Zeroes of analytic functions – 1

UNIVERSITY OF TORONTO

November 4th, 2020 and November 6th, 2020

Reviews from Oct 16 – Zeroes

Definition: order of a zero

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic. Let $z_0 \in U$ be such that $f(z_0) = 0$.
We define the **order of vanishing of f at z_0** by $m_f(z_0) \coloneqq \min\left\{n \in \mathbb{N} \ : \ f^{(n)}(z_0) \neq 0\right\}$.
Note that $m_f(z_0) > 0$ since $f(z_0) = 0$.

Proposition

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic. Let $z_0 \in U$ be such that $f(z_0) = 0$.

Denote the power series expansion of f at z_0 by $f(z) = \displaystyle\sum_{n=0}^{+\infty} a_n (z - z_0)^n$.

Then $m_f(z_0) = \min\left\{n \in \mathbb{N} \ : \ a_n \neq 0\right\}$.

Proposition

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic. Then z_0 is a zero of order n of f if and only if there exists $g : U \to \mathbb{C}$ holomorphic such that $f(z) = (z - z_0)^n g(z)$ and $g(z_0) \neq 0$.

Reviews from Oct 16 – Analytic continuation

Theorem

Let $U \subset \mathbb{C}$ be a **domain** and $f : U \to \mathbb{C}$ be a holomorphic/analytic function.
If there exists $z_0 \in U$ such that $\forall n \in \mathbb{N}_{\geq 0},\ f^{(n)}(z_0) = 0$ then $f \equiv 0$ on U.

Corollary

Let $U \subset \mathbb{C}$ be a **domain** and $f, g : U \to \mathbb{C}$ be holomorphic/analytic functions.
If f and g coincide in the neighborhood of a point,

$$\text{i.e. } \exists z_0 \in U,\ \exists r > 0,\ \forall z \in D_r(z_0) \cap U,\ f(z) = g(z),$$

then they coincide on U,

$$\text{i.e. } \forall z \in U,\ f(z) = g(z).$$

Isolated zeroes – 1

It is actually possible to strengthen the previous results.

Theorem

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be a holomorphic/analytic function.
Then either $f \equiv 0$ or the zeroes of f are isolated[1]:
if $f(z_0) = 0$ then there exists $r > 0$ such that $D_r(z_0) \subset U$ and $\forall z \in D_r(z_0) \setminus \{z_0\},\ f(z) \neq 0$.

[1] Otherwise stated, if you attend MAT327, either f is constant equal to 0 or $\{z \in U : f(z) = 0\}$ is discrete.

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Proof. Assume that z_0 is a non-isolated zero of f.

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Proof. Assume that z_0 is a non-isolated zero of f.

We know that f admits a power series expansion $f(z) = \displaystyle\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ in a neighborhood of z_0.

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Assume by contradiction that there exists a smallest $n \in \mathbb{N}_{\geq 0}$ such that $a_n \neq 0$ then
$f(z) = (z - z_0)^n g(z)$ where g is holomorphic and $g(z_0) \neq 0$.

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For every $n \in \mathbb{N}_{>0}$, $\exists w_n \in \left(D_{\frac{1}{n}}(z_0) \cap U\right) \setminus \{z_0\},\ f(w_n) = 0$.

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For every $n \in \mathbb{N}_{>0}$, $\exists w_n \in \left(D_{\frac{1}{n}}(z_0) \cap U \right) \setminus \{z_0\},\ f(w_n) = 0$. But then $g(w_n) = 0$ since $w_n \neq z_0$

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For every $n \in \mathbb{N}_{>0}$, $\exists w_n \in \left(D_{\frac{1}{n}}(z_0) \cap U\right) \setminus \{z_0\},\ f(w_n) = 0$. But then $g(w_n) = 0$ since $w_n \neq z_0$
Then, since $w_n \xrightarrow[n \to +\infty]{} z_0$, by continuity $g(z_0) = \lim_{n \to +\infty} g(w_n) = 0$.

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For every $n \in \mathbb{N}_{>0}$, $\exists w_n \in \left(D_{\frac{1}{n}}(z_0) \cap U \right) \setminus \{z_0\},\ f(w_n) = 0$. But then $g(w_n) = 0$ since $w_n \neq z_0$
Then, since $w_n \xrightarrow[n \to +\infty]{} z_0$, by continuity $g(z_0) = \lim_{n \to +\infty} g(w_n) = 0$. Which leads to a contradiction.
Hence $\forall n \in \mathbb{N}_{\geq 0},\ f^{(n)}(z_0) = n! a_n = 0$ and $f \equiv 0$ on U by the theorem on Slide 3. ∎

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Isolated zeroes – 2

Corollary

Let $U \subset \mathbb{C}$ be a domain and $f, g : U \to \mathbb{C}$ be holomorphic/analytic functions.
If $f - g$ admits a non-isolated zero

$$\text{i.e. } \exists z_0 \in U, \forall r > 0, \exists z \in \left(U \cap D_r(z_0)\right) \setminus \{z_0\}, \ f(z) - g(z) = f(z_0) - g(z_0) = 0$$

then f and g coincide on U,

$$\text{i.e. } \forall z \in U, \ f(z) = g(z).$$

Corollary

Let $U \subset \mathbb{C}$ be a domain and $f, g : U \to \mathbb{C}$ be holomorphic/analytic functions.
Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of terms in U which is convergent to $\tilde{z}$ in U and such that $\forall n \in \mathbb{N},\ f(z_n) = 0$.
Then $f \equiv 0$ on U.

Remark

The fact that the limit $\tilde{z} \in U$ is very important.
Indeed, let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be defined by $f(z) = \sin\left(\frac{\pi}{z}\right)$.
Then $f\left(\frac{1}{n}\right) = 0$ but $f \not\equiv 0$ on $\mathbb{C} \setminus \{0\}$.

Hence, it is possible for the zeroes of f to accumulate at a point of the boundary of the domain (including ∞, see for instance $z_n = \pi n$ for $f = \sin$).

Isolated zeroes – 4

Homework

Let $U \subset \mathbb{C}$ be a domain and $f, g : U \to \mathbb{C}$ be holomorphic/analytic on U.
Prove that if $fg \equiv 0$ on U then either $f \equiv 0$ or $g \equiv 0$.

Homework

Let $U = D_1(0)$. Find all the holomorphic functions $f : U \to \mathbb{C}$ satisfying respectively:

1. $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$
2. $f\left(\frac{1}{2n}\right) = f\left(\frac{1}{2n+1}\right) = \frac{1}{n}$