University of Toronto – MAT246H1-S – LEC0201/9201

Concepts in Abstract Mathematics

Homework questions – Week 1

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January 18th, 2021 to January 22nd, 2021

Exercise 1.
Using only the definition of the multiplication, the properties of the addition and Peano axioms, prove that:
1. $\forall a \in \mathbb{N},\ 0 \times a = a \times 0 = 0$
2. $\forall a \in \mathbb{N},\ a \times 1 = a$

Exercise 2.
Given $m \in \mathbb{N}$, we define inductively the function $m^\bullet : \begin{array}{ccc} \mathbb{N} & \to & \mathbb{N} \\ n & \mapsto & m^n \end{array}$ by $m^0 = 1$ and $\forall n \in \mathbb{N},\ m^{s(n)} = m^n \times m$.
Prove that:
1. $\forall m \in \mathbb{N},\ m^1 = m$
2. $\forall a, b, n \in \mathbb{N},\ (a \times b)^n = a^n \times b^n$
3. $\forall a, m, n \in \mathbb{N},\ a^{m+n} = a^m \times a^n$
4. $\forall n \in \mathbb{N} \setminus \{0\},\ 0^n = 0$
5. $\forall n \in \mathbb{N},\ 1^n = 1$

Exercise 3.
For each of the followings, is the binary relation $\mathcal{R}$ an order on E? If so, is it total?
1. $E = \mathbb{Z}$ and $\forall x, y \in \mathbb{Z},\ x\mathcal{R}y \Leftrightarrow x = -y$
2. $E = \mathbb{R}$ and $\forall x, y \in \mathbb{R},\ x\mathcal{R}y \Leftrightarrow \cos^2 x + \sin^2 y = 1$
3. $E = \mathcal{P}(S)$ is the set of subsets of a fixed set S and $\forall A, B \in \mathcal{P}(S),\ A\mathcal{R}B \Leftrightarrow A \subset B$

Exercise 4.
We define a binary relation $\mathcal{R}$ on $\mathbb{N}$ by $\forall x, y \in \mathbb{N},\ x\mathcal{R}y \Leftrightarrow \exists p, q \in \mathbb{N} \setminus \{0\},\ y = px^q$.
1. Prove that $\mathcal{R}$ is an order.
2. Is it a total order?

Exercise 5.
We define a binary relation $\prec$ on $\mathbb{N}^2$ by $(x_1, y_1) \prec (x_2, y_2) \Leftrightarrow \left(x_1 \le x_2 \text{ and } y_1 \le y_2\right)$.
1. Prove that $\prec$ is an order.
2. Is it a total order?

Exercise 6.
Prove that
1. $\forall a, b, c, d \in \mathbb{N},\ \left(a \le b \text{ and } c \le d\right) \Rightarrow a + c \le b + d$
2. $\forall a, b, c, d \in \mathbb{N},\ \left(a \le b \text{ and } c \le d\right) \Rightarrow ac \le bd$

Exercise 7.
For which $c \in \mathbb{N}$, do we have $\forall a, b \in \mathbb{N},\ ac \le bc \implies a \le b$?

Exercise 8.
Using the well-ordering principle, find an alternative proof of: *there is no natural number n between* 0 *and* 1.

Sample solutions to Exercise 1.

1. Given $a \in \mathbb{N}$, we already know that $a \times 0 = 0$ by definition of the multiplication. So we only need to prove that $\forall a \in \mathbb{N}$, $0 \times a = 0$.
   Set $A = \{a \in \mathbb{N} \ : \ 0 \times a = 0\}$, then
   - $A \subset \mathbb{N}$
   - $0 \in A$ since $0 \times 0 = 0$ by definition of the multiplication.
   - $s(A) \subset A$. Indeed, let $m \in s(A)$, then $m = s(a)$ for some $a \in A$. Then
     $$\begin{aligned} 0 \times m &= 0 \times s(a) \\ &= 0 \times a + 0 \text{ by definition of the multiplication} \\ &= 0 + 0 \text{ since } a \in A \\ &= 0 \end{aligned}$$
     Thus $m \in A$.

   Therefore, by the induction principle, $A = \mathbb{N}$. So $\forall a \in \mathbb{N}$, $0 \times a = 0$.

2. Let $a \in \mathbb{N}$. Then
   $$\begin{aligned} a \times 1 &= a \times s(0) \text{ since } 1 = s(0) \\ &= a \times 0 + a \text{ by definition of the multiplication} \\ &= 0 + a \text{ by definition of the multiplication} \\ &= a \end{aligned}$$

Sample solutions to Exercise 2.

1. Let $m \in \mathbb{N}$ then $m^1 = m^{s(0)} = m^0 \times m = 1 \times m = m$.

2. Let $a, b \in \mathbb{N}$. Set $A = \left\{n \in \mathbb{N} \ : \ (ab)^n = a^n b^n\right\}$.
   - $A \subset \mathbb{N}$
   - $0 \in A$: indeed, $(ab)^0 = 1$ and $a^0 b^0 = 1 \times 1 = 1$.
   - $s(A) \subset A$: let $m \in s(A)$ then $m = s(n)$ for some $n \in A$. Next
     $$\begin{aligned} (ab)^m &= (ab)^{s(n)} \text{ since } m = s(n) \\ &= (ab)^n (ab) \text{ by definition of } (ab)^\bullet \\ &= a^n b^n ab \text{ since } n \in A \\ &= (a^n a)(b^n b) \text{ by properties of the product} \\ &= a^{s(n)} b^{s(n)} \text{ by definition of } a^\bullet \text{ and } b^\bullet \\ &= a^m b^m \text{ since } m = s(n) \end{aligned}$$
     Hence $m \in A$.

   Therefore, by the induction principle, $A = \mathbb{N}$. So for all $n \in \mathbb{N}$, $(ab)^n = a^n b^n$.

3. Let $a, m \in \mathbb{N}$. Set $A = \left\{n \in \mathbb{N} \ : \ a^{m+n} = a^m a^n\right\}$. Then
   - $A \subset \mathbb{N}$
   - $0 \in A$: indeed, $a^{m+0} = a^m = a^m \times 1 = a^m \times a^0$
   - $s(A) \subset A$: let $k \in s(A)$ then $k = s(n)$ for some $n \in A$. Next
     $$\begin{aligned} a^{m+k} &= a^{m+s(n)} \text{ since } k = s(n) \\ &= a^{s(m+n)} \text{ by definition of the addition} \\ &= a^{m+n} \times a \text{ by definition of } a^\bullet \\ &= a^m a^n a \text{ since } n \in A \\ &= a^m a^{s(n)} \text{ by definition of } a^\bullet \\ &= a^m a^k \text{ since } k = s(n) \end{aligned}$$

Hence $k \in A$.

Therefore, by the induction principle, $A = \mathbb{N}$. So for all $n \in \mathbb{N}$, $a^{m+n} = a^m a^n$.

4. Let $n \in \mathbb{N} \setminus \{0\}$. Then there exists $m \in \mathbb{N}$ such that $n = s(m)$. Thus $0^n = 0^{s(m)} = 0^m \times 0 = 0$.

5. Set $A = \left\{n \in \mathbb{N} \ : \ 1^n = 1\right\}$. Then
   - $A \subset \mathbb{N}$
   - $0 \in A$: $1^0 = 1$ by definition of $1^\bullet$.
   - $s(A) \subset A$: let $m \in s(A)$ then $m = s(n)$ for some $n \in A$. Next

$$\begin{aligned} 1^m &= 1^{s(n)} \text{ since } m = s(n) \\ &= 1^n \times 1 \text{ by definition of } 1^\bullet \\ &= 1 \times 1 \text{ since } n \in A \\ &= 1 \end{aligned}$$

   Hence $m \in A$.

   Therefore, by the induction principle, $A = \mathbb{N}$. So for all $n \in \mathbb{N}$, $1^n = 1$.

Sample solutions to Exercise 3.

1. This binary relation is not an order since it is not reflexive.
   Indeed, $1\mathcal{R}1$ is false since $1 \neq -1$.

2. This binary relation is not an order since it is not antisymmetric.
   Indeed, $0\mathcal{R}(2\pi)$ and $(2\pi)\mathcal{R}0$ are true but $0 \neq 2\pi$.

3. The inclusion is an order on $\mathcal{P}(S)$. Indeed
   - $\forall A \in \mathcal{P}(S)$, $A \subset A$ (reflexivity).
   - $\forall A, B \in \mathcal{P}(S)$, $\left(A \subset B \text{ and } B \subset A\right) \implies A = B$ (antisymmetry).
   - $\forall A, B, C \in \mathcal{P}(S)$, $\left(A \subset B \text{ and } B \subset C\right) \implies A \subset C$ (transitivity).

   If $S = \varnothing$ then $\mathcal{P}(S) = \{\varnothing\}$: the order is obviously total.
   If $S = \{*\}$ has only one element then $\mathcal{P}(S) = \{\varnothing, \{*\}\}$: the order is obviously total.
   If S contains at least two elements a, b then the order is not total.
   Indeed, set $A = S \setminus \{a\}$ and $B = S \setminus \{b\}$.
   Then $A \not\subset B$ since $b \in A$ but $b \notin B$, and, $B \not\subset A$ since $a \in B$ but $a \notin A$.
   Thus, if S contains at least two elements, then $\subset$ is not a total order on $\mathcal{P}(S)$.

Sample solutions to Exercise 4.

1.
   - Reflexivity. Let $x \in \mathbb{N}$. Then $x = 1 \times x^1$. Hence $x\mathcal{R}x$.
   - Antisymmetry. Let $x, y \in \mathbb{N}$ be such that $x\mathcal{R}y$ and $y\mathcal{R}x$. Then $x \leq y$ and $y \leq x$. Thus $x = y$.
   - Transitivity. Let $x, y, z \in \mathbb{N}$ be such that $x\mathcal{R}y$ and $y\mathcal{R}z$. Then $y = px^q$ and $z = ry^s$ for some $p, q, r, s \in \mathbb{N} \setminus \{0\}$. Hence $z = ry^s = rp^s x^{qs}$ with $rp^s, qs \in \mathbb{N} \setminus \{0\}$. Thus $x\mathcal{R}z$.

2. This order is not total since $0\mathcal{R}1$ and $1\mathcal{R}0$ are both false.

Sample solutions to Exercise 5.

1.
   - *Reflexivity.* Let $(x, y) \in \mathbb{N}^2$, then $x \leq x$ and $y \leq y$ hence $(x, y) \prec (x, y)$.
   - *Antisymmetry.* Assume that $(x_1, y_1) \prec (x_2, y_2)$ and that $(x_2, y_2) \prec (x_1, y_1)$.
     Then $x_1 \leq x_2$, $y_1 \leq y_2$, $x_2 \leq x_1$ and $y_2 \leq y_1$.
     Since $\leq$ is an order on $\mathbb{N}$, we get that $x_1 = x_2$ and $y_1 = y_2$. Thus $(x_1, y_1) = (x_2, y_2)$.
   - *Transitivity.* Assume that $(x_1, y_1) \prec (x_2, y_2)$ and that $(x_2, y_2) \prec (x_3, y_3)$.
     Then $x_1 \leq x_2$, $y_1 \leq y_2$, $x_2 \leq x_3$ and $y_2 \leq y_3$.
     Since $\leq$ is an order on $\mathbb{N}$, we get that $x_1 \leq x_3$ and $y_1 \leq y_3$. Thus $(x_1, y_1) \prec (x_3, y_3)$.

2. Note that $(1,0) \prec (0,1)$ and $(0,1) \prec (1,0)$ are both false. Hence $\prec$ is not a total order on $\mathbb{N}^2$.

Sample solutions to Exercise 6.
Method 1: using the definition.
1. Let $a,b,c,d \in \mathbb{N}$. Assume that $a \leq b$ and $c \leq d$.
   Then there exist $k,l \in \mathbb{N}$ such that $b = a+k$ and $d = c+l$.
   Hence $b+d = a+k+c+l = (a+c)+(k+l)$ with $k+l \in \mathbb{N}$.
   Thus $a+c \leq b+d$.
2. Let $a,b,c,d \in \mathbb{N}$. Assume that $a \leq b$ and $c \leq d$.
   Then there exist $k,l \in \mathbb{N}$ such that $b = a+k$ and $d = c+l$.
   Hence $bd = (a+k)(c+l) = ac + (al+kc+kl)$ with $al+kc+kl \in \mathbb{N}$.
   Thus $ac \leq bd$.

Method 2: using the properties proved in class.
1. Let $a,b,c,d \in \mathbb{N}$ be such that $a \leq b$ and $c \leq d$.
   Then $a \leq b \implies a+c \leq b+c$ and $c \leq d \implies b+c \leq b+d$.
   Finally $\begin{cases} a+c \leq b+c \\ b+c \leq b+d \end{cases} \implies a+c \leq b+d.$
2. Let $a,b,c,d \in \mathbb{N}$ be such that $a \leq b$ and $c \leq d$.
   Then $a \leq b \implies ac \leq bc$ and $c \leq d \implies bc \leq bd$.
   Finally $\begin{cases} ac \leq bc \\ bc \leq bd \end{cases} \implies ac \leq bd.$

Sample solutions to Exercise 7.
- The statement is false for $c = 0$, indeed, $2 \times 0 \leq 1 \times 0$ but $2 \leq 1$ is false.
- The statement is true for $c \neq 0$. We are going to prove the contrapositive, $\forall a,b \in \mathbb{N},\ b < a \implies bc < ac$.
  Let $a,b \in \mathbb{N}$ be such that $b < a$. Then $b \leq a$ and hence $bc \leq ac$.
  Assume by contradiction that $bc = ac$ then $b = a$ since $c \neq 0$. Hence $bc < ac$ as expected.

Sample solutions to Exercise 8.
Assume by contradiction that the set $E = \{n \in \mathbb{N} \ :\ 0 < n < 1\}$ is not empty.
Then, by the well-ordering principle, E admits a least element, i.e. there exists $l \in E$ such that $\forall n \in \mathbb{N},\ l \leq n$.
Since $l \in E$, we get that $l < 1$. Note that $0 \notin E$, so $l \neq 0$. Hence $l < 1 \implies l^2 < l$.
We know that if $0 = l^2 = l \times l$ then $l = 0$. Hence l^2 is positive.
Finally $0 < l^2 < l < 1$. So $l^2 \in E$ which contradicts the fact that l is the least element of E.