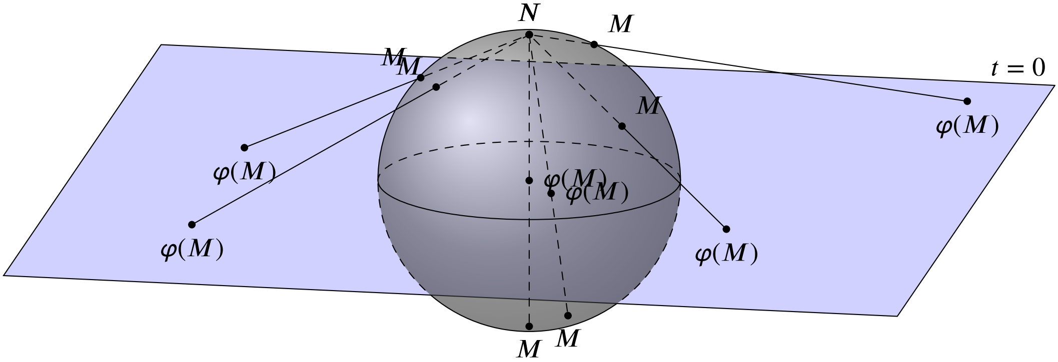
THE RIEMANN SPHERE

September 21$^{\text{st}}$, 2020

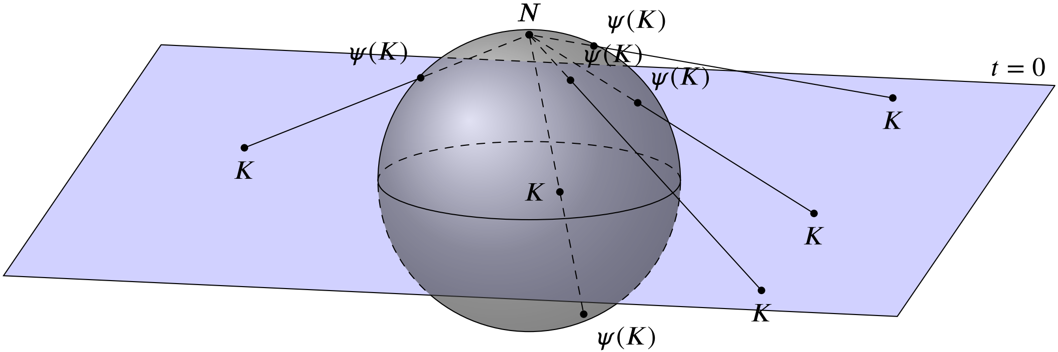
We set $S^2 := \left\{(r,s,t) \in \mathbb{R}^3 \ : \ r^2+s^2+t^2=1\right\}$ and $N=(0,0,1)$ (the *north pole* of S^2).
We identify $\mathbb{C}$ with the equatorial plane $P=\{t=0\}$.
We define the stereographic projection with respect to N:

$$\varphi : \left\{ \begin{array}{ccc} S^2 \setminus \{N\} & \to & P \\ M & \mapsto & (NM) \cap P \end{array} \right.$$



We define the

$$\psi : \begin{cases} P & \to & S^2 \setminus \{N\} \\ K & \mapsto & (NK) \cap S^2 \end{cases}$$



We can see that φ and ψ are inverse of each other *(go to my notes for a formal proof with the formulae)*.

Hence the stereographic projection with respect to N:

$$\varphi : \begin{cases} S^2 \setminus \{N\} & \to & P \\ M & \mapsto & (NM) \cap P \end{cases}$$

is a bijection (even a homeomorphism) from the sphere minus the north pole $S^2 \setminus \{N\}$ to $\mathbb{C}$.
Every point of $S^2 \setminus \{N\}$ corresponds exactly to a unique point of $\mathbb{C}$.

You may see it as the complex plane wraping up the sphere minus the north pole $S^2 \setminus \{N\}$.

Hence the sphere S^2 may be seen as the complex plane with an additional point, the point at infinity. Indeed, the closer M is to N, the farther $\varphi(M)$ is to the origin.

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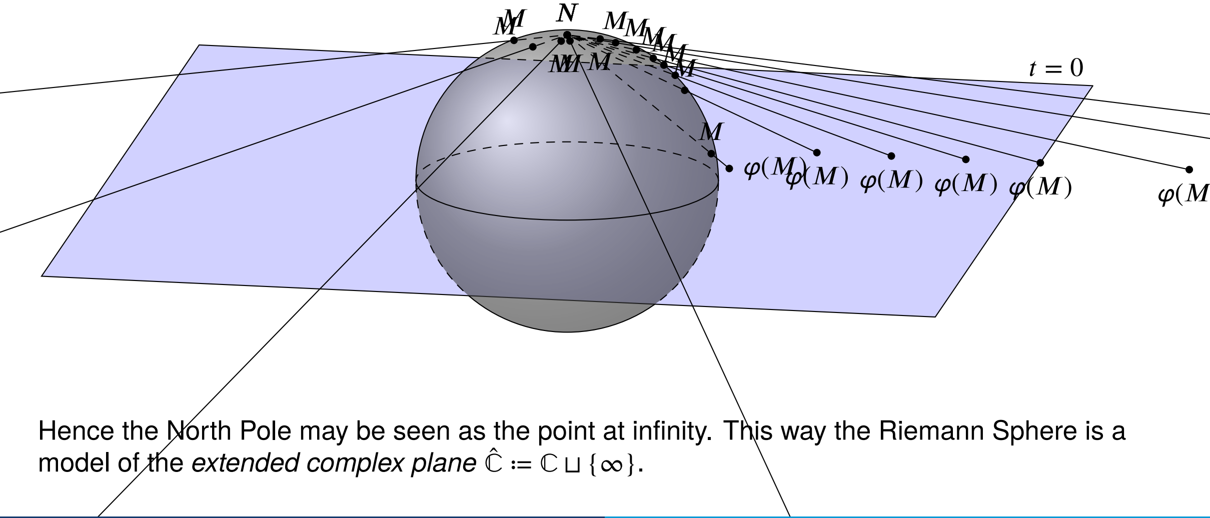
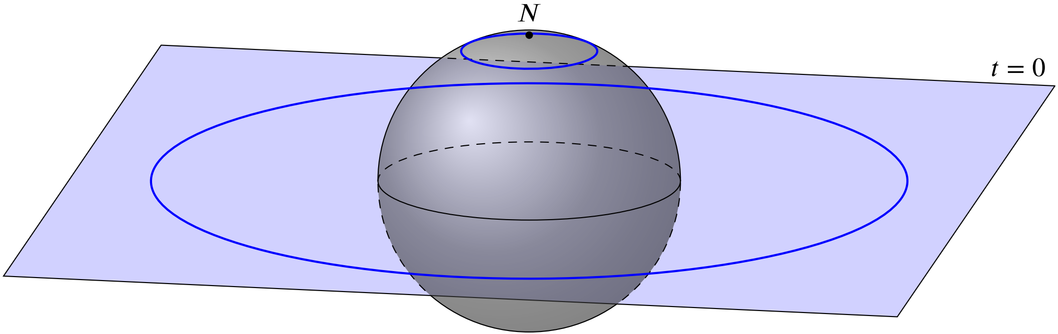


Hence the North Pole may be seen as the point at infinity. This way the Riemann Sphere is a model of the *extended complex plane* $\hat{\mathbb{C}} := \mathbb{C} \sqcup \{\infty\}$.

A neighborhood of N in S^2 is mapped by φ to the complement of a bounded set in $\mathbb{C}$.
Conversely, the complement of a bounded set in $\mathbb{C}$ is mapped by ψ to a neighborhood of N.



Thus, we define:

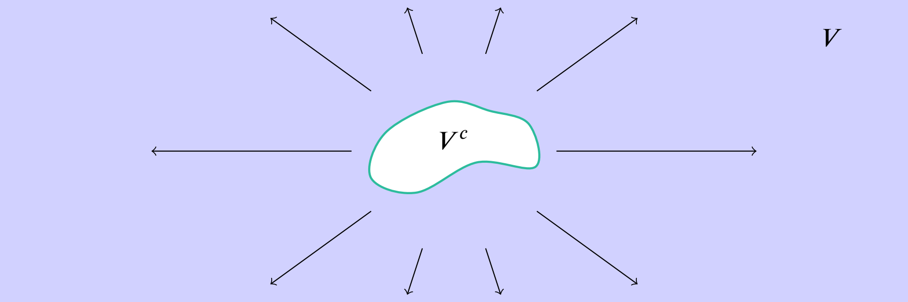
Definition: Neighborhood of the ∞

We say that $V \subset \mathbb{C}$ is *a neighborhood of* ∞ if $V^c := \mathbb{C} \setminus V$ is bounded.

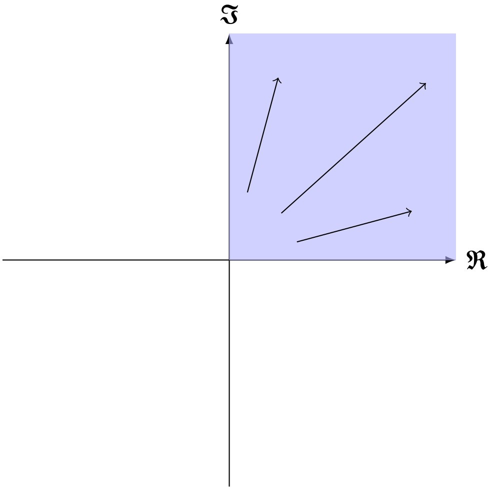
Proposition

$V \subset \mathbb{C}$ is a a neighborhood of ∞ if and only if $\exists R \in \mathbb{R}_{>0}, \{z \in \mathbb{C} \ : \ |z| > R\} \subset V$.

Geometrically, it means that we can approach ∞ from all the possible directions:



The first quadrant $\{z \in \mathbb{C} \;:\; \Re(z) > 0,\ \Im(z) > 0\}$ is not a neighborhood of ∞ (despite being unbounded):



Remember that a set is open if and only if it is a neighborhood of each of its points. Hence, we defined a topology on $\hat{\mathbb{C}}$. It makes $\varphi : S^2 \to \hat{\mathbb{C}}$ a homeomorphism.

Definition: Open sets of $\hat{\mathbb{C}}$

A subset $S \subset \hat{\mathbb{C}}$ is open if

- $S \subset \mathbb{C}$ is open or
- $S = \{\infty\} \cup U$ where $U = K^c \subset \mathbb{C}$ is the complement of $K \subset \mathbb{C}$ closed and bounded (compact).

Beware: the Riemann sphere is only one model of $\hat{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$ among others (e.g. complex projective line). So, what you need to remember is that:

- $\hat{\mathbb{C}} := \mathbb{C} \sqcup \{\infty\}$ is $\mathbb{C}$ extended by a unique additional point at infinity,
- definition of a neighborhood of ∞,
- open sets of $\hat{\mathbb{C}}$.

NOT part of MAT334: there are also other ways to compactify $\mathbb{C}$: with $\hat{\mathbb{C}}$, all the directions tend to the same point at ∞, but, for instance, it is also possible to compactify $\mathbb{C}$ with a circle at infinity to keep track of the directions.

In MAT334, we will only work with $\hat{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$ with a unique point at infinity.

The extended inversion

We may extend the inversion[a] to $\hat{\mathbb{C}}$ by $\operatorname{inv} : \begin{cases} \hat{\mathbb{C}} & \to & \hat{\mathbb{C}} \\ z & \mapsto & z^{-1} & \text{if } z \in \mathbb{C} \setminus \{0\} \\ 0 & \mapsto & \infty \\ \infty & \mapsto & 0 \end{cases}$

[a] Actually, it is possible to define division by 0, what is **not** possible is to define a multiplicative inverse of 0.

Remark

The inversion $\operatorname{inv} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ maps a neighborhood of 0 to a neighborhood of ∞ and vice-versa.

In some sense, it swaps 0 and ∞.