# 15 - The Maximum Modulus Principle \& the Mean Value Property 

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## 1 The open mapping theorem

Theorem 1 (The open mapping theorem).
Let $U \subset \mathbb{C}$ be a domain and $f: U \rightarrow \mathbb{C}$ be a non-constant holomorphic/analytic function.
Then its image $f(U)$ is a domain (i.e. it is path-connected and open).
Proof. Since $f$ is holomorphic, it is continuous. Hence $f(U)$ is path-connected as the image of a pathconnected set by a continuous function.
Let's prove that $f(U)$ is open. Let $b \in f(U)$ then $b=f(a)$ for some $a \in U$.
Note that $a$ is a zero of $z \mapsto f(z)-f(a)$ but since this function is non-constant, by the isolated zero theorem there exists $r>0$ such that $\overline{D_{r}(a)} \subset U$ and $\forall z \in \overline{D_{r}(a)} \backslash\{a\}, f(z) \neq f(a)$.
Set $m=\min _{|z-a|=r}|f(z)-f(a)|>0$. Let's prove that $D_{m}(b) \subset f(U)$.
Let $w \in D_{m}(b)$. Let $z$ be such that $|z-a|=r$ then

$$
|(f(z)-w)-(f(z)-b)|=|w-b|<m \leq|f(z)-b|
$$

By Rouchés theorem, $f(z)-w$ and $f(z)-b$ have exactly the same number of zeroes on $D_{r}(a)$ (counted with multiplicities) so $f(z)-w=0$ for some $z \in D_{r}(a)$ and $w \in f\left(D_{r}(a)\right) \subset f(U)$.

## 2 The maximum modulus principle

We saw on October 7 that if the range of a holomorphic function defined on a domain lies on a horizontal line, or on a vertical line, or on a circle, then the function is constant. We are now going to give a stronger version of this result.

Theorem 2 (The maximum modulus principle).
Let $U \subset \mathbb{C}$ be a domain and $f: U \rightarrow \mathbb{C}$ be holomorphic/analytic.
If $|f|$ has a local maximum on $U$ then $f$ is constant on $U$.
Proof. Assume that $z_{0}$ is a local maximum of $|f|$ then there exists $r>0$ such that $D_{r}\left(z_{0}\right) \subset U$ and $\forall z \in D_{r}\left(z_{0}\right),|f(z)| \leq\left|f\left(z_{0}\right)\right|$.
Assume by contradiction that $f$ is not constant, then, by the previous theorem, $f\left(D_{r}\left(z_{0}\right)\right)$ is open so $\exists \delta>$ $0, D_{\delta}\left(f\left(z_{0}\right)\right) \subset f\left(D_{r}\left(z_{0}\right)\right)$.
Set $w=\left(1+\frac{\delta}{2\left|f\left(z_{0}\right)\right|}\right) f\left(z_{0}\right)$ then $w \in D_{\delta}\left(f\left(z_{0}\right)\right) \subset f\left(D_{r}\left(z_{0}\right)\right)$ but $|w|>\left|f\left(z_{0}\right)\right|$.
Contradiction.
Remark 3. Once again this phenomenon is specific to complex calculus: it is possible for a real differentiable function $f$ to be non-constant whereas $|f|$ has local maxima.

Corollary 4. Let $U \subset \mathbb{C}$ be a domain and $f: U \rightarrow \mathbb{C}$ be a non-constant holomorphic/analytic function.

1. $\mathfrak{R}(f)$ has no local extremum on $U$.
2. $\mathfrak{J}(f)$ has no local extremum on $U$.

Proof.

1. Define $g=e^{f}$, then $g$ and $1 / g$ are holomorphic and non-constant.

Hence, by the maximum modulus principle, neither $|g|=e^{\mathfrak{R}(f)}$ nor $\left|\frac{1}{g}\right|=e^{-\mathfrak{R}(f)}$ have a local maximum.
Since the real exponential is increasing, we get that neither $\Re(f)$ nor $-\Re(f)$ have a local maximum.
So $\Re(f)$ has no local maximum and no local minimum.
2. Apply the first item to $-i f$.

Lemma 5 (Schwarz's lemma). Assume that $f: D_{1}(0) \rightarrow \mathbb{C}$ is holomorphic/analytic.
If

1. $f(0)=0$
2. $\forall z \in D_{1}(0),|f(z)| \leq 1 \quad$ then
3. $\forall z \in D_{1}(0),|f(z)| \leq|z|$.
4. $\left|f^{\prime}(0)\right| \leq 1$

Moreover, if there exists $z_{0} \neq 0$ such that $\left|f\left(z_{0}\right)\right|=\left|z_{0}\right|$ or if $\left|f^{\prime}(0)\right|=1$ then $f(z)=\lambda z$ where $|\lambda|=1$.
Proof.
We define $g: D_{1}(0) \rightarrow \mathbb{C}$ by $g(z)=\left\{\begin{array}{cl}\frac{f(z)}{z} & \text { if } z \neq 0 \\ f^{\prime}(0) & \text { otherwise }\end{array}\right.$. Then $g$ is holomorphic/analytic.
By the maximum modulus principle, for $r \in(0,1), \max _{|z| \leq r}|g|=\max _{|z|=r}|g| \leq \frac{1}{r}$.
As $r \rightarrow 1$, we see that $|g(z)| \leq 1$, i.e. $|f(z)| \leq|z|$. It follows that $\left|f^{\prime}(0)\right| \leq 1$
Assume that there exists $z_{0} \neq 0$ such that $\left|f\left(z_{0}\right)\right|=\left|z_{0}\right|$, then $z_{0}$ is local max of $g$. Hence, by the maximum modulus principle, $g$ is constant. Besides it is of modulus 1, i.e. $g(z)=\lambda$ where $|\lambda|=1$. Hence $f(z)=\lambda z$.
Assume that $\left|f^{\prime}(0)\right|=1$ then $|g(0)|=\left|f^{\prime}(0)\right|=1 \mid$ and 0 is a local max of $g$. Hence we may conclude as in the above case.

Remark 6. Once again, Schwarz's lemma doesn't hold for real differentiable functions.
For instance, define $u(x)=\frac{2 x}{x^{2}+1}$ on $[-1,1]$ :
it is $C^{1}, u(0)=0,|u(x)| \leq 1$ but $|u(x)|>|x|$ on $[-1,1] \backslash\{0\}$.
In the above proof we used the fact that if $f$ is defined on $\bar{U}$ and holomorphic on $U$, then the local maximum of $|f|$, it there are some, are located on $\partial U$.
Corollary 7. Let $U \subset \mathbb{C}$ be a domain and $f: \bar{U} \rightarrow \mathbb{C}$ a function.
Assume that $f$ is holomorphic/analytic and non-constant on $U$.
Then the possible

1. local extrema of $\Re(f)$,
2. local extrema of $\mathfrak{J}(f)$, and,
3. local maxima of $|f|$
are located on $\partial U$.
Corollary 8. Let $U \subset \mathbb{C}$ be a bounded domain and $f: \bar{U} \rightarrow \mathbb{C}$ a continuous function.
Assume that $f$ is holomorphic/analytic. If $f_{\mid \partial U}=0$ then $f=0$.
Proof. $|f|$ is continuous on the compact set $\bar{U}$ (closed and bounded), hence it admits a maximum.
By the above corollary, either the function is constant equal to 0 on $\bar{U}$ or the local maxima of $|f|$ are located on $\partial U$ (and so are the global maxima).
In either case the maximum of $|f|$ is reached on the boundary of $U$ so that it has to be 0 .
Hence $\forall z \in \bar{U}, f(z)=0$.

## 3 The mean value property

Theorem 9 (The mean value property for holomorphic functions).
Let $U \subset \mathbb{C}$ be open and $f: U \rightarrow \mathbb{C}$ be holomorphic/analytic.
Let $z_{0} \in U$ and $r>0$ be such that $\overline{D_{r}\left(z_{0}\right)} \subset U$. Then

$$
f\left(z_{0}\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} f\left(z_{0}+r e^{i t}\right) \mathrm{d} t
$$

Proof. Define $\gamma:[0,2 \pi] \rightarrow \mathbb{C}$ by $\gamma(t)=z_{0}+r e^{i t}$ then, by Cauchy's theorem,

$$
f\left(z_{0}\right)=\frac{1}{2 i \pi} \int_{\gamma} \frac{f(w)}{w-z_{0}} \mathrm{~d} w=\frac{1}{2 i \pi} \int_{0}^{2 \pi} \frac{f\left(z_{0}+r e^{i t}\right)}{r e^{i t}} r i e^{i t} \mathrm{~d} t=\frac{1}{2 \pi} \int_{0}^{2 \pi} f\left(z_{0}+r e^{i t}\right) \mathrm{d} t
$$

Theorem 10 (The mean value property for harmonic functions).
Let $\mathscr{U} \subset \mathbb{R}^{2}$ be open and simply connected. Let $u: \mathscr{U} \rightarrow \mathbb{R}$ be harmonic.
Let $p_{0}=\left(x_{0}, y_{0}\right) \in \mathscr{U}$ and $r>0$ be such that $\overline{D_{r}\left(p_{0}\right)} \subset \mathcal{U}$. Then

$$
u\left(p_{0}\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} u\left(x_{0}+r \cos t, y_{0}+r \sin t\right) \mathrm{d} t
$$

Proof. Since $u$ is harmonic on $\mathscr{U}$ simply connected, we know that $u$ is the real part of a holomorphic function $f=u+i v$. Then, by the previous theorem

$$
f\left(x_{0}+i y_{0}\right)=\frac{1}{2 \pi} \int_{0}^{2 \pi} f\left(x_{0}+i y_{0}+r e^{i t}\right) \mathrm{d} t
$$

We conclude by taking the real part of both sides.

