Zeroes of analytic functions – 2

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Reviews from Oct 23 – Poles

Theorem

Let $U \subset \mathbb{C}$ be open and $z_0 \in U$. Assume that $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic/analytic. Then TFAE:

1. z_0 is a pole of f, i.e. $\lim\limits_{z \to z_0} |f(z)| = +\infty$.
2. There exist $n \in \mathbb{N}_{>0}$ and $g : U \to \mathbb{C}$ analytic such that $g(z_0) \neq 0$ and $f(z) = \dfrac{g(z)}{(z-z_0)^n}$ on $U \setminus \{z_0\}$.
3. z_0 is not a removable singularity of f and there exists $n \in \mathbb{N}_{>0}$ such that $\lim\limits_{z \to z_0} (z-z_0)^{n+1} f(z) = 0$.

Definition: order of a pole

The integer $n \in \mathbb{N}_{>0}$ in 2 is uniquely defined and we say that f admits a **pole of order n at** z_0.

Proposition

The order of the pole z_0 is also:

- The order of vanishing of $1/f$ at z_0.
- The smallest n such that $\lim\limits_{z \to z_0} (z-z_0)^{n+1} f(z) = 0$.

Logarithmic residue

Lemma

- If z_0 is an isolated zero of f then $\operatorname{Res}\left(\frac{f'}{f}, z_0\right)$ is the order of z_0.
- If z_0 is an isolated pole of f then $-\operatorname{Res}\left(\frac{f'}{f}, z_0\right)$ is the order of z_0.

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Proof.

- Assume that $f(z) = (z - z_0)^m g(z)$ in a neighborhood of z_0 where g is analytic and $g(z_0) \neq 0$.
  Then $\frac{f'(z)}{f(z)} = m(z - z_0)^{-1} + \frac{g'(z)}{g(z)}$.
  We conclude using that $\frac{g'}{g}$ is holomorphic in a neighborhood of z_0.

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  Then $\frac{f'(z)}{f(z)} = m(z - z_0)^{-1} + \frac{g'(z)}{g(z)}$.
  We conclude using that $\frac{g'}{g}$ is holomorphic in a neighborhood of z_0.
- z_0 is a pole of order m of f if and only if it is a zero of order m of $\frac{1}{f}$. We conclude using that

$$\operatorname{Res}\left(\frac{(1/f)'}{(1/f)}, z_0\right) = -\operatorname{Res}\left(\frac{f'}{f}, z_0\right)$$

∎

The argument principle – Statement

Theorem: the argument principle

Let $U \subset \mathbb{C}$ be open. Let $S \subset U$ be finite. Let $f : U \setminus S \to \mathbb{C}$ be holomorphic/analytic.
Let $\gamma : [a,b] \to \mathbb{C}$ be piecewise smooth positively oriented simple closed curve on U which doesn't pass through a zero or a pole of f and such that its inside is entirely included in U.
Then

$$\frac{1}{2i\pi}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = Z_{f,\gamma} - P_{f,\gamma}$$

where

- $Z_{f,\gamma}$ is the number of zeroes of f enclosed in γ counted with their multiplicites/orders,
- $P_{f,\gamma}$ is the number of poles of f enclosed in γ counted with their multiplicites/orders.

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Proof. We apply Cauchy's residue theorem to $\frac{f'}{f}$ and then we use the above lemma:

$$\frac{1}{2i\pi} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \sum_{z \in \mathrm{Inside}(\gamma)} \mathrm{Res}\left(\frac{f'}{f}, z\right) = \sum_{z \text{ zero of } f} \mathrm{Res}\left(\frac{f'}{f}, z_0\right) + \sum_{z \text{ pole of } f} \mathrm{Res}\left(\frac{f'}{f}, z_0\right) = Z_{f,\gamma} - P_{f,\gamma}$$

∎

The argument principle – Interpretation

The value $\frac{1}{2i\pi}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z$ involved in the previous slide is equal to the number of counterclockwise turns made by $f(z)$ as z goes through γ.

Indeed, if we set $\tilde\gamma(t) = f\circ\gamma$ then $\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \int_{\tilde\gamma}\frac{1}{w}\mathrm{d}w$.

Assume for instance that $\tilde\gamma : [0,1]\to\mathbb{C}$ is defined by $\tilde\gamma(t) = z_0 + re^{2i\pi nt}$ where $n\in\mathbb{Z}$.
Then $\frac{1}{2i\pi}\int_{\tilde\gamma}\frac{1}{w}\mathrm{d}w = n$ which is the number of counterclockwise turns made by $\tilde\gamma$ around z_0.

Then the conclusion of the previous statement can be rewritten as

$$\frac{\text{changes of } \arg(f(z)) \text{ as } z \text{ goes through } \gamma}{2\pi} = Z_{f,\gamma} - P_{f,\gamma}$$

That's why it is called *the argument principle*.

Generalization at infinity

The previous lemma holds at ∞:

Lemma

- If ∞ is an isolated zero of f then $\operatorname{Res}\left(\frac{f'}{f},\infty\right)$ is the order of ∞.
- If ∞ is an isolated pole of f then $-\operatorname{Res}\left(\frac{f'}{f},\infty\right)$ is the order of ∞.

Proof. ∞ is an isolated zero (resp. pole) of order m of f if and only if 0 is an isolated zero (resp. pole) of order m of $g(z)=f(1/z)$.
Then $m=\operatorname{Res}\left(\frac{g'}{g},0\right)=\operatorname{Res}\left(\frac{-1}{z^2}\frac{f'(1/z)}{f(1/z)},0\right)=\operatorname{Res}\left(\frac{f'}{f},\infty\right)$. ∎

Rouché's theorem – 1

Rouché's theorem – version 1

Let $U \subset \mathbb{C}$ be open, $f, g : U \to \mathbb{C}$ be two holomorphic/analytic functions on U, and $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth simple closed curve on U whose inside is also included in U.
Assume that

$$\forall t \in [a,b],\ |g(\gamma(t))| < |f(\gamma(t))|$$

Then f and $f+g$ have the same number of zeroes inside γ, counted with multiplicities.

Proof. For $t \in [0,1]$, set $\varphi_t(z) = f(z) + (1-t)g(z)$ and $h(t) = \frac{1}{2i\pi}\int_\gamma \frac{\varphi_t'(z)}{\varphi_t(z)}\mathrm{d}z$.
The function h is continuous since φ_t doesn't vanish on γ, indeed for $z \in \gamma$

$$|\varphi_t(z)| \geq |f(z)| + (1-t)|g(z)| \geq |f(z)| - |g(z)| > 0$$

Hence h is a continuous function taking values in $\mathbb{Z}$ (by the principle argument), so it is constant.
Hence $h(0) = h(1)$, i.e. $Z_{f+g,\gamma} - P_{f+g,\gamma} = Z_{f,\gamma} - P_{f,\gamma}$ by the principle argument.
But these functions have no poles in the inside of γ, hence $Z_{f+g,\gamma} = Z_{f,\gamma}$. ∎

Rouché's theorem – 2

Rouché's theorem – version 2

Let $U \subset \mathbb{C}$ be open, $f, g : U \to \mathbb{C}$ be two holomorphic/analytic functions on U, and $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth simple closed curve on U whose inside is also included in U.
Assume that

$$\forall z \in \gamma,\ |f(z) - g(z)| < |f(z)|$$

Then f and g have the same number of zeroes inside γ, counted with multiplicities.

Proof. That's an immediate consequence of the previous version since z_0 is a zero of order n of g iff it is a zero of order n of $-g$. ■

Rouché's theorem – version 3

Let $U \subset \mathbb{C}$ be open, $f, g : U \to \mathbb{C}$ be two holomorphic/analytic functions on U, and $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth simple closed curve on U whose inside is also included in U.
Assume that

$$\forall z \in \gamma,\ |f(z) + g(z)| < |f(z)|$$

Then f and g have the same number of zeroes inside γ, counted with multiplicities.

Proof. Since z_0 is a zero of order n of g iff it is a zero of order n of $-g$. ■

The fundamental theorem of algebra – *bis repetita placent*

We already proved the FTA (or d'Alembert–Gauss theorem) using Liouville's theorem (Oct 21): a non-constant complex polynomial admits at least one root.
Here is another proof using Rouché's theorem.

Theorem

A complex polynomial of degree n has exactly n complex roots (counted with multiplicity).

Proof. Assume that $P(z) = a_n z^n + Q(z)$ where Q is a polynomial of degree $< n$ and $a_n \neq 0$.
If we take $R > 0$ big enough then $|Q(z)| < |a_n z^n|$ on $\gamma : [0,1] \to \mathbb{C}$ defined by $\gamma(t) = Re^{2i\pi t}$.
By Rouché's theorem, $P(z) = a_n z^n + Q(z)$ and $a_n z^n$ have the same number of zeroes counted with multiplicity. ∎