## CARDINALITY: INFINITE SETS

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## Infinite sets - 1

## Definition: infinite set

We say that a set is infinite if it is not finite.

## Infinite sets - 2

## Theorem

$\mathbb{N}$ is infinite.
Proof. Assume by contradiction that $\mathbb{N}$ is finite. Then $\mathbb{N} \backslash\{0\} \subset \mathbb{N}$ so $\mathbb{N} \backslash\{0\}$ is finite too. We define $f: \mathbb{N} \rightarrow \mathbb{N} \backslash\{0\}$ by $f(n)=n+1$.
Note that $f$ is bijective with inverse $f^{-1}: \mathbb{N} \backslash\{0\} \rightarrow \mathbb{N}$ defined by $f^{-1}(n)=n-1$.
Thus $|\mathbb{N}|=|\mathbb{N} \backslash\{0\}|=|\mathbb{N}|-|\{0\}|=|\mathbb{N}|-1$, i.e. $0=1$.
Hence a contradiction.

## Cardinality of infinite sets - 1

## Definition: cardinality

We say that two sets $E$ and $F$ have same cardinality, denoted by $|E|=|F|$, if there exists a bijection $f: E \rightarrow F$.
We also say that $E$ and $F$ are equinumerous or equipotent.

## Remark

We proved that $|\mathbb{N} \backslash\{0\}|=|\mathbb{N}|$ although $\mathbb{N} \backslash\{0\} \subsetneq \mathbb{N}$.
That's a first quirkiness about infinite cardinals.

## Cardinality of infinite sets - 2

## Proposition

(1) If $E$ is a set then $|E|=|E|$.
(2) Given two sets $E$ and $F$, if $|E|=|F|$ then $|F|=|E|$.
(3) Given three sets $E, F$ and $G$, if $|E|=|F|$ and $|F|=|G|$ then $|E|=|G|$.

## Proof.

(1) id $: E \rightarrow E$ is a bijection.
(2) Assume that $|E|=|F|$, i.e. that there exists a bijection $f: E \rightarrow F$. Then $f^{-1}: F \rightarrow E$ is a bijection, so $|F|=|E|$.
(3) Assume that $|E|=|F|$ and $|F|=|G|$, i.e. that there exist bijections $f: E \rightarrow F$ and $g: F \rightarrow G$. Then $g \circ f: E \rightarrow G$ is a bijection, thus $|E|=|G|$.

## Remark

Equipotence seems to be an equivalence relation since it satisfies reflexivity, symmetry and transitivity. Nonetheless, recall that an equivalence relation is a binary relation on a set whereas the set of all sets doesn't exist (we will prove this fact later).

## Cardinality of infinite sets - 3

## Theorem

A set $E$ is infinite if and only if for every $n \in \mathbb{N}$ there exists $S \subset E$ such that $|S|=n$.

## Proof.

$\Rightarrow$ Assume that $E$ is infinite.
We are going to prove by induction that for every $n \in \mathbb{N}$ there exists $S \in \mathcal{P}(E)$ such that $|S|=n$.

- Base case at $n=0: \varnothing \subset E$ satisfies $|\varnothing|=0$.
- Induction step. Assume that for some $n \in \mathbb{N}$ there exists $T \subset E$ such that $|T|=n$. Note that $E \backslash T \neq \varnothing$ (otherwise $E=T$, which is impossible since $E$ is infinite).
Therefore there exists $x \in E \backslash T$. Define $S:=T \sqcup\{x\}$, then $S \subset E$ is finite and $|S|=|T|+1=n+1$. Which ends the induction step.
$\Leftarrow$ Let $E$ be a set such that for every $n \in \mathbb{N}$ there exists $S \subset E$ such that $|S|=n$.
Assume by contradiction that $E$ is finite. Then there exists $k \in \mathbb{N}$ such that $|E|=k$.
Since $k+1 \in \mathbb{N}$, there exists $S \subset E$ such that $|S|=k+1$.
Since $S \subset E$, we get $k+1=|S| \leq|E|=k$. Hence a contradiction.


## Corollary

A set $E$ is infinite if and only if for every $n \in \mathbb{N}$ there exists an injective function $\{0,1,2, \ldots, n-1\} \rightarrow E$.

