University of Toronto – MAT246H1-S – LEC0201/9201

Concepts in Abstract Mathematics

Problem Set n°1

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Except otherwise stated, you can only use the material covered from Jan 12 to Jan 26 (i.e. Chapter 1 & Chapter 2 up to §3).

Exercise 1.
We define the binary relation $\prec$ on $\mathbb{N}^2$ by $(x_1, y_1) \prec (x_2, y_2) \Leftrightarrow \big(x_1 < x_2 \text{ or } \big(x_1 = x_2 \text{ and } y_1 \leq y_2\big)\big)$.
Is it an order? If so, is it total?

Exercise 2.
Prove that given $n \in \mathbb{N} \setminus \{0\}$ there exist finitely many $\alpha_1, \ldots, \alpha_m \in \mathbb{N}$ pairwise distinct such that

$$n = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_m}$$

Exercise 3.
Solve $4x(x+1) = y(y+1)$ for $(x, y) \in \mathbb{N}^2$.

Your answer can only rely on the properties of $\mathbb{N}$ proved in Chapter 1.
Particularly, your proof should not involve negative integers, rationals, calculus...
Hint: *compare $2x$ and y.*

Exercise 4.
Prove that for every $n \geq 3$, there exist $x_1, \ldots, x_n \in \mathbb{N} \setminus \{0\}$ pairwise distinct such that

$$1 = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$$

In this exercise, you may assume that you already know $\mathbb{Q}$ or $\mathbb{R}$ so that $\frac{1}{x_i}$ is well-defined.
Hint: $1 = \frac{1}{2} + \frac{1}{2}$.

Sample solution to Exercise 1.

We are going to prove that $\prec$ is a total order on $\mathbb{N}^2$. It is actually called the *lexicographic order*.
It is the one used in dictionaries: you compare the first letter, if it is the same, then you look at the next one...

- *Reflexivity.* Let $(x, y) \in \mathbb{N}^2$. Then $x = x$ and $y \leq y$. Thus $(x, y) \prec (x, y)$.
- *Antisymmetry.* Let $(x_1, y_1), (x_2, y_2) \in \mathbb{N}^2$ satisfying $(x_1, y_1) \prec (x_2, y_2)$ and $(x_2, y_2) \prec (x_1, y_1)$.
  Assume by contradiction that $x_1 < x_2$, then $(x_2, y_2) \nprec (x_1, y_1)$. Which is a contradiction.
  Assume by contradiction that $x_2 < x_1$, then $(x_1, y_1) \nprec (x_2, y_2)$. Which is a contradiction.
  Thus $x_1 = x_2$.
  Since $(x_1, y_1) \prec (x_2, y_2)$, we know that $y_1 \leq y_2$. Since $(x_2, y_2) \prec (x_1, y_1)$, we know that $y_2 \leq y_1$.
  Thus $y_1 = y_2$.
  We proved that $(x_1, y_1) = (x_2, y_2)$.
- *Transitivity.* Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{N}^2$ satisfying $(x_1, y_1) \prec (x_2, y_2)$ and $(x_2, y_2) \prec (x_3, y_3)$.
  - Case 1: $x_1 = x_2$ and $x_2 = x_3$.
    Then $x_1 = x_3$. Furthemore $y_1 \leq y_2$ and $y_2 \leq y_3$, so $y_1 \leq y_3$.
    Hence $(x_1, y_1) \prec (x_3, y_3)$.
  - Case 2: $x_1 = x_2$ and $x_2 < x_3$.
    Then $x_1 < x_3$. Hence $(x_1, y_1) \prec (x_3, y_3)$.
  - Case 3: $x_1 < x_2$ and $x_2 = x_3$.
    Then $x_1 < x_3$. Hence $(x_1, y_1) \prec (x_3, y_3)$.
  - Case 4: $x_1 < x_2$ and $x_2 < x_3$.
    Then $x_1 < x_3$. Hence $(x_1, y_1) \prec (x_3, y_3)$.
- *$\prec$ is a total order.* Let $(x_1, y_1), (x_2, y_2) \in \mathbb{N}^2$. According to the lectures, exactly one of the follows occurs.
  - *Case 1:* $x_1 < x_2$. Then $(x_1, y_1) \prec (x_2, y_2)$.
  - *Case 2:* $x_2 < x_1$. Then $(x_2, y_2) \prec (x_1, y_1)$.
  - *Case 3:* $x_1 = x_2$. Since $\leq$ is a total order on $\mathbb{N}$ then
    * either $y_1 \leq y_2$ and then $(x_1, y_1) \prec (x_2, y_2)$
    * or $y_2 \leq y_1$ and then $(x_2, y_2) \prec (x_1, y_1)$.

Sample solution to Exercise 2.

That's the existence of the positional numeral system with base 2 (*binary numeral system*).

Method 1:

We are going to prove by strong induction that for every $n \geq 1$, there exist finitely many $\alpha_1, \dots, \alpha_m \in \mathbb{N}$ pairwise distinct such that $n = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_m}$.

- *Base case at $n = 1$.* $1 = 2^0$.
- *Induction step.* Assume that the statement holds for $1, 2, \dots, n$ where $n \geq 1$.
  By Euclidean division, $n + 1 = 2q + r$ where $q \in \mathbb{N}$ and $r \in \{0, 1\}$.
  Note that $q \neq 0$ since otherwise $1 < n + 1 = r \leq 1$.
  Hence $1 \leq q < 2q + r = n + 1$.
  Thus, by the induction hypothesis, $q = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_m}$ where $\alpha_1 > \alpha_2 > \cdots > \alpha_m$ are natural numbers.
  Therefore $n + 1 = 2q + r = 2^{\alpha_1 + 1} + 2^{\alpha_2 + 1} + \cdots + 2^{\alpha_m + 1} + r2^0$.
  Note that $\alpha_1 + 1 > \alpha_2 + 1 > \cdots > \alpha_m + 1 > 0$. Hence the exponents are pairwise distinct (it is possible for 2^0 to not appear if $r = 0$).
  Which ends the induction step.

Method 2:
We are going to prove by strong induction that for every $n \geq 1$, there exist finitely many $\alpha_1, \dots, \alpha_m \in \mathbb{N}$ pairwise distinct such that $n = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_m}$.

- *Base case at $n = 1$.* $1 = 2^0$.
- *Induction step.* Assume that the statement holds for $1, 2, \dots, n$ where $n \geq 1$.
  (i) First case: $n + 1$ is even. Then $n + 1 = 2k$ for some $k \in \mathbb{N}$.
  Note that $k \neq 0$ since otherwise $1 \leq n + 1 = 2k = 0$.
  Since $k \neq 0$, we get that $k < 2k = n + 1$, i.e. $k \leq n$.
  Thus, by the induction hypothesis, $k = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_m}$ where the $\alpha_1, \dots, \alpha_m \in \mathbb{N}$ are pairwise distinct.
  So $n + 1 = 2 \times (2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_m}) = 2^{\alpha_1+1} + 2^{\alpha_2+1} + \cdots + 2^{\alpha_m+1}$.
  Assume by contradiction that there exist $i \neq j$ such that $\alpha_i + 1 = \alpha_j + 1$. Then, by the cancellation rule, $\alpha_i = \alpha_j$. Which is a contradiction since the α_i are pairwise distinct.
  Therefore the $\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_m + 1$ are pairwise distinct as requested.
  (ii) Second case: $n + 1$ is odd. Then $n + 1 = 2k + 1$ for some $k \in \mathbb{N}$.
  Note that $k \neq 0$ since otherwise $n + 1 = 2 \times 0 + 1 = 1 \implies n = 0$. Hence, as above, $k < 2k = n$.
  Thus, by the induction hypothesis, $k = 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_m}$ where the $\alpha_1, \dots, \alpha_m \in \mathbb{N}$ are pairwise distinct.
  Hence $n + 1 = 1 + 2k = 2^0 + 2 \times (2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_m}) = 2^0 + 2^{\alpha_1+1} + 2^{\alpha_2+1} + \cdots + 2^{\alpha_m+1}$.
  As above, the $\alpha_i + 1$ are pairwise distinct. Moreover $\alpha_i + 1 > 0$. Therefore the $0, \alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_m + 1$ are pairwise distinct, as requested.

  Which ends the induction step.

Sample solution to Exercise 3.
Let $(x, y) \in \mathbb{N}^2$ be such that $4x(x + 1) = y(y + 1)$.

1. *First case: assume that $y \leq 2x$.* Then
$$y(y + 1) \leq 2x(2x + 1) \leq 2x(2x + 2) = 4x(x + 1) = y(y + 1)$$
   Hence $2x(2x + 1) \leq 2x(2x + 2)$ and $2x(2x + 2) = y(y + 1) \leq 2x(2x + 1)$.
   Thus $2x(2x + 1) = 2x(2x + 2)$, from which we get that $x(2x + 1) = x(2x + 2)$.
   - Either $x = 0$ and then $y \leq 0$ so $y = 0$.
   - Or $x \neq 0$ and then, by cancellation, we get $2x + 1 = 2x + 2$.
     We derive from the previous equality that $1 = 2$, which is impossible.

   Thus the only possible solution in this case is $(x, y) = (0, 0)$.
2. *Second case: assume that $2x < y$, i.e. $2x + 1 \leq y$.* Then
$$y(y + 1) \geq (2x + 1)(2x + 2) \geq 2x(2x + 2) = y(y + 1)$$
   Hence, as above, $(2x + 1)(2x + 2) = 2x(2x + 2)$.
   Note that $2x + 2 \neq 0$ since $2x + 2 \geq 2 > 0$.
   So, by cancellation, $2x = 2x + 1$ and hence $0 = 1$, which is impossible.
   Therefore there is no solution $(x, y) \in \mathbb{N}^2$ satisfying $2x < y$.

We proved that the only possible solution is $(x, y) = (0, 0)$.
We have to check that conversely it is a solution, which is the case since then $4x(x + 1) = 0 = y(y + 1)$.
So the only solution is $(x, y) = (0, 0)$.

Sample solution to Exercise 4.
Method 1 (using my hint):
We are going to prove the statement by induction on n.

- *Base case at* $n = 3$. Note that $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$
- *Induction step.* Assume that the statement holds for some $n \geq 3$.
  By the induction hypothesis, there exist $x_1 < \ldots < x_n$ in $\mathbb{N} \setminus \{0\}$ such that $1 = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$.
  Note that $x_1 \neq 1$ since otherwise $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1 + \frac{1}{x_2} + \cdots + \frac{1}{x_n} > 1$. Thus $x_1 > 1$.
  Hence $1 = \frac{1}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) = \frac{1}{2} + \frac{1}{2x_1} + \frac{1}{2x_2} + \cdots + \frac{1}{2x_n}$.
  Besides, since $1 < x_1 < x_2 < \cdots < x_n$, we get that $2 < 2x_1 < 2x_2 < \cdots < 2x_n$.
  So the $n + 1$ denominators are pairwise distinct.

Method 2:
We are going to prove the following stronger statement by induction on n: for $n \geq 3$, there exist $1 < x_1 < x_2 < \cdots < x_n$ such that $1 = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$ and x_n is even.

- *Base case at* $n = 3$. Note that $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$
- *Induction step.* Assume that the statement holds for some $n \geq 3$.
  By the induction hypothesis, there exist $1 < x_1 < \ldots < x_n$ in $\mathbb{N}$ such that $1 = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$ and x_n is even.
  Hence $x_n = 2k$ for some $k \in \mathbb{N} \setminus \{0\}$.
  Note that $\frac{1}{x_n} = \frac{1}{2k} = \frac{1}{3k} + \frac{1}{6k}$.
  Hence
  $$1 = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}} + \frac{1}{3k} + \frac{1}{6k}$$
  Besides $6k$ is even and $1 < x_1 < x_2 < \cdots < x_n = 2k < 3k < 6k$.
  So the $n + 1$ denominators are pairwise distinct.

Method 3:
We are going to prove the statement by induction on n.

- *Base case at* $n = 3$. Note that $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$
- *Induction step.* Assume that the statement holds for some $n \geq 3$.
  By the induction hypothesis, there exist $x_1 < \ldots < x_n$ in $\mathbb{N} \setminus \{0\}$ such that $1 = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$.
  Note that $x_1 \neq 1$ since otherwise $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1 + \frac{1}{x_2} + \cdots + \frac{1}{x_n} > 1$. Thus $x_1 > 1$.
  Note that for $x \neq 0$, $\frac{1}{x(x+1)} + \frac{1}{x+1} = \frac{x+1}{x(x+1)} = \frac{1}{x}$.
  Therefore
  $$1 = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}} + \frac{1}{x_n} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_{n-1}} + \frac{1}{x_n + 1} + \frac{1}{x_n(x_n + 1)}$$
  Since $1 < x_n$ and $0 < x_n + 1$, we get $x_n + 1 < x_n(x_n + 1)$.
  Therefore $1 < x_1 < x_2 < \cdots < x_{n-1} < x_n < x_n + 1 < x_n(x_n + 1)$.
  So the $n + 1$ denominators are pairwise distinct.