University of Toronto – MAT334H1-F – LEC0101 Complex Variables

9 - Cauchy's Integral Formula

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1 Simple connectedness

Definition 1. A hole of $S \subset \mathbb{C}$ is a bounded connected component of $\mathbb{C} \setminus S$.

Definition 2. We say that $S \subset \mathbb{C}$ is **simply connected** if it is path connected and has no hole.

Remark 3. The above definition is not informal since we formally defined what is a hole.



Theorem 4. $S \subset \mathbb{C}$ is simply connected if and only if it is path-connected and for any simple closed curve included in *S*, its inside \star is also included in *S*.



Remark 5. An important property [†] of simply connected sets is that any closed curve on it can be *continuously deformed* to a constant curve without leaving the set:



^{*} See Jordan curve theorem from September 28.

[†] That's actually the usual definition of simple connectedness: a path connected set is simply connected if any closed curve on it is homotopic to a point.

2 Cauchy's integral theorem

Theorem 6 (Cauchy's integral theorem – version 1).

Let $U \subset \mathbb{C}$ *be an open subset and* $f : U \to \mathbb{C}$ *be holomorphic/analytic. Let* $\gamma : [a, b] \to \mathbb{C}$ *be a piecewise smooth simple closed curve on* U *whose inside is also entirely included in* U*, then*

$$\int_{\gamma} f(z) \mathrm{d}z = 0$$

Proof. There is an easy proof with the extra assumption that f' is continuous \star :

$$\int_{\gamma} f = i \iint_{\gamma \cup \text{Inside}} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$
by Green's theorem
= $i \iint_{\gamma \cup \text{Inside}} 0$ by the Cauchy–Riemann equations
= 0

Remark 7. In the next corollaries we assume that the domain is simply connected. It ensures that for a simple closed curve on *U*, its inside is also included on *U*, so that we can use Cauchy's integral theorem.

Corollary 8. Let $U \subset \mathbb{C}$ be an open simply connected subset and $f : U \to \mathbb{C}$ be holomorphic/analytic. Then there exists $F : U \to \mathbb{C}$ holomorphic/analytic such that F' = f. We say that F is a (complex) antiderivative/primitive of f on U.

Proof. Fix $z_0 \in U$ and for $z \in U$ set $F(z) = \int_{\gamma} f(z) dz$ where γ is a polygonal curve from z_0 to z.

We need to check that *F* is well defined, i.e. that *F* doesn't depend on the choice of γ . For that, we check that if η is another such curve then

$$\int_{\gamma} f(z) dz - \int_{\eta} f(z) dz = \int_{\gamma - \eta} f(z) dz = 0$$

Indeed the integrals cancel each other on common edges with reversed orientation, and by the previous theorem, each integral around a simple closed polygonal curve is 0, see the drawing below.



^{*} It is always the case: the derivative of a holomorphic/analytic function is always continuous but you don't know that yet. To avoid circular arguments, it would be better to give a proof without this assumption. There is such a proof (due to Goursat), but it is far more technical. So I am cheating a little bit here.

Finally

$$\begin{aligned} \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &= \left| \frac{\int_{[z,z+h]} f(w) \mathrm{d}w}{h} - f(z) \right| = \left| \frac{\int_{[z,z+h]} f(w) \mathrm{d}w - hf(z)}{h} \right| \\ &= \left| \frac{\int_{[z,z+h]} f(w) \mathrm{d}w - \int_{[z,z+h]} f(z) \mathrm{d}w}{h} \right| \\ &= \left| \int_{[z,z+h]} \frac{f(w) - f(z)}{h} \mathrm{d}w \right| \\ &\leq \left(\max_{w \in [z,z+h]} |f(w) - f(z)| \right) \frac{\mathscr{L}([z,z+h])}{|h|} \\ &= \left(\max_{w \in [z,z+h]} |f(w) - f(z)| \right) \xrightarrow[h \to 0]{} \end{aligned}$$

Remark 9. Let *F* be a primitive of *f* on *U* and $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth curve on *U* then

$$\int_{\gamma} f(z) dz = \int_{a}^{b} f(\gamma(t))\gamma'(t) dt = \int_{a}^{b} (F \circ \gamma)'(t) dt = F(\gamma(b)) - F(\gamma(a))$$

The two following corollaries (path independence of the line integral of a holomorphic function) are consequences of the previous remark together with the fact that a holomorphic function on a simply connected domain admits a primitive.

Corollary 10. Let $U \subset \mathbb{C}$ be an open simply connected subset and $f : U \to \mathbb{C}$ be holomorphic/analytic. Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth closed curve included \star in U, then

$$\int_{\gamma} f(z) \mathrm{d}z = 0$$

Corollary 11. Let $U \subset \mathbb{C}$ be an open simply connected subset and $f : U \to \mathbb{C}$ be holomorphic/analytic. Let $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ be two piecewise smooth curves included \star in U with same endpoints \dagger , then

$$\int_{\gamma_1} f(z) \mathrm{d}z = \int_{\gamma_2} f(z) \mathrm{d}z$$

When the domain is **simply connected**, we proved:

Theorem 12 (Cauchy's integral theorem – version 2). Let $U \subset \mathbb{C}$ be an open simply connected subset and $f : U \to \mathbb{C}$ be holomorphic/analytic. Then

• For γ : $[a, b] \rightarrow \mathbb{C}$ a piecewise smooth closed curve included in U,

$$\int_{\gamma} f(z) \mathrm{d}z = 0$$

• For $\gamma_i : [a_i, b_i] \to \mathbb{C}$, i = 1, 2 two piecewise smooth curves included in U with same endpoints,

$$\int_{\gamma_1} f(z) \mathrm{d}z = \int_{\gamma_2} f(z) \mathrm{d}z$$

• f admits a primitive/antiderivative $\ddagger F$ on U, i.e. there exists $F : U \to \mathbb{C}$ holomorphic/analytic such that F' = f.

[†] i.e. $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$.

^{*} i.e. $\forall t \in [a, b], \gamma(t) \in U$.

[‡] Actually a function $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is simply connected, is holomorphic if and only if it admits a (complex) antiderivative: indeed, we will see soon that the derivative of a holomorphic function is holomorphic too.

Remark 13. The simple connectedness assumption of the domain is essential. Indeed $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by f(z) = 1/z is holomorphic, but :

- For $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(t) = e^{it}$, we have $\int_{\gamma} \frac{1}{z} dz = 2i\pi \neq 0$
- f has no antiderivative on $\mathbb{C} \setminus \{0\}$.

The following (improper) integrals are difficult to compute using only "real" methods. We present an easy computation relying on Cauchy's integral formula.

Example 14 (Frenel's integrals).

$$\int_{0}^{+\infty} \cos(t^{2}) dt = \int_{0}^{+\infty} \sin(t^{2}) dt = \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

Proof.



• $\int_{\gamma_1} e^{-z^2} dz = \int_0^r e^{-t^2} dt \xrightarrow[r \to +\infty]{} \frac{\sqrt{\pi}}{2}.$

•
$$\left| \int_{\gamma_2} e^{-z^2} dz \right| = \left| \int_0^1 ir e^{(t^2 - 1)r^2} e^{2itr^2} dt \right| \le r e^{-r^2} \int_0^1 e^{t^2 r^2} dt \le r e^{-r^2} \int_0^1 e^{tr^2} dt = \frac{1 - e^{-r^2}}{r} \xrightarrow[r \to +\infty]{} 0$$

•
$$\int_{\gamma_3} e^{-z^2} dz = \int_0^r (1+i)e^{((1+i)t)^2} dt = (1+i)\int_0^r e^{-2it^2} dt = e^{i\frac{\pi}{4}}\int_0^{\frac{1}{\sqrt{2}}} e^{-it^2} dt$$

By Cauchy's integral theorem $0 = \int_{\gamma_1 + \gamma_2 - \gamma_3} e^{-z^2} dz = \int_{\gamma_1} e^{-z^2} dz + \int_{\gamma_2} e^{-z^2} dz - \int_{\gamma_3} e^{-z^2} dz.$

So, by taking the limit when $r \to +\infty$ in the above equality, $\int_0^{+\infty} e^{-it^2} dt = e^{-i\frac{\pi}{4}} \frac{\sqrt{\pi}}{2}$, that we may rewrite

$$\int_{0}^{+\infty} \cos(t^{2}) dt - i \int_{0}^{+\infty} \sin(t^{2}) dt = \frac{1}{2} \sqrt{\frac{\pi}{2}} - i \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

We conclude by identifying the real and imaginary parts.

3 Cauchy's integral formula

Theorem 15. Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic. Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth positively oriented simple closed curve on U whose inside $\Omega := \text{Inside}(\gamma)$ is also included in U then

$$\forall z \in \Omega, \ f(z) = \frac{1}{2i\pi} \int_{\gamma} \frac{f(w)}{w - z} \mathrm{d}w$$

Proof. Define $g : U \to \mathbb{C}$ by

$$g(w) = \begin{cases} \frac{f(w) - f(z)}{w - z} & \text{if } w \neq z \\ f'(z) & \text{if } w = z \end{cases}$$

Then *g* is holomorphic on $U \setminus \{z\}$ and continuous on *U*. By Cauchy's integral theorem $\int g(w)dw = 0$, thence

$$\int_{\gamma} \frac{f(w)}{w-z} \mathrm{d}w = \int_{\gamma} \frac{f(z)}{w-z} \mathrm{d}w = f(z) \int_{\gamma} \frac{1}{w-z} \mathrm{d}w$$

From the next lemma, we have
$$\int_{\gamma} \frac{1}{w-z} dw = 2i\pi$$
, which ends the proof.

Lemma 16. Let $U \subset \mathbb{C}$ be open. Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth positively oriented simple closed curve on U whose inside $\Omega :=$ Inside (γ) is also included in U then

$$\forall z \in \Omega, \ \int_{\gamma} \frac{1}{w - z} \mathrm{d}w = 2i\pi$$

Proof. Let $z \in \Omega$. There exists $\varepsilon > 0$ such that $\overline{D_{\varepsilon}}(z) \subset \Omega$.

We can't directly apply Cauchy's integral theorem because $w \mapsto \frac{1}{w-z}$ is not defined at *z* contained in the inside of γ . So we divide $\Omega \setminus D_{\varepsilon}(z)$ into two simply connected pieces which don't contain *z* in their insides, as in the following drawing.



Then

$$0 = 0 + 0$$

= $\int_{\text{orange}} \frac{1}{w - z} dw + \int_{\text{red}} \frac{1}{w - z} dw$ by Cauchy's integral theorem
= $\int_{\gamma} \frac{1}{w - z} dw + \int_{\sigma} \frac{1}{w - z} dw$ since the segment lines are counted twice with reversed orientation
= $\int_{\gamma} \frac{1}{w - z} dw - 2i\pi$ since $\sigma(t) = z + e^{-it}, t \in [0, 2\pi]$

Corollary 17. Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic. Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth positively oriented simple closed curve on U whose inside is also included in U.

• If $z \in U$ is in the inside of γ then

$$\int_{\gamma} \frac{f(w)}{w-z} \mathrm{d}w = 2i\pi f(z)$$

• If $z \in U$ is in the outside of γ then

$$\int_{\gamma} \frac{f(w)}{w-z} \mathrm{d}w = 0$$

Proof. First case: it is Cauchy's integral formula.

Second case: it is a consequence of Cauchy's integral theorem.

Remark 18. We don't say anything then $z \in \gamma$ (the integrand $w \mapsto \frac{f(w)}{w-z}$ is not defined at z).

Remark 19. The above corollary could be improved using "winding numbers".

Next lecture, we will see that Cauchy's integral formula has deep consequences concerning properties of holomorphic functions. In the meantime, let's use it to compute a difficult real improper integral.

Example 20.

 $\int_{-\infty}^{+\infty} \frac{\cos(t)}{1+t^2} \mathrm{d}t = \frac{\pi}{e}$

Proof.



• Assume that r > 1, then

$$\int_{\gamma_1+\gamma_2} \frac{e^{iz}}{1+z^2} dz = \frac{1}{2i} \left(\int_{\gamma_1+\gamma_2} \frac{e^{iz}}{z-i} dz - \int_{\gamma_1+\gamma_2} \frac{e^{iz}}{z+i} dz \right) \text{ since } \frac{1}{1+z^2} = \frac{1}{2i} \left(\frac{1}{z-i} - \frac{1}{z+i} \right)$$
$$= \frac{1}{2i} \left(2i\pi e^{i^2} - 0 \right) \text{ by Cauchy's integral formula and theorem}$$
$$= \frac{\pi}{e}$$

•
$$\left| \int_{\gamma_2} \frac{e^{iz}}{1+z^2} dz \right| \le \text{Length}(\gamma_2) \frac{1}{r^2-1} = \frac{\pi r}{r^2-1} \xrightarrow[r \to +\infty]{} 0 \quad \left(\begin{array}{c} |e^{iz}| \le 1 \text{ since } \Im(z) \ge 0\\ |1+z^2| \ge r^2-1 \end{array} \right).$$

• $\int_{\gamma_1} \frac{e^{iz}}{1+z^2} dz = \int_{-r}^r \frac{e^{it}}{t^2+1} dt = \int_{-r}^r \frac{\cos(t)+i\sin(t)}{t^2+1} dt = \int_{-r}^r \frac{\cos(t)}{t^2+1} dt$ since sin is odd.

Since $\left|\frac{\cos(t)}{1+t^2}\right| \le \frac{1}{1+t^2}, \int_{-\infty}^{+\infty} \frac{\cos(t)}{1+t^2} dt$ is absolutely convergent. Hence $\int_{-\infty}^{+\infty} \frac{\cos(t)}{1+t^2} dt = \lim_{r \to +\infty} \int_{-r}^{r} \frac{\cos(t)}{1+t^2} dt$ (we can use the same variable for both bounds). Therefore $\frac{\pi}{e} = \lim_{r \to +\infty} \int_{\gamma_1+\gamma_2} \frac{e^{iz}}{1+z^2} dz = \int_{-\infty}^{+\infty} \frac{\cos(t)}{1+t^2} dt + 0.$