to the third color (for instance, if a green chameleon and a red chameleon meet, then they both become blue).

Cherge, one of the chameleons, is a retired mathematician who likes funny mathematical riddles and tongue twisters. While he stands at the highest place on the island, he is able to see all the chameleons: 17 of them are blue, 15 are green and 13 are red (including himself).

Then he wonders about a new riddle "Could the island become monochromatic?". What do you think?

#### Sample solutions to Exercise 1.

- 1. Note that  $2^2 = 4 \equiv -1 \pmod{5}$  and that  $3^2 = 9 \equiv -1 \pmod{5}$ . Therefore, for  $n \in \mathbb{N}$ , we have  $2^{2n+1} + 3^{2n+1} = (2^2)^n \times 2 + (3^2)^n \times 3 \equiv (-1)^n \times 2 + (-1)^n \times 3 \pmod{5} \equiv (-1)^n \times 5 \pmod{5} \equiv 0 \pmod{5}.$
- 2. Let  $n \in \mathbb{N}$ , then  $2^{7n+1} + 3^{2n+1} + 5^{10n+1} + 7^{6n+1} \equiv 9^n \times 2 + 9^n \times 3 + 9^n \times 5 + 9^n \times 7 \pmod{17} \equiv 9^n \times 17 \pmod{17} \equiv 0 \pmod{17}$

### Sample solutions to Exercise 2.

We first compute  $x^2 + 3 \pmod{7}$  in terms of  $x \pmod{7}$ :

<i>x</i> (mod 7)	0	1	2	3	4	5	6
$x^2 \pmod{7}$	0	1	4	2	2	4	1
$x^2 + 3 \pmod{7}$	3	4	0	5	5	0	4

Let  $x \in \mathbb{Z}$ . Then  $x^2 + 3 \equiv 0 \pmod{7}$  if and only if  $x \equiv 2 \pmod{7}$  or  $x \equiv 5 \pmod{7}$ if and only if  $x \in \{2 + 7k : k \in \mathbb{Z}\} \cup \{5 + 7k : k \in \mathbb{Z}\}.$ 

### Sample solutions to Exercise 3.

1. We first look for the least  $k \in \mathbb{N} \setminus \{0\}$  such that  $2^k \equiv 1 \pmod{5}$ :

- $2^1 \equiv 2 \pmod{5}$
- $2^2 \equiv 4 \pmod{5}$
- $2^3 \equiv 3 \pmod{5}$
- $2^4 \equiv 1 \pmod{5}$

Hence it is 4.

We perform the Euclidean division of  $n \in \mathbb{N}$  by 4: n = 4q + r where  $0 \le r < 4$ . Then  $2^n = 2^{4q+r} = (2^4)^q 2^r \equiv 1^q 2^r \pmod{5} \equiv 2^r \pmod{5}$ . Thus

- If  $n \equiv 0 \pmod{4}$  then  $2^n \equiv 2^0 \pmod{5} \equiv 1 \pmod{5}$ , so the remainder is 1.
- If *n* ≡ 1 (mod 4) then 2<sup>*n*</sup> ≡ 2<sup>1</sup> (mod 5) ≡ 2 (mod 5), so the remainder is 2.
  If *n* ≡ 2 (mod 4) then 2<sup>*n*</sup> ≡ 2<sup>2</sup> (mod 5) ≡ 4 (mod 5), so the remainder is 4.
- If  $n \equiv 3 \pmod{4}$  then  $2^n \equiv 2^3 \pmod{5} \equiv 3 \pmod{5}$ , so the remainder is 3.
- 2. Note that  $1357 = 1355 + 2 \equiv 2 \pmod{5}$ . Therefore  $1357^{2021} \equiv 2^{2021} \pmod{5}$ . Since  $2021 = 505 \times 4 + 1 \equiv 1 \pmod{4}$  we get that the remainder of  $1357^{2021}$  by 5 is 2.

#### Sample solutions to Exercise 4.

1. Note that  $10 \equiv 0 \pmod{5}$ , hence

$$5|\overline{a_r a_{r-1} \dots a_0}^{10} \Leftrightarrow \overline{a_r a_{r-1} \dots a_0}^{10} \equiv 0 \pmod{5}$$
$$\Leftrightarrow \sum_{k=0}^r a_k 10^k \equiv 0 \pmod{5}$$
$$\Leftrightarrow a_0 \equiv 0 \pmod{5}$$

Therefore  $5|\overline{a_r a_{r-1} \dots a_0}^{10}$  if and only if  $a_0 = 0$  or  $a_0 = 5$ .

2. Note that  $10^3 = 175 \times 8 \equiv 0 \pmod{8}$ , hence

$$8|\overline{a_r a_{r-1} \dots a_0}^{10} \Leftrightarrow \overline{a_r a_{r-1} \dots a_0}^{10} \equiv 0 \pmod{8}$$
$$\Leftrightarrow \sum_{k=0}^r a_k 10^k \equiv 0 \pmod{8}$$
$$\Leftrightarrow 10^2 a_2 + 10a_1 + a_0 \equiv 0 \pmod{8}$$
$$\Leftrightarrow 4a_2 + 2a_1 + a_0 \equiv 0 \pmod{8}$$

Therefore  $8|\overline{a_r a_{r-1} \dots a_0}^{10}$  if and only if  $8|(4a_2 + 2a_1 + a_0)$ .

Note that 8|958547 if and only if  $8|4 \times 5 + 2 \times 4 + 7 = 35 = 8 \times 4 + 3$ . Therefore  $8 \nmid 958547$ . Note that 8|123456789336 if and only if  $8|4 \times 3 + 2 \times 3 + 6 = 24 = 8 \times 3$ . Therefore 8|123456789336.

3. Note that  $10 \equiv -1 \pmod{11}$ , hence

$$11|\overline{a_r a_{r-1} \dots a_0}^{10} \Leftrightarrow \overline{a_r a_{r-1} \dots a_0}^{10} \equiv 0 \pmod{11}$$
$$\Leftrightarrow \sum_{k=0}^r a_k 10^k \equiv 0 \pmod{11}$$
$$\Leftrightarrow \sum_{k=0}^r (-1)^k a_k \equiv 0 \pmod{11}$$

Therefore  $11|\overline{a_r a_{r-1} \dots a_0}^{10}$  if and only if  $11|(-1)^r a_r + (-1)^{r-1} a_{r-1} + \dots + a_2 - a_1 + a_0$ .

Note that 11|123456789 if and only if 11|9 - 8 + 7 - 6 + 5 - 4 + 3 - 2 + 1 = 5. Therefore  $118 \neq 123456789$ . Note that 11|715 if and only if 11|5 - 1 + 7 = 11. Therefore 11|715.

### Sample solutions to Exercise 5.

1. If  $(x, y) \in \mathbb{Z}^2$  is a solution then  $x^2 \equiv 3 \pmod{5}$ . But

- if  $x \equiv 0 \pmod{5}$  then  $x^2 \equiv 0 \pmod{5}$ ,
- if  $x \equiv \pm 1 \pmod{5}$  then  $x^2 \equiv 1 \pmod{5}$ ,
- if  $x \equiv \pm 2 \pmod{5}$  then  $x^2 \equiv 4 \pmod{5}$ .

Thus the equation has no integer solution.

2. Assume that  $(x, y) \in \mathbb{Z}^2$  is a solution, then taking congruences modulo 3, the equation becomes

$$0x^2 - (-1)y^2 \equiv 0 \pmod{3}$$

i.e.  $y^2 \equiv 0 \pmod{3}$ .

<i>y</i> (mod 3)	0	1	2
$y^2 \pmod{3}$	0	1	1

Therefore  $y \equiv 0 \pmod{3}$ , i.e. y = 3k for some  $k \in \mathbb{Z}$ , and the equation becomes  $15x^2 - 63k^2 = 9$ . Dividing by 3, we get  $5x^2 - 21k^2 = 3$ . Taking congruences modulo 3, we obtain  $-x^2 \equiv 0 \pmod{3}$ . As above, the only possibility is that  $x \equiv 0 \pmod{3}$ , i.e. x = 3l for some  $l \in \mathbb{Z}$ . Then the equation becomes  $45l^2 - 21k^2 = 3$ , and dividing by 3, we get  $15l^2 - 7k^2 = 1$ . Modulo 3, we finally get  $-k^2 \equiv 1 \pmod{3}$ , i.e.  $k^2 \equiv -1 \pmod{3} \equiv 2 \pmod{3}$ . Which is impossible (a square modulo 3 is either congruent to 0 or 1, according to the above array). Thus the equation has no integer solution.

3. Below are the possible values for  $x^2 \pmod{4}$  depending on  $x \pmod{4}$ .

<i>x</i> (mod 4)	0	1	2	3
$x^2 \pmod{4}$	0	1	0	1

Therefore either  $x^2 \equiv 0 \pmod{4}$  or  $x^2 \equiv 1 \pmod{4}$  and similarly either  $y^2 \equiv 0 \pmod{4}$  or  $y^2 \equiv 1 \pmod{4}$ . Thus either  $x^2 + y^2 \equiv 0 \pmod{4}$ , or  $x^2 + y^2 \equiv 1 \pmod{4}$ , or  $x^2 + y^2 \equiv 2 \pmod{4}$ . Since  $4003 = 4 \times 1000 + 3 \equiv 3 \pmod{4}$ , there is no integer solutions.

#### Sample solutions to Exercise 6.

Note that  $3^3 \equiv 1 \pmod{13}$ . Since  $126 = 3 \times 42$ , we get  $3^{126} \equiv (3^3)^{42} \pmod{13} \equiv 1^{42} \pmod{13} \equiv 1 \pmod{13}$ . Note that  $5^4 \equiv 1 \pmod{13}$ . Since  $126 = 4 \times 31 + 2$ , we get  $5^{126} \equiv (5^4)^{31} \times 5^2 \pmod{13} \equiv 1^{31} \times 25 \pmod{13} \equiv -1 \pmod{13}$ . Therefore  $3^{126} + 5^{126} \equiv 0 \pmod{13}$ .

#### Sample solutions to Exercise 7.

1. Let  $n \in \mathbb{N}$ .

- If *n* is even, i.e. n = 2k, then  $3^n + 4n + 1 = 9^k + 8k + 1 \equiv 1^k + 0 + 1 \pmod{8} \equiv 2 \pmod{8}$ .
- If *n* is odd, i.e. n = 2k+1, then  $3^n + 4n + 1 = 9^k \times 3 + 8k + 4 + 1 \equiv 1^k \times 3 + 0 + 4 + 1 \pmod{8} \equiv 0 \pmod{8}$ .

Therefore  $8|3^n + 4n + 1$  if and only if *n* is odd.

2. Let  $n \in \mathbb{N}$ . Note that  $2^6 = 64 = 21 \times 3 + 1 \equiv 1 \pmod{21}$ .

Therefore, if n = 6q + r with  $0 \le r < 6$ , we have that  $2^n = (2^6)^q \times 2^r \equiv 2^r \pmod{21}$ . Thus  $2^n \pmod{21}$  depends only on  $n \pmod{6}$ . Let's study the cases separately.

<i>n</i> (mod 6)	0	1	2	3	4	5
$2^{n} \pmod{21}$	1	2	4	8	16	11
$2^{2^n} \pmod{21}$	2	4	-5	4	16	-10
$2^{2^n} + 2^n + 1 \pmod{21}$	4	7	0	13	12	2

Therefore  $21|2^{2^n} + 2^n + 1$  if and only if  $n \equiv 2 \pmod{6}$ .

#### Sample solutions to Exercise 8.

1. For  $a, b \in \mathbb{Z}$ , we compute  $a^2 + b^2 \pmod{3}$  depending on  $a \pmod{3}$  and  $b \pmod{3}$ :

<i>b</i> (mod 3) <i>a</i> (mod 3)	0	1	2
0	0	1	1
1	1	2	2
2	1	2	2
1 1			

We see that  $a^2 + b^2 \equiv 0 \pmod{3}$  if and only if  $a \equiv 0 \pmod{3}$  and  $b \equiv 0 \pmod{3}$ .

- 2. Same as above.
- 3. Let  $a, b \in \mathbb{Z}$ . Assume that  $21|a^2 + b^2$ . Then  $3|a^2 + b^2$ , thus 3|a and 3|b from the first question. Similarly 7|a and 7|b from the second question. Therefore the least common divisor of 3 and 7 divides *a* and *b*, i.e. 21|a and 21|b. Hence a = 21k and b = 21l, so  $a^2 + b^2 = 441(k^2 + l^2)$ .

#### Sample solutions to Exercise 9.

Note that  $2^4 \equiv 1 \pmod{15}$ . Since  $445 = 4 \times 111 + 1$  we get

$$2^{445} + 7 = (2^4)^{111} \times 2 + 7 \equiv 1^{111} \times 2 + 7 \pmod{15} \equiv 9 \pmod{15}$$

Therefore, there exists  $k \in \mathbb{Z}$  such that  $2^{445} + 7 = 15k + 9$ . Finally gcd  $(2^{445} + 7, 15) = \text{gcd}(15k + 9, 15) = \text{gcd}(9, 15) = 3$ .

#### Sample solutions to Exercise 10.

Note that 2 doesn't work and that 3 works. Assume that *p* is a prime number greater than 3, then

$$2^{p} + p^{2} \equiv (-1)^{p} + (\pm 1)^{2} \pmod{3} \equiv -1 + 1 \pmod{3} \equiv 0 \pmod{3}$$

so  $3|2^p + p^2$  and thus  $2^p + p^2$  is not prime. The only prime number *p* such that  $2^p + p^2$  is also prime is p = 3.

### Sample solutions to Exercise 11.

Note that  $7^2 \equiv -1 \pmod{10}$  so  $7^4 \equiv 1 \pmod{10}$ . Therefore if n = 4q + r with  $0 \le r < 4$ , we get that  $7^n = (7^4)^q \times 7^r \equiv 1^q \times 7^r \pmod{10} \equiv 7^r \pmod{10}$ . Hence it is enough to compute  $3^{8^4} \pmod{4}$ . Note that  $3^2 = 9 \equiv 1 \pmod{4}$ , therefore

$$3^{8^4} = 3^{8^3 \times 8} = (3^2)^{8^3 \times 4} \equiv 1^{8^3 \times 4} \pmod{4} \equiv 1 \pmod{4}$$

and  $3^{8^4} = 4q + 1$  for some  $q \in \mathbb{N}$  Therefore  $7^{3^{8^4}} = 7^{4q+1} \equiv 7 \pmod{10}$ . So the last digit in the decimal expansion of  $7^{3^{8^4}}$  is 7.

### Sample solutions to Exercise 12.

- 1.  $\textcircled{AT} = 57 \times 60^3 + 42 \times 60^2 + 3 \times 60 + 11 \times 1 = 12463391$
- 2. We perform successive Euclidean division by 60:

$$42137 = 702 \times 60 + 17$$
  
= (11 × 60 + 42) × 60 + 17  
= 11 × 60<sup>2</sup> + 42 × 60 + 17  
= 4T 4T 4F

- 3.  $\overline{F42C}^{16} = 15 \times 16^4 + 4 \times 16^3 + 2 \times 16^2 + 0 \times 16 + 12 = 999948$
- 4. We perform successive Euclidean division by 16:

$$11211 = 700 \times 16 + 11$$
  
= (43 × 16 + 12) × 16 + 11  
= ((2 × 16 + 11) × 16 + 12) × 16 + 11  
= 2 × 16<sup>3</sup> + 11 × 16<sup>2</sup> + 12 × 16 + 11  
=  $\overline{2BCB}^{16}$ 

5. 
$$\frac{{}^{1}_{9AB7}}{{}^{+3C0D}_{=D6C4}}$$

6. 
$$\overline{9AB7}^{16} = 9 \times 16^{3} + 10 \times 16^{2} + 11 \times 16 + 7 = 39607$$
$$\overline{3C0D}^{16} = 3 \times 16^{3} + 12 \times 16^{2} + 0 \times 16 + 13 = 15373$$
$$39607 + 15373 = 54980$$
$$58820 = 3436 \times 16 + 4 = (214 \times 16 + 12) \times 16 + 4 = ((13 \times 16 + 6) \times 16 + 12) \times 16 + 4 = 13 \times 16^{3} + 6 \times 16^{2} + 12 \times 16 + 4$$
Therefore  $\overline{9AB7}^{16} + \overline{3C0D}^{16} = \overline{D6C4}^{16}$  I think it is easier to directly compute in base 16!

#### Sample solutions to Exercise 13.

Let's denote the number of blue, green, and red chameleons respectively by *b*, *g* and *r*.

- If a blue and a green chameleons meet, the new repartition bescomes b' = b-1, g' = g-1 and r' = r+2. Therefore b' - g' = b - g, b' - r' = b - r - 3 and g' - r' = g - r - 3.
- Similarly, if a blue and a red chameleons meet, we have b' = b 1, g' = g + 2 and r' = r 1. Therefore b' - g' = b - g - 3, b' - r' = b - r and g' - r' = g - r + 3.
- Finally, if a green and a red chameleons meet, we have b' = b + 2, g' = g 1 and r' = r 1. Therefore b' - g' = b - g + 3, b' - r' = b - r + 3 and g' - r' = g - r.

Note that in all the cases we have

$$b' - g' \equiv b - g \pmod{3}$$
  $b' - r' \equiv b - r \pmod{3}$   $g' - r' \equiv g - r \pmod{3}$ 

Therefore these three quantities modulo 3 don't change when the chameleons meet, they always stay constant, mathematically we say that they are invariant. At the beginning, we have

 $b-g \equiv 2 \pmod{3}$   $b-r \equiv 1 \pmod{3}$   $g-r \equiv 2 \pmod{3}$ 

Assume by contradiction that all the chameleons become blue after several meetings (i.e. b = 45, g = 0 and r = 0), then

$$b-g \equiv 0 \pmod{3}$$
  $b-r \equiv 0 \pmod{3}$   $g-r \equiv 0 \pmod{3}$ 

Since these quantities don't change when chameleons meet, we obtain a contradiction. Therefore, it is not possible to obtain an island with only blue chameleons from the initial situation.

We conclude similarly for the other colors. Thus, it is not possible to obtain a monochromatic island!



"Suis-je bien chez ce cher Serge ?"