MAT246H1-S – LEC0201/9201 Concepts in Abstract Mathematics

CARDINALITY: INFINITE SETS



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Definition: infinite set

We say that a set is *infinite* if it is not finite.

Theorem

 \mathbb{N} is infinite.

Proof. Assume by contradiction that \mathbb{N} is finite. Then $\mathbb{N} \setminus \{0\} \subset \mathbb{N}$ so $\mathbb{N} \setminus \{0\}$ is finite too. We define $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ by f(n) = n + 1. Note that f is bijective with inverse $f^{-1} : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ defined by $f^{-1}(n) = n - 1$. Thus $|\mathbb{N}| = |\mathbb{N} \setminus \{0\}| = |\mathbb{N}| - |\{0\}| = |\mathbb{N}| - 1$, i.e. 0 = 1. Hence a contradiction.

Definition: cardinality

We say that two sets *E* and *F* have same *cardinality*, denoted by |E| = |F|, if there exists a bijection $f : E \to F$. We also say that *E* and *F* are *equinumerous* or *equipotent*.

Remark

We proved that $|\mathbb{N} \setminus \{0\}| = |\mathbb{N}|$ although $\mathbb{N} \setminus \{0\} \subseteq \mathbb{N}$. That's a first quirkiness about infinite cardinals.

Cardinality of infinite sets - 2

Proposition

1 If E is a set then |E| = |E|.

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2 Given two sets E and F, if |E| = |F| then |F| = |E|.
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3 Given three sets E, F and G, if |E| = |F| and |F| = |G| then |E| = |G|.
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Proof.

- **1** $id : E \to E$ is a bijection.
- **2** Assume that |E| = |F|, i.e. that there exists a bijection $f : E \to F$. Then $f^{-1} : F \to E$ is a bijection, so |F| = |E|.

 3 Assume that |E| = |F| and |F| = |G|, i.e. that there exist bijections f : E → F and g : F → G. Then g ∘ f : E → G is a bijection, thus |E| = |G|.

Remark

Equipotence seems to be an equivalence relation since it satisfies reflexivity, symmetry and transitivity. Nonetheless, recall that an equivalence relation is a binary relation on a set whereas the set of all sets doesn't exist (we will prove this fact later).

Cardinality of infinite sets - 3

Theorem

A set *E* is infinite if and only if for every $n \in \mathbb{N}$ there exists $S \subset E$ such that |S| = n.

Proof.

 \Rightarrow Assume that *E* is infinite.

We are going to prove by induction that for every $n \in \mathbb{N}$ there exists $S \in \mathcal{P}(E)$ such that |S| = n.

- Base case at n = 0: $\emptyset \subset E$ satisfies $|\emptyset| = 0$.
- Induction step. Assume that for some $n \in \mathbb{N}$ there exists $T \subset E$ such that |T| = n. Note that $E \setminus T \neq \emptyset$ (otherwise E = T, which is impossible since E is infinite). Therefore there exists $x \in E \setminus T$. Define $S := T \sqcup \{x\}$, then $S \subset E$ is finite and |S| = |T| + 1 = n + 1. Which ends the induction step.

⇐ Let *E* be a set such that for every $n \in \mathbb{N}$ there exists $S \subset E$ such that |S| = n. Assume by contradiction that *E* is finite. Then there exists $k \in \mathbb{N}$ such that |E| = k. Since $k + 1 \in \mathbb{N}$, there exists $S \subset E$ such that |S| = k + 1. Since $S \subset E$, we get $k + 1 = |S| \le |E| = k$. Hence a contradiction.

Corollary

A set *E* is infinite if and only if for every $n \in \mathbb{N}$ there exists an injective function $\{0, 1, 2, ..., n-1\} \rightarrow E$.