University of Toronto – MAT334H1-F – LEC0101 Complex Variables

15 – The Maximum Modulus Principle & the Mean Value Property

Jean-Baptiste Campesato

November 18th, 2020 and November 20th, 2020

1 The open mapping theorem

Theorem 1 (The open mapping theorem).

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be a non-constant holomorphic/analytic function. Then its image f(U) is a domain (i.e. it is path-connected and open).

Proof. Since f is holomorphic, it is continuous. Hence f(U) is path-connected as the image of a path-connected set by a continuous function.

Let's prove that f(U) is open. Let $b \in f(U)$ then b = f(a) for some $a \in U$. Note that a is a zero of $z \mapsto f(z) - f(a)$ but since this function is non-constant, by the isolated zero theorem there exists r > 0 such that $\overline{D_r(a)} \subset U$ and $\forall z \in \overline{D_r(a)} \setminus \{a\}, f(z) \neq f(a)$. Set $m = \min_{|z-a|=r} |f(z) - f(a)| > 0$. Let's prove that $D_m(b) \subset f(U)$. Let $w \in D_m(b)$. Let z be such that |z - a| = r then

 $|(f(z) - w) - (f(z) - b)| = |w - b| < m \le |f(z) - b|$

By Rouché's theorem, f(z) - w and f(z) - b have exactly the same number of zeroes on $D_r(a)$ (counted with multiplicities) so f(z) - w = 0 for some $z \in D_r(a)$ and $w \in f(D_r(a)) \subset f(U)$.

2 The maximum modulus principle

We saw on October 7 that if the range of a holomorphic function defined on a domain lies on a horizontal line, or on a vertical line, or on a circle, then the function is constant. We are now going to give a stronger version of this result.

Theorem 2 (The maximum modulus principle). Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be holomorphic/analytic. If |f| has a local maximum on U then f is constant on U.

Proof. Assume that z_0 is a local maximum of |f| then there exists r > 0 such that $D_r(z_0) \subset U$ and $\forall z \in D_r(z_0), |f(z)| \le |f(z_0)|$.

Assume by contradiction that *f* is not constant, then, by the previous theorem, $f(D_r(z_0))$ is open so $\exists \delta > 0$, $D_{\delta}(f(z_0)) \subset f(D_r(z_0))$.

Set
$$w = \left(1 + \frac{\delta}{2|f(z_0)|}\right) f(z_0)$$
 then $w \in D_{\delta}(f(z_0)) \subset f(D_r(z_0))$ but $|w| > |f(z_0)|$.
Contradiction.

Remark 3. Once again this phenomenon is specific to complex calculus: it is possible for a real differentiable function f to be non-constant whereas |f| has local maxima.

Corollary 4. Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be a non-constant holomorphic/analytic function.

- 1. $\Re(f)$ has no local extremum on U.
- 2. $\mathfrak{F}(f)$ has no local extremum on U.

Proof.

1. Define $g = e^{f}$, then g and 1/g are holomorphic and non-constant.

Hence, by the maximum modulus principle, neither $|g| = e^{\Re(f)}$ nor $\left|\frac{1}{g}\right| = e^{-\Re(f)}$ have a local maximum. Since the real exponential is increasing, we get that neither $\Re(f)$ nor $-\Re(f)$ have a local maximum. So $\Re(f)$ has no local maximum and no local minimum.

2. Apply the first item to -if.

Lemma 5 (Schwarz's lemma). Assume that $f : D_1(0) \to \mathbb{C}$ is holomorphic/analytic.

1. f(0) = 02. $\forall z \in D_1(0), |f(z)| \le 1$ then 1. $\forall z \in D_1(0), |f(z)| \le |z|.$ 2. $|f'(0)| \le 1$ If

Moreover, if there exists $z_0 \neq 0$ such that $|f(z_0)| = |z_0|$ or if |f'(0)| = 1 then $f(z) = \lambda z$ where $|\lambda| = 1$. Proof.

We define $g : D_1(0) \to \mathbb{C}$ by $g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0\\ f'(0) & \text{otherwise} \end{cases}$. Then g is holomorphic/analytic.

By the maximum modulus principle, for $r \in (0, 1)$, $\max_{|z| \le r} |g| = \max_{|z|=r} |g| \le \frac{1}{r}$.

As $r \to 1$, we see that $|g(z)| \le 1$, i.e. $|f(z)| \le |z|$. It follows that $|f'(0)| \le 1$

Assume that there exists $z_0 \neq 0$ such that $|f(z_0)| = |z_0|$, then z_0 is local max of g. Hence, by the maximum modulus principle, g is constant. Besides it is of modulus 1, i.e. $g(z) = \lambda$ where $|\lambda| = 1$. Hence $f(z) = \lambda z.$

Assume that |f'(0)| = 1 then |g(0)| = |f'(0)| = 1| and 0 is a local max of g. Hence we may conclude as in the above case.

Remark 6. Once again, Schwarz's lemma doesn't hold for real differentiable functions. For instance, define $u(x) = \frac{2x}{x^2 + 1}$ on [-1, 1]: it is C^1 , u(0) = 0, $|u(x)| \le 1$ but |u(x)| > |x| on $[-1, 1] \setminus \{0\}$.

In the above proof we used the fact that if f is defined on \overline{U} and holomorphic on U, then the local maximum of |f|, it there are some, are located on ∂U .

Corollary 7. Let $U \subset \mathbb{C}$ be a domain and $f : \overline{U} \to \mathbb{C}$ a function. Assume that f is holomorphic/analytic and non-constant on U. *Then the possible*

- 1. local extrema of $\Re(f)$,
- 2. local extrema of $\mathfrak{T}(f)$, and,
- 3. local maxima of |f|

are located on ∂U .

Corollary 8. Let $U \subset \mathbb{C}$ be a bounded domain and $f : \overline{U} \to \mathbb{C}$ a continuous function. Assume that f is holomorphic/analytic. If $f_{|\partial U} = 0$ then f = 0.

Proof. |f| is continuous on the compact set \overline{U} (closed and bounded), hence it admits a maximum. By the above corollary, either the function is constant equal to 0 on \overline{U} or the local maxima of |f| are located on ∂U (and so are the global maxima).

In either case the maximum of |f| is reached on the boundary of U so that it has to be 0. Hence $\forall z \in \overline{U}, f(z) = 0.$

3 The mean value property

Theorem 9 (The mean value property for holomorphic functions). Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic. Let $z_0 \in U$ and r > 0 be such that $\overline{D_r(z_0)} \subset U$. Then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt$$

Proof. Define $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = z_0 + re^{it}$ then, by Cauchy's theorem,

$$f(z_0) = \frac{1}{2i\pi} \int_{\gamma} \frac{f(w)}{w - z_0} dw = \frac{1}{2i\pi} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} rie^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt$$

Theorem 10 (The mean value property for harmonic functions). Let $\mathcal{U} \subset \mathbb{R}^2$ be open and simply connected. Let $u : \mathcal{U} \to \mathbb{R}$ be harmonic. Let $p_0 = (x_0, y_0) \in \mathcal{U}$ and r > 0 be such that $\overline{D_r(p_0)} \subset \mathcal{U}$. Then

$$u(p_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(x_0 + r\cos t, y_0 + r\sin t\right) dt$$

Proof. Since *u* is harmonic on \mathcal{U} simply connected, we know that *u* is the real part of a holomorphic function f = u + iv. Then, by the previous theorem

$$f(x_0 + iy_0) = \frac{1}{2\pi} \int_0^{2\pi} f(x_0 + iy_0 + re^{it}) dt$$

We conclude by taking the real part of both sides.