University of Toronto – MAT334H1-F – LEC0101

Complex Variables

16 – Linear fractional transformations

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1 Definition and first properties

Definition 1. A **linear fractional transformation** (or **Möbius transformation**, or **homography**) is a function of the form

$$T : \begin{array}{ccc} \widehat{\mathbb{C}} & \to & \widehat{\mathbb{C}} \\ z & \mapsto & \dfrac{az+b}{cz+d} \end{array}$$

where $a, b, c, d \in \mathbb{C}$ satisfy $ad - bc \neq 0$.

By convention $T\left(-\frac{d}{c}\right) = \infty$ and $T(\infty) = \frac{a}{c}$ where the latter is ∞ if $c = 0$ (note that a and c can't be simultaneously 0).

Remark 2. Note that $ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence the condition $ad - bc \neq 0$ means that (a, b) and (c, d) are linearly independent, i.e. that T is not constant.

Proposition 3.
1. *The composition $T_1 \circ T_2$ of two linear fractional transformations is a linear fractional transformation.*
2. *A linear fraction transformation $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is bijective and its inverse $T^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a linear fractional transformation too.*
3. *The inversion $z \mapsto \frac{1}{z}$ and the identity $z \mapsto z$ are linear fractional transformations.*

Proof.
(1) Assume that $T_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$ and that $T_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$. Then

$$T_1(T_2(z)) = \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} = \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)}$$

$$\text{and} \quad (a_1 a_2 + b_1 c_2)(c_1 b_2 + d_1 d_2) - (a_1 b_2 + b_1 d_2)(c_1 a_2 + d_1 c_2) = (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) \neq 0 \qquad (\star)$$

(2) $w = \dfrac{az+b}{cz+d} \Leftrightarrow w(cz+d) = (az+b) \Leftrightarrow z(cw-a) = (-dw+b) \Leftrightarrow z = \dfrac{-dw+b}{cw-a}$.
It is compatible with $T^{-1}(\infty) = -\frac{d}{c}$ and $T^{-1}\left(\frac{a}{c}\right) = \infty$.

(3) Note that

$$z = \frac{1 \cdot z + 0}{0 \cdot z + 1} \quad \text{and} \quad \infty \mapsto \infty$$
$$\frac{1}{z} = \frac{0 \cdot z + 1}{1 \cdot z + 0} \quad \text{and} \quad 0 \mapsto \infty \quad \& \quad \infty \mapsto 0$$

∎

Remark 4. We saw in the proof that if $T(z) = \frac{az+b}{cz+d}$ then $T^{-1}(z) = \frac{dz-b}{-cz+a}$.

2 Matrix representation

Definition 5. To a 2×2 invertible matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we associate the linear fractional transformation

$$T_M(z) = \frac{az+b}{cz+d}.$$

Remark 6. The function

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbb{C}) & \to & \{\text{linear fractional transformations}\} \\ M & \mapsto & T_M \end{array}$$

is surjective : any linear fractional transformation comes from a 2×2 invertible matrix.

But it is not injective : two different matrices may be mapped to the same linear fractional transformation. Indeed, for $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ we have $T_M = T_N$.

Proposition 7.

1. $T_{I_2} = \mathrm{id}$
2. $T_{MN} = T_M \circ T_N$
3. $T_{M^{-1}} = (T_M)^{-1}$

Proof. (1) Trivial.
(2) $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_1 & c_1b_2 + d_1d_2 \end{pmatrix}$ and compare with the proof of 3.(1).
(3) $M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ hence $T_{M^{-1}}(z) = \frac{dz-b}{-cz+a} = (T_M)^{-1}(z)$ by the proof of 3.(2). ■

Remark 8. We recover $(\star)$ since $\det(MN) = \det(M)\det(N)$.

3 Fixed points

Proposition 9.

1. *A linear fractional transformation admits at least one fixed point (i.e. a $z_0 \in \widehat{\mathbb{C}}$ such that $T(z_0) = z_0$).*
2. *If a linear fractional transformation admits more than 2 fixed points then it is the identity.*

*Particularly, a linear fractional transformation which is not the identity has either 1 or 2 fixed points.
So if a linear fractional transformation has 3 fixed points then it is the identity.*

Remark 10. It may be that ∞ is the only fixed point, i.e. there is no $z \in \mathbb{C}$ such that $T(z) = z$ (e.g. $T(z) = z+1$).

Proof. Let $T(z) = \frac{az+b}{cz+d}$.
First case: ∞ is a fixed point, i.e. $c = 0$.
Let's see if there is another fixed point $z \in \mathbb{C}$:

$$T(z) = z \Leftrightarrow (d-a)z - b = 0$$

Either $d = a$ and $b = 0$ then T is the identity.
Or the above polynomial is of degree at most 1 and hence has at most 1 root (so, with ∞, T has at least 1 fixed point and at most 2 fixed points).

Second case: $c \neq 0$ (i.e. ∞ is not a fixed point).
Let's see if there is a fixed point $z \in \mathbb{C}$:

$$T(z) = z \Leftrightarrow cz^2 + (d-a)z - b = 0$$

By the FTA, this polynomial has either 1 double root or 2 simple roots (so T has either 1 or 2 fixed points). ■

4 Triples

Proposition 11. *Let $T, S : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be two linear fractional transformations.*
Let $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ be three distinct points.
If $T(z_i) = S(z_i)$, $i = 1, 2, 3$, then $T = S$.

Proof. Then $S^{-1} \circ T$ is a linear fractional transformations (since the inverse of a LFT is a LFT and the composition of LFTs is a LFT) with three distinct fixed points z_1, z_2, z_3 hence $S^{-1} \circ T = \mathrm{id}$ and $S = T$. ■

Proposition 12. *Let $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ be three distinct points. There exists a unique linear fractional transformation $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $T(z_1) = 0$, $T(z_2) = 1$ and $T(z_3) = \infty$.*

Proof. Uniqueness derives from the previous result, so it is enough to show the existence.
But $T(z) = \left(\frac{z-z_1}{z-z_3}\right)\left(\frac{z_2-z_3}{z_2-z_1}\right)$ is a suitable linear fractional transformation (check it). ■

Proposition 13. *Let $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ be three distinct points and let $w_1, w_2, w_3 \in \widehat{\mathbb{C}}$ be another triple of distinct points. Then there exists a unique linear fractional transformation $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $T(z_1) = w_1$, $T(z_2) = w_2$ and $T(z_3) = w_3$.*

Proof. Once again, we already know that if such a linear fractional transformation exists then it is unique. Hence it is enough to show the existence.
Define $S : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as the unique linear fractional transformation such that $S(z_1) = 0$, $S(z_2) = 1$ and $S(z_3) = \infty$, and $R : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as the unique linear fractional transformation such that $R(w_1) = 0$, $R(w_2) = 1$ and $R(w_3) = \infty$.
Then $T = R^{-1} \circ S$ is a suitable linear fractional transformation. ■

5 Circles & Lines

Theorem 14. *A linear fractional transformation maps $\{$lines and circles of $\mathbb{C}\}$ to $\{$lines and circles of $\mathbb{C}\}$.*

Remark 15. Careful: a circle may be mapped to a line and vice-versa.

Proof. Let $T(z) = \dfrac{az+b}{cz+d}$. Note that $T(z) = \dfrac{a}{c} - \dfrac{ad-bc}{c}\dfrac{1}{cz+d}$.
Hence T is given by a composition of translations, scalings and inversions, but we know that {lines,circles} is invariant for such maps (see lecture from Sep 16). ■

Remark 16. Remember that a line in $\mathbb{C}$ is a circle on $\widehat{\mathbb{C}}$ passing through ∞. Hence we may simplify the above statement by saying that linear fraction transformations preserve circles of $\widehat{\mathbb{C}}$.

6 Cross ratio

Definition 17. The **cross-ratio** of four distinct elements $z_0, z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ is

$$[z_0, z_1, z_2, z_3] = \frac{z_0 - z_1}{z_0 - z_2}\frac{z_3 - z_2}{z_3 - z_1}$$

Proposition 18. *Let $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ be three distinct points and $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a linear fractional transformation. Then $[z, z_1, z_2, z_3] = [T(z), T(z_1), T(z_2), T(z_3)]$.*

Proof. We may assume that $T(z_1) = 0$, $T(z_2) = 1$ and $T(z_3) = \infty$, i.e. $T(z) = \dfrac{z-z_1}{z-z_3}\dfrac{z_2-z_3}{z_2-z_1}$.

Then $[T(z), T(z_1), T(z_2), T(z_3)] = \frac{T(z)}{T(z)-1} = \frac{\frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1}}{\frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1}-1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_1)(z_2-z_3)-(z-z_3)(z_2-z_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_2)(z_1-z_3)} = [z, z_1, z_2, z_3]$.
■

Proposition 19. *Let $z_0, z_1, z_2, z_3 \in \mathbb{C}$ be four distinct complex numbers.*
Then $[z_0, z_1, z_2, z_3] \in \mathbb{R}$ if and only if either the z_i lie on a line or they lie on a circle.

Proof. Let $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the unique linear fractional transformation such that $T(z_1) = 1$, $T(z_2) = 0$ and $T(z_3) = -1$.
Then $[z_0, z_1, z_2, z_3] = [T(z_0), 1, 0, -1] = \frac{T(z_0)-1}{T(z_0)}$.
Hence $[z_0, z_1, z_2, z_3] \in \mathbb{R}$ iff $\frac{T(z_0)-1}{T(z_0)} \in \mathbb{R}$ iff $T(z_0) \in \mathbb{R}$ iff the $T(z_i)$ lie on the real axis.
But linear fractional transformations preserve {lines, circles} hence $T(z_0), T(z_1), T(z_2), T(z_3)$ are on the real axis iff z_0, z_1, z_2, z_3 are either on a line or on a circle. ■

7 Automorphisms of the unit disk

Theorem 20. *The biholomorphic maps $T : D_1(0) \to D_1(0)$ are exactly the linear fractional transformations*

$$T(z) = \lambda \frac{a - z}{1 - \bar{a}z}$$

where $|\lambda| = 1$ and $|a| < 1$.

Proof.
Step 1: for $h_a(z) = \dfrac{a - z}{1 - \bar{a}z}$ with $|a| < 1$, we have $h_a(D_1(0)) = D_1(0)$, $h_a(a) = 0$ and $h_a(0) = a$.
Indeed $|h_a(z)|^2 = \dfrac{|a - z|^2}{|1 - \bar{a}z|^2} = \dfrac{|a|^2 - 2\mathfrak{R}(\bar{a}z) + |z|^2}{1 - 2\mathfrak{R}(\bar{a}z) + |a|^2|z|^2}$ so that

$$|h_a(z)|^2 < 1 \Leftrightarrow |a|^2 + |z|^2 < 1 + |a|^2|z|^2 \Leftrightarrow (1 - |a|^2)(1 - |z|^2) > 0 \Leftrightarrow |z| < 1$$

Step 2: let $f : D_1(0) \to D_1(0)$ be a biholomorphic map.
There exists a unique $a \in D_1(0)$ such that $f(a) = 0$.
Then $g = f \circ h_a : D_1(0) \to D_1(0)$ is a biholomorphic map such that $g(0) = 0$.
By Schwarz lemma, $|g'(0)| \leq 1$ and similarly $\frac{1}{|g'(0)|} = |(g^{-1})'(0)| \leq 1$.
Hence $|g'(0)| = 1$ and $g(z) = \lambda z$ where $|\lambda| = 1$. ■