University of Toronto – MAT334H1-F – LEC0101
Complex Variables

14 - Zeroes of analytic functions

Jean-Baptiste Campesato

November 4th, 2020

Reviews from Oct 16 – Zeroes

Definition 1.
Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic. Let $z_0 \in U$ be such that $f(z_0) = 0$.
We define the **order of vanishing of** f **at** z_0 by $m_f(z_0) \coloneqq \min\left\{n \in \mathbb{N} \;:\; f^{(n)}(z_0) \neq 0\right\}$.
Note that $m_f(z_0) > 0$ *since* $f(z_0) = 0$.

Proposition 2. *Let* $U \subset \mathbb{C}$ *be open and* $f : U \to \mathbb{C}$ *be holomorphic/analytic. Let* $z_0 \in U$ *be such that* $f(z_0) = 0$.
Denote the power series expansion of f *at* z_0 *by* $f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$.
Then $m_f(z_0) = \min\left\{n \in \mathbb{N} \;:\; a_n \neq 0\right\}$.

Proposition 3. *Let* $U \subset \mathbb{C}$ *be open and* $f : U \to \mathbb{C}$ *be holomorphic/analytic. Then* z_0 *is a zero of order* n *of* f *if and only if there exists* $g : U \to \mathbb{C}$ *holomorphic such that* $f(z) = (z - z_0)^n g(z)$ *and* $g(z_0) \neq 0$.

Reviews from Oct 16 – Analytic continuation

Theorem 4. *Let* $U \subset \mathbb{C}$ *be a* ***domain*** *and* $f : U \to \mathbb{C}$ *be a holomorphic/analytic function.*
If there exists $z_0 \in U$ *such that* $\forall n \in \mathbb{N}_{\geq 0},\ f^{(n)}(z_0) = 0$ *then* $f \equiv 0$ *on* U.

Corollary 5. *Let* $U \subset \mathbb{C}$ *be a* ***domain*** *and* $f, g : U \to \mathbb{C}$ *be holomorphic/analytic functions.*
If f *and* g *coincide in the neighborhood of a point,*

$$\text{i.e. } \exists z_0 \in U,\ \exists r > 0,\ \forall z \in D_r(z_0) \cap U,\ f(z) = g(z),$$

then they coincide on U*,*

$$\textit{i.e.}\ \forall z \in U,\ f(z) = g(z).$$

Isolated zeroes
It is actually possible to strengthen the previous results.

Theorem 6. *Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be a holomorphic/analytic function.*
Then either $f \equiv 0$ or the zeroes of f are isolated [⋆] *:*
if $f(z_0) = 0$ then there exists $r > 0$ such that $D_r(z_0) \subset U$ and $\forall z \in D_r(z_0) \setminus \{z_0\},\ f(z) \neq 0$.

Proof. Assume that z_0 is a non-isolated zero of f.

We know that f admits a power series expansion $f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$ in a neighborhood of z_0.
Assume by contradiction that there exists a smallest $n \in \mathbb{N}_{\geq 0}$ such that $a_n \neq 0$ then $f(z) = (z - z_0)^n g(z)$ where g is holomorphic and $g(z_0) \neq 0$.
For every $n \in \mathbb{N}_{>0}$, $\exists w_n \in \left(D_{\frac{1}{n}}(z_0) \cap U\right) \setminus \{z_0\},\ f(w_n) = 0$. But then $g(w_n) = 0$ since $w_n \neq z_0$
Then, since $w_n \xrightarrow[n \to +\infty]{} z_0$, by continuity $g(z_0) = \lim_{n \to +\infty} g(w_n) = 0$. Which leads to a contradiction.
Hence $\forall n \in \mathbb{N}_{\geq 0},\ f^{(n)}(z_0) = n! a_n = 0$ and $f \equiv 0$ on U by Theorem 4. ∎

Corollary 7. *Let $U \subset \mathbb{C}$ be a domain and $f, g : U \to \mathbb{C}$ be holomorphic/analytic functions.*
If $f - g$ admits a non-isolated zero

$$\text{i.e. } \exists z_0 \in U,\ \forall r > 0,\ \exists z \in \big(U \cap D_r(z_0)\big) \setminus \{z_0\},\ f(z) - g(z) = f(z_0) - g(z_0) = 0$$

then f and g coincide on U,

$$\text{i.e. } \forall z \in U,\ f(z) = g(z).$$

Corollary 8. *Let $U \subset \mathbb{C}$ be a domain and $f, g : U \to \mathbb{C}$ be holomorphic/analytic functions.*
Let $(z_n)_{n \in \mathbb{N}}$ be a sequence of terms in U which is convergent to $\tilde{z}$ in U and such that $\forall n \in \mathbb{N},\ f(z_n) = 0$.
Then $f \equiv 0$ on U.

Remark 9. The fact that the limit $\tilde{z} \in U$ is very important.
Indeed, let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be defined by $f(z) = \sin\left(\frac{\pi}{z}\right)$.
Then $f\left(\frac{1}{n}\right) = 0$ but $f \not\equiv 0$ on $\mathbb{C} \setminus \{0\}$.

Hence, it is possible for the zeroes of f to accumulate at a point of the boundary of the domain (including ∞, see for instance $z_n = \pi n$ for $f = \sin$).

Homework 10. Let $U \subset \mathbb{C}$ be a domain and $f, g : U \to \mathbb{C}$ be holomorphic/analytic on U.
Prove that if $fg \equiv 0$ on U then either $f \equiv 0$ or $g \equiv 0$.

Homework 11. Let $U = D_1(0)$. Find all the holomorphic functions $f : U \to \mathbb{C}$ satisfying respectively:

1. $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$
2. $f\left(\frac{1}{2n}\right) = f\left(\frac{1}{2n+1}\right) = \frac{1}{n}$

[⋆] Otherwise stated, if you attend MAT327, either f is constant equal to 0 or $\{z \in U : f(z) = 0\}$ is discrete.

Reviews from Oct 23 – Poles

Theorem 12. *Let $U \subset \mathbb{C}$ be open and $z_0 \in U$. Assume that $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic/analytic. Then TFAE:*

1. *z_0 is a pole of f, i.e. $\lim_{z \to z_0} |f(z)| = +\infty$.*
2. *There exist $n \in \mathbb{N}_{>0}$ and $g : U \to \mathbb{C}$ analytic such that $g(z_0) \neq 0$ and $f(z) = \frac{g(z)}{(z-z_0)^n}$ on $U \setminus \{z_0\}$.*
3. *z_0 is not a removable singularity of f and there exists $n \in \mathbb{N}_{>0}$ such that $\lim_{z \to z_0} (z-z_0)^{n+1} f(z) = 0$.*

Definition 13. The integer $n > 0$ in (2) is uniquely defined and we say that f admits a **pole of order n at** z_0.

Proposition 14. *The order of the pole z_0 is also:*

- *The order of vanishing of $1/f$ at z_0.*
- *The smallest n such that $\lim_{z \to z_0} (z-z_0)^{n+1} f(z) = 0$.*

The argument principle

Lemma 15 (Logarithmic residue)**.**

- *If z_0 is an isolated zero of f then $\operatorname{Res}\left(\frac{f'}{f}, z_0\right)$ is the order of z_0.*
- *If z_0 is an isolated pole of f then $-\operatorname{Res}\left(\frac{f'}{f}, z_0\right)$ is the order of z_0.*

Proof.

- Assume that $f(z) = (z-z_0)^m g(z)$ in a neighborhood of z_0 where g is analytic and $g(z_0) \neq 0$. Then $\frac{f'(z)}{f(z)} = m(z-z_0)^{-1} + \frac{g'(z)}{g(z)}$.
We conclude using that $\frac{g'}{g}$ is holomorphic in a neighborhood of z_0.
- z_0 is a pole of order m of f if and only if it is a zero of order m of $\frac{1}{f}$. We conclude using that

$$\operatorname{Res}\left(\frac{(1/f)'}{(1/f)}, z_0\right) = -\operatorname{Res}\left(\frac{f'}{f}, z_0\right)$$

∎

The previous lemma holds at ∞:

Lemma 16.

- *If ∞ is an isolated zero of f then $\operatorname{Res}\left(\frac{f'}{f}, \infty\right)$ is the order of ∞.*
- *If ∞ is an isolated pole of f then $-\operatorname{Res}\left(\frac{f'}{f}, \infty\right)$ is the order of ∞.*

Proof. ∞ is an isolated zero (resp. pole) of order m of f if and only if 0 is an isolated zero (resp. pole) of order m of $g(z) = f(1/z)$.
Then $m = \operatorname{Res}\left(\frac{g'}{g}, 0\right) = \operatorname{Res}\left(\frac{-1}{z^2}\frac{f'(1/z)}{f(1/z)}, 0\right) = \operatorname{Res}\left(\frac{f'}{f}, \infty\right)$. ∎

Theorem 17 (The argument principle)**.**
Let $U \subset \mathbb{C}$ be open. Let $S \subset U$ be finite. Let $f : U \setminus S \to \mathbb{C}$ be holomorphic/analytic.
Let $\gamma : [a,b] \to \mathbb{C}$ be piecewise smooth positively oriented simple closed curve on U which doesn't pass through a zero or a pole of f and such that its inside is entirely included in U.
Then

$$\frac{1}{2i\pi}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = Z_{f,\gamma} - P_{f,\gamma}$$

where

- *$Z_{f,\gamma}$ is the number of zeroes of f enclosed in γ counted with their multiplicites/orders,*
- *$P_{f,\gamma}$ is the number of poles of f enclosed in γ counted with their multiplicites/orders.*

Proof. We apply Cauchy's residue theorem to $\frac{f'}{f}$ and then we use the above lemma:

$$\frac{1}{2i\pi}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \sum_{z\in\mathrm{Inside}(\gamma)} \mathrm{Res}\left(\frac{f'}{f}, z\right) = \sum_{z \text{ zero of } f} \mathrm{Res}\left(\frac{f'}{f}, z_0\right) + \sum_{z \text{ pole of } f} \mathrm{Res}\left(\frac{f'}{f}, z_0\right) = Z_{f,\gamma} - P_{f,\gamma}$$ ■

Remark 18. The value $\frac{1}{2i\pi}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z$ involved in the previous slide is equal to the number of counterclockwise turns made by $f(z)$ as z goes through γ.

Indeed, if we set $\tilde\gamma(t) = f \circ \gamma$ then $\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \int_{\tilde\gamma} \frac{1}{w}\mathrm{d}w$.

Assume for instance that $\tilde\gamma : [0,1] \to \mathbb{C}$ is defined by $\tilde\gamma(t) = z_0 + re^{2i\pi nt}$ where $n \in \mathbb{Z}$.
Then $\frac{1}{2i\pi}\int_{\tilde\gamma} \frac{1}{w}\mathrm{d}w = n$ which is the number of counterclockwise turns made by $\tilde\gamma$ around z_0.

Then the conclusion of the previous statement can be rewritten as

$$\frac{\text{changes of } \arg(f(z)) \text{ as } z \text{ goes through } \gamma}{2\pi} = Z_{f,\gamma} - P_{f,\gamma}$$

That's why it is called *the argument principle*.

Rouché's theorem

Theorem 19 (Rouché's theorem – version 1)**.**
Let $U \subset \mathbb{C}$ be open, $f, g : U \to \mathbb{C}$ be two holomorphic/analytic functions on U, and $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth simple closed curve on U whose inside is also included in U.
Assume that

$$\forall t \in [a,b],\ |g(\gamma(t))| < |f(\gamma(t))|$$

Then f and $f+g$ have the same number of zeroes inside γ, counted with multiplicities.

Proof. For $t \in [0,1]$, set $\varphi_t(z) = f(z) + (1-t)g(z)$ and $h(t) = \frac{1}{2i\pi}\int_\gamma \frac{\varphi_t'(z)}{\varphi_t(z)}\mathrm{d}z$.
The function h is continuous since φ_t doesn't vanish on γ, indeed for $z \in \gamma$

$$|\varphi_t(z)| \geq |f(z)| + (1-t)|g(z)| \geq |f(z)| - |g(z)| > 0$$

Hence h is a continuous function taking values in $\mathbb{Z}$ (by the principle argument), so it is constant.
Hence $h(0) = h(1)$, i.e. $Z_{f+g,\gamma} - P_{f+g,\gamma} = Z_{f,\gamma} - P_{f,\gamma}$ by the principle argument.
But these functions have no poles in the inside of γ, hence $Z_{f+g,\gamma} = Z_{f,\gamma}$. ■

Theorem 20 (Rouché's theorem – version 2).
Let $U \subset \mathbb{C}$ be open, $f, g : U \to \mathbb{C}$ be two holomorphic/analytic functions on U, and $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth simple closed curve on U whose inside is also included in U.
Assume that

$$\forall z \in \gamma,\ |f(z) - g(z)| < |f(z)|$$

Then f and g have the same number of zeroes inside γ, counted with multiplicities.

Proof. That's an immediate consequence of the previous version since z_0 is a zero of order n of g iff it is a zero of order n of $-g$. ■

Theorem 21 (Rouché's theorem – version 3).
Let $U \subset \mathbb{C}$ be open, $f, g : U \to \mathbb{C}$ be two holomorphic/analytic functions on U, and $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth simple closed curve on U whose inside is also included in U.
Assume that

$$\forall z \in \gamma,\ |f(z) + g(z)| < |f(z)|$$

Then f and g have the same number of zeroes inside γ, counted with multiplicities.

Proof. Since z_0 is a zero of order n of g iff it is a zero of order n of $-g$. ■

We already proved the Fundamental Theorem of Algebra (or d'Alembert–Gauss theorem) using Liouville's theorem (Oct 21): a non-constant complex polynomial admits at least one root.
Here is another proof using Rouché's theorem.

Theorem 22. *A complex polynomial of degree n has exactly n complex roots (counted with multiplicity).*

Proof. Assume that $P(z) = a_n z^n + Q(z)$ where Q is a polynomial of degree $< n$ and $a_n \neq 0$.
If we take $R > 0$ big enough then $|Q(z)| < |a_n z^n|$ on $\gamma : [0, 1] \to \mathbb{C}$ defined by $\gamma(t) = Re^{2i\pi t}$.
By Rouché's theorem, $P(z) = a_n z^n + Q(z)$ and $a_n z^n$ have the same number of zeroes counted with multiplicity.
■