University of Toronto – MAT246H1-S – LEC0201/9201

Concepts in Abstract Mathematics

Problem Set n°5

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Due on April 12th, 2021

Write your solutions concisely but without skipping important steps.
Make sure that your submission is readable on Crowdmark.

Exercise 1.
Prove that $\forall x \in \mathbb{R} \setminus \mathbb{Q},\ \forall a, b, c, d \in \mathbb{Q},\ ad - bc \neq 0 \implies \frac{ax+b}{cx+d} \notin \mathbb{Q}$.

Remark: note that $cx + d \neq 0$ under the given assumptions.
Either $c = 0$ but then $cx + d = d \neq 0$ since $ad - bc \neq 0$. Or $c \neq 0$ but then $cx \in \mathbb{R} \setminus \mathbb{Q}$ and $-d \in \mathbb{Q}$ thus $cx + d \neq 0$.

Exercise 2.
Let E be a finite set. Express

$$|\{(A, B) \in \mathcal{P}(E) \times \mathcal{P}(E)\ :\ A \cup B = E\}|$$

in terms of $|E|$.

Hint: you may start studying the case where the cardinality of A is fixed.

Exercise 3.
The following questions are independent.

1. Does it exist a set E such that $|\mathcal{P}(E)| = \aleph_0$?
2. Prove that $|[0, 1]| = |(0, 1)|$.

Exercise 4.
We set

$$S = \left\{x \in \mathbb{R}\ :\ \exists n \in \mathbb{N},\ \exists a_0, a_1, \dots, a_n \in \mathbb{Z},\ a_n \neq 0 \text{ and } a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0\right\}$$

What is $|S|$?

Remark: you can use basic facts concerning polynomials and their roots.

Sample solution to Exercise 1.
Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Let $a, b, c, d \in \mathbb{Q}$ be such that $ad - bc \neq 0$.
Assume by contradiction that $q \coloneqq \frac{ax+b}{cx+d} \in \mathbb{Q}$, then

$$\frac{ax + b}{cx + d} = q \Leftrightarrow x(a - qc) = qd - b$$

- First case: if $a \neq qc$ then $x = \frac{qd-b}{a-qc} \in \mathbb{Q}$, so there is a contradiction.
- Second case: if $a = qc$ then $qd - b = x(a - qc) = 0$, i.e. $b = qd$.
  Therefore $ad - bc = qcd - qdc = 0$, so there is a contradiction.

Sample solution to Exercise 2.
Set $n \coloneqq |E|$.
Let $i = 0, \dots, n$. Set $\Omega_i \coloneqq \{(A, B) \in \mathcal{P}(E) \times \mathcal{P}(E) \ : \ A \cup B = E \text{ and } |A| = i\}$.
There are $\binom{n}{i}$ subsets $A \in \mathcal{P}(E)$ such that $|A| = i$ (See Q10E09).
For a given A as above, in order to have $A \cup B = E$, B must be of the form $B = A^c \sqcup C$ where $C \subset A$.
There are $2^i = |\mathcal{P}(A)|$ choices for such a subset C, and hence for B (See Q10E10).
Therefore $|\Omega_i| = \binom{n}{i} 2^i$.

Finally

$$\begin{aligned} |\{(A, B) \in \mathcal{P}(E) \times \mathcal{P}(E) \ : \ A \cup B = E\}| &= \left| \bigsqcup_{i=0}^{n} \Omega_i \right| = \sum_{i=0}^{n} |\Omega_i| \\ &= \sum_{i=0}^{n} \binom{n}{i} 2^i = \sum_{i=0}^{n} \binom{n}{i} 2^i 1^{n-i} = (2 + 1)^n = 3^n = 3^{|E|} \end{aligned}$$

Sample solution to Exercise 3.

1. Let E be a set, then:

   - Either E is finite and then $\mathcal{P}(E)$ is finite too by Q10E10, so that $|\mathcal{P}(E)| < \aleph_0$.
   - Or E is infinite and then $\aleph_0 \leq |E| < |P(E)|$ by Cantor's theorem.

   In both cases $|\mathcal{P}(E)| \neq \aleph_0$, so there is no set E such that $|\mathcal{P}(E)| = \aleph_0$.

2. **Method 1.**
   Note that $(0, 1) \subset [0, 1]$, therefore $|(0, 1)| \leq |[0, 1]|$.
   Define $f \ : \ [0, 1] \to (0, 1)$ by $f(x) = \frac{x+1}{3}$.
   Note that f is well-defined since if $0 \leq x \leq 1$ then $0 < \frac{1}{3} \leq \frac{x+1}{3} \leq \frac{2}{3} < 1$.
   Besides f is injective since if $x, y \in [0, 1]$ satisfy $f(x) = f(y)$, then $\frac{x+1}{3} = \frac{y+1}{3}$ which implies $x = y$.
   Therefore $|[0, 1]| \leq |(0, 1)|$.
   By Cantor–Schröder–Bernstein theorem, we conclude that $|[0, 1]| = |(0, 1)|$.

   Method 2.
   We know that $(0, 1) \subset [0, 1] \subset \mathbb{R}$ and that $|(0, 1)| = |\mathbb{R}|$ (see Q11E08).
   Therefore $|[0, 1]| = |(0, 1)|$ (see Q11E01).

Sample solution to Exercise 4.
Comment: *a complex number that is the root of a non-zero polynomial with integers (or rational, it is equivalent) coefficients is said to be an* algebraic number. *Complex numbers which are not roots of such polynomials are called* transcendental numbers.
The field of algebraic real numbers is quite often denoted by

$$\mathbb{R}_{\text{alg}} \coloneqq \left\{ x \in \mathbb{R} \;:\; \exists n \in \mathbb{N},\ \exists a_0, a_1, \dots, a_n \in \mathbb{Z},\ a_n \neq 0 \text{ and } a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 \right\}$$

The goal of this exercise was to prove that $|\mathbb{R}_{\text{alg}}| = \aleph_0$, *i.e. there are infinitely countably many algebraic real numbers, so that* almost all *real numbers are transcendental (but it is usually quite difficult to prove that a number is transcendental: we still don't know whether* $\pi + e$ *or* πe *are transcendental or not). This was first proved by Cantor in is famous article* Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen *published 1874.*

Claim. $\forall n \in \mathbb{N} \setminus \{0\},\ |\mathbb{Z}^n| = \aleph_0$.
Proof by induction on $n \geq 1$.
Base case at $n = 1$: $|\mathbb{Z}^1| = |\mathbb{Z}| = \aleph_0$ (from the lecture notes).
Induction step. Assume that $|\mathbb{Z}^n| = \aleph_0$ for some $n \geq 1$.
Since $|\mathbb{Z}^n| = |\mathbb{N}|$ and $|\mathbb{Z}| = |\mathbb{N}|$, we have $|\mathbb{Z}^{n+1}| = |\mathbb{Z}^n \times \mathbb{Z}| = |\mathbb{N} \times \mathbb{N}| = \aleph_0$. □

Method 1.
For $n \in \mathbb{N}$ and $a_0, a_1, \dots, a_n \in \mathbb{Z}$ with $a_n \neq 0$, the set

$$\{x \in \mathbb{R} \;:\; a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0\}$$

is finite since a polynomial of degree n has at most n roots.
For $n \in \mathbb{N}$, we set

$$A_n = \bigcup_{(a_0, a_1, \dots, a_n) \in \mathbb{Z}^n \times (\mathbb{Z} \setminus \{0\})} \{x \in \mathbb{R} \;:\; a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0\}$$

Since $|\mathbb{Z}^n| = |\mathbb{N}|$ and $|\mathbb{Z} \setminus \{0\}| = |\mathbb{N}|$, we have that $|\mathbb{Z}^n \times (\mathbb{Z} \setminus \{0\})| = |\mathbb{N} \times \mathbb{N}| = \aleph_0$.
Therefore A_n is countable as a countable union of finite sets.
Hence $S = \bigcup_{n \in \mathbb{N}} A_n$ is countable as a countable union of countable sets.
Note that $\mathbb{Z} \subset S$, since $m \in \mathbb{Z}$ is a root of $x - m = 0$.
Therefore S is countably infinite, i.e. $|S| = \aleph_0$.

Method 2.
For $n \in \mathbb{N}$, we denote by P_n the set of polynomials of degree n with integer coefficients.
Note that $|P_n| = |\mathbb{Z}^n \times \mathbb{Z} \setminus \{0\}|$ since a polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ of degree n is characterized by its coefficients $(a_0, a_1, \dots, a_n) \in \mathbb{Z}^n \times (\mathbb{Z} \setminus \{0\})$.
Since $|\mathbb{Z}^n| = |\mathbb{N}|$ and $|\mathbb{Z} \setminus \{0\}| = |\mathbb{N}|$, we have that $|P_n| = |\mathbb{Z}^n \times (\mathbb{Z} \setminus \{0\})| = |\mathbb{N} \times \mathbb{N}| = \aleph_0$.
Hence the set $P = \bigcup\limits_{n \in \mathbb{N}} P_n$ of non-zero polynomials with integer coefficients is countable as a countable union of countable sets.
Given $f \in P$, $f^{-1}(\{0\}) = \{x \in \mathbb{R} \;:\; f(x) = 0\}$ is finite since a polynomial of degree n has at most n roots.
Therefore $S = \bigcup\limits_{f \in P} f^{-1}(0)$ is countable as a countable union of finite sets.
Note that $\mathbb{Z} \subset S$, since $m \in \mathbb{Z}$ is a root of $x - m = 0$.
Therefore S is countably infinite, i.e. $|S| = \aleph_0$.