Power Series

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Power series: definition

Definition: power series

A **(complex) power series centered at** $z_0 \in \mathbb{C}$ is a series of the form

$$S(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$$

where $a_n \in \mathbb{C}$.

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Examples

$$\sum_{n\geq 1} \frac{(z-1)^n}{n}, \sum_{n\geq 0} (n^2+1)(z-2i)^n, \sum_{n\geq 0} \frac{z^n}{n!}, \dots$$

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A first natural question is: for which $z \in \mathbb{C}$ is the series $S(z)$ convergent?

Radius of convergence – 1

Theorem: radius of convergence

Given a power series $S(z) = \sum_{n=0}^{+\infty} a_n(z-z_0)^n$ there exists a unique $R \in [0,+\infty]$ such that

$$\forall z \in \mathbb{C}, \begin{cases} |z-z_0| < R \implies S(z) = \sum_{n=0}^{+\infty} a_n(z-z_0)^n \text{ is absolutely convergent} \\ |z-z_0| > R \implies S(z) = \sum_{n=0}^{+\infty} a_n(z-z_0)^n \text{ is divergent} \end{cases}$$

We say that R is the **radius of convergence** of the power series $S(z)$.

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Below is the meaning of R for the extremal values 0 and $+\infty$:

- $R=0$ means that $S(z)$ is convergent if and only if $z=z_0$.
- $R=+\infty$ means that $S(z)$ is absolutely convergent for all $z\in\mathbb{C}$.

Radius of convergence – 2

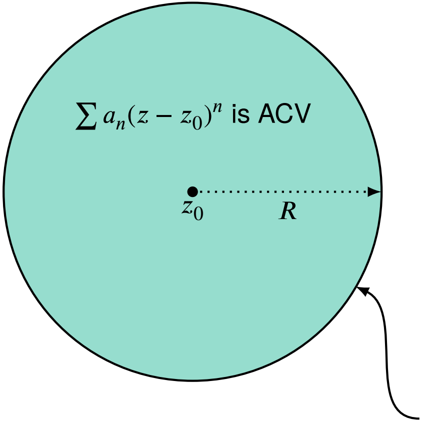
Careful

We can't conclude anything when $|z - z_0| = R$.

Indeed, the radius of convergence of the following power series is $R = 1$, but:

- $\sum_{n=0}^{+\infty} z^n$ is divergent for all $z \in \mathbb{C}$ such that $|z| = 1$.
- $\sum_{n=0}^{+\infty} \frac{z^n}{n}$ is divergent for $z = 1$ but (semi)-convergent for all $z \in \mathbb{C}$ such that $|z| = 1$ and $z \neq 1$.
- $\sum_{n=0}^{+\infty} \frac{z^n}{n^2}$ is absolutely convergent for all $z \in \mathbb{C}$ such that $|z| = 1$.

$\sum a_n(z-z_0)^n$ is DV, actually $(a_n(z-z_0)^n)_n$ is not even bounded.



Circle of indeterminacy

For z in the circle of indeterminacy $|z-z_0| = R$, the power series $S(z)$ may be divergent, semi-convergent or absolutely convergent: we can't conclude...

Radius of convergence – 4

Proof: existence of R for $S(z) = \sum a_n (z - z_0)^n$.

Note that $E = \left\{ r \in [0, +\infty) \ : \ (a_n r^n)_n \text{ is bounded} \right\}$ is non-empty since $0 \in E$.
Hence $R = \sup E$ is well-defined (possibly $+\infty$ if E is unbounded).

- Assume that $|z - z_0| < R$ then there exists ρ such that $|z - z_0| < \rho < R$ and $\exists M > 0, \forall n \in \mathbb{N}, \left| a_n \rho^n \right| < M$.

  Then $\displaystyle\sum_{n \geq 0} \left| a_n (z - z_0)^n \right| = \sum_{n \geq 0} \left| a_n \rho^n \right| \left(\frac{|z - z_0|}{\rho} \right)^n \leq M \sum_{n \geq 0} \left(\frac{|z - z_0|}{\rho} \right)^n$.

  The last series is convergent (geometric series with $\frac{|z - z_0|}{\rho} < 1$).

  Hence $\displaystyle S(z) = \sum_{n \geq 0} a_n (z - z_0)^n$ is absolutely convergent.

- Assume that $|z - z_0| > R$ then $\displaystyle\lim_{n \to +\infty} a_n (z - z_0)^n \neq 0$ since $\left(a_n (z - z_0)^n \right)_n$ is not bounded.

  Hence $\displaystyle S(z) = \sum_{n \geq 0} a_n (z - z_0)^n$ is divergent.

$\blacksquare$

Radius of convergence – 5

Hence we have that the radius of convergence of $S(z) = \sum a_n(z-z_0)^n$ is given by

$$\begin{aligned} R &= \sup\left\{r \in [0,+\infty) \,:\, (a_n r^n)_n \text{ is bounded}\right\} \\ &= \sup\left\{ |z-z_0| \,:\, \sum_{n=0}^{+\infty} a_n(z-z_0)^n \text{ is convergent}\right\} \\ &= \inf\left\{ |z-z_0| \,:\, \sum_{n=0}^{+\infty} a_n(z-z_0)^n \text{ is divergent}\right\} \end{aligned}$$

By a theorem of Hadamard, we also have that $R = \dfrac{1}{\limsup\limits_{n\to+\infty} \sqrt[n]{|a_n|}}$

Two useful methods to compute the radius of convergence

Theorem: d'Alembert ratio test for power series

Let $S(z) = \sum_{n=0}^{+\infty} a_n(z-z_0)^n$ be a power series such that $a_n \neq 0$ for n big enough.

If the limit $\ell := \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists or is $+\infty$ then the radius of convergence of $S(z)$ is $R = \frac{1}{\ell}$.

Theorem: root test for power series

Let $S(z) = \sum_{n=0}^{+\infty} a_n(z-z_0)^n$ be a power series.

If the limit $\ell := \lim_{n\to+\infty} \sqrt[n]{|a_n|}$ exists or is $+\infty$ then the radius of convergence of $S(z)$ is $R = \frac{1}{\ell}$.

In the two above tests we use the convention $\frac{1}{\infty} = 0$ *and* $\frac{1}{0} = \infty$.

Homework: geometric series

We define $S(z) = \sum_{n=n_0}^{+\infty} (z - z_0)^n$.

1. What is the radius of convergence R of $S(z)$?
2. Study $S(z)$ of z such that $|z - z_0| = R$.
3. Find an expression for $S(z)$ when $z \in D_R(z_0)$.

Homework – 1

What are the radii of convergence of the following power series:

1. $\displaystyle\sum_{n=0}^{+\infty} a^n z^n$ (where $a \in \mathbb{C}$)
2. $\displaystyle\sum_{n=0}^{+\infty} e^{-\sqrt{n}} z^n$
3. $\displaystyle\sum_{n=0}^{+\infty} \sin(n) z^n$
4. $\displaystyle\sum_{n=0}^{+\infty} \frac{n^2 - n + 3}{2n^3 + n + \pi} z^n$
5. $\displaystyle\sum_{n=0}^{+\infty} \ln(n!) z^n$
6. $\displaystyle\sum_{n=0}^{+\infty} \binom{5n}{2n} z^n$
7. $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{n^2 + 1} z^{2n}$ (be careful with this one, why?)
8. $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n3^n} z^n$

Hint: for the first three, you can simply use that $R = \sup\left\{ r \in [0, +\infty) \ : \ (a_n r^n)_n \text{ is bounded} \right\}$.

Homework – 2

For question 7 in the previous slide, you can NOT apply the ratio test for power series.

Indeed, $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{n^2+1} z^{2n} = 1 + 0z + \frac{1}{2}z^2 + 0z^3 + \frac{1}{5}z^4 + 0z^5 + \frac{1}{10}z^6 + 0z^7 + \cdots$.

If we want to rewrite $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{n^2+1} z^{2n} = \sum_{n=0}^{+\infty} a_n z^n$ then $a_n = \begin{cases} \frac{1}{(n/2)^2+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$.

Hence, since $a_n = 0$ for n odd, we can't compute $\displaystyle\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

One way is to directly work with the usual ratio test (for series, not power series): if $z \neq 0$ then

$$\lim_{n\to+\infty} \left| \frac{\frac{1}{(n+1)^2+1} z^{2(n+1)}}{\frac{1}{n^2+1} z^{2n}} \right| = \lim_{n\to+\infty} \left| \frac{n^2+1}{(n+1)^2+1} z^2 \right| = |z|^2$$

Hence, by the usual ratio test, the series is absolutely convergent when $|z| < 1$ and divergent when $|z| > 1$. Therefore the radius of convergence is 1.

Sum of two power series – 1

Theorem

Let $S_A = \sum_{n=0}^{+\infty} a_n(z-z_0)^n$ and $S_B = \sum_{n=0}^{+\infty} b_n(z-z_0)^n$ be two power series of radii R_A and R_B.

We define $S_{A+B}(z) \coloneqq \sum_{n=0}^{+\infty}(a_n+b_n)(z-z_0)^n$ and denote its radius of convergence by R_{A+B}.

Then

1. $R_{A+B} \geq \min(R_A, R_B)$.
2. If $R_A \neq R_b$ then $R_{A+B} = \min(R_A, R_B)$.
3. If $|z-z_0| < \min(R_A, R_B)$ then $S_{A+B}(z) = S_A(z) + S_B(z)$,

   i.e. $\sum_{n=0}^{+\infty}(a_n+b_n)(z-z_0)^n = \sum_{n=0}^{+\infty} a_n(z-z_0)^n + \sum_{n=0}^{+\infty} b_n(z-z_0)^n$.

Sum of two power series – 2

Careful

When $R_A = R_B$ we may have $R_{A+B} = R_A = R_B$ or $R_{A+B} > R_A = R_B$.

1. $S_A(z) = \sum_{n=0}^{+\infty} z^n$ and $S_B(z) = \sum_{n=0}^{+\infty} nz^n$ are of radii 1, and $S_{A+B}(z) = \sum_{n=0}^{+\infty}(1+n)z^n$ is of radius 1.
2. $S_A(z) = \sum_{n=0}^{+\infty} z^n$ and $S_B(z) = \sum_{n=0}^{+\infty}(2^{-n}-1)z^n$ are of radii 1, and $S_{A+B}(z) = \sum_{n=0}^{+\infty} 2^{-n}z^n$ is of radius 2.
3. $S_A(z) = \sum_{n=0}^{+\infty} z^n$ and $S_B(z) = \sum_{n=0}^{+\infty} -z^n$ are of radii 1, and $S_{A+B}(z) = \sum_{n=0}^{+\infty} 0z^n$ is of radius $+\infty$.

Product of two power series – 1

Theorem

Let $S_A = \sum_{n=0}^{+\infty} a_n(z-z_0)^n$ and $S_B = \sum_{n=0}^{+\infty} b_n(z-z_0)^n$ be two power series of radii R_A and R_B.

We define $S_{AB}(z) \coloneqq \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)(z-z_0)^n$ and denote its radius of convergence by R_{AB}.

Then

1. $R_{A+B} \geq \min(R_A, R_B)$.
2. If $|z-z_0| < \min(R_A, R_B)$ then $S_{AB}(z) = S_A(z)S_B(z)$,

   i.e. $\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)(z-z_0)^n = \left(\sum_{n=0}^{+\infty} a_n(z-z_0)^n\right)\left(\sum_{n=0}^{+\infty} b_n(z-z_0)^n\right)$.

Product of two power series – 1

Theorem

Let $S_A = \sum_{n=0}^{+\infty} a_n(z-z_0)^n$ and $S_B = \sum_{n=0}^{+\infty} b_n(z-z_0)^n$ be two power series of radii R_A and R_B.

We define $S_{AB}(z) := \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z-z_0)^n$ and denote its radius of convergence by R_{AB}.

Then

1. $R_{A+B} \geq \min(R_A, R_B)$.
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$$\text{i.e. } \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) (z-z_0)^n = \left(\sum_{n=0}^{+\infty} a_n(z-z_0)^n \right) \left(\sum_{n=0}^{+\infty} b_n(z-z_0)^n \right).$$

Careful

$S_{AB}(z) \neq \sum_{n=0}^{+\infty} (a_n b_n)(z-z_0)^n$: we need to use the *Cauchy product* as defined above!

Product of two power series – 2

Careful

We may have $R_{AB} = \min(R_A, R_B)$ or $R_{AB} > \min(R_A, R_B)$, even when $R_A \neq R_B$.

$S_A(z) = \sum_{n=0}^{+\infty} z^n$ is of radius 1 and $S_B(z) = 1 - z$ is of radius $+\infty$ but $S_{AB}(z) = 1$ is of radius $+\infty$.

In this example, intuitively: $\frac{1}{1-z}(1-z) = 1$

Differentiability of power series – 1

Theorem

Let $S(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$ be a power series of radius of convergence R.
Then S is holomorphic/analytic[1]/$\mathbb{C}$-differentiable on $D_R(z_0)$ and moreover

$$\forall z \in D_R(z_0),\ S'(z) = \sum_{n=1}^{+\infty} n a_n (z - z_0)^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} (z - z_0)^n$$

whose radius of convergence is also R.

[1] I really don't like the choice in the textbook...

Differentiability of power series – 2

The previous result contains several things:

1. The power series $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$ and $\sum_{n=1}^{+\infty} na_n(z-z_0)^{n-1}$ have same radius of convergence,
2. The function $S : D_R(z_0) \to \mathbb{C}$ defined by $S(z) = \sum_{n=0}^{+\infty} a_n(z-z_0)^n$ is holomorphic on $D_R(z_0)$,
3. For all $z \in D_R(z_0)$, $S'(z) = \sum_{n=1}^{+\infty} na_n(z-z_0)^{n-1}$.

Corollary

Let $S(z) = \sum_{n=0}^{+\infty} a_n(z - z_0)^n$ be a power series of radius of convergence R.
Then S is infinitely many times $\mathbb{C}$-differentiable on $D_R(z_0)$ and moreover

$$\forall z \in D_R(z_0),\ S^{(k)}(z) = \sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} a_n(z - z_0)^{n-k}$$

whose radius of convergence is also R.

Differentiability of power series – 3

Corollary

Let $S(z) = \sum_{n=0}^{+\infty} a_n(z-z_0)^n$ be a power series of radius of convergence R.
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$$\forall z \in D_R(z_0),\ S^{(k)}(z) = \sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} a_n (z-z_0)^{n-k}$$

whose radius of convergence is also R.

(Important) Corollary

If $S(z) = \sum_{n=0}^{+\infty} a_n(z-z_0)^n$ is a power series with radius of convergence $R > 0$, then $a_n = \dfrac{S^{(n)}(z_0)}{n!}$.

Differentiability of power series – 4

Example: power series expansion of the complex exponential

1. The radius of convergence of $S(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$ is $+\infty$.

   Indeed, we may conclude with the ratio test since $\lim_{n\to+\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n\to+\infty} \frac{1}{n+1} = 0$.

2. For all $z \in \mathbb{C}$, $S'(z) = S(z)$.

   Indeed, $S'(z) = \sum_{n=1}^{+\infty} n\frac{z^{n-1}}{n!} = \sum_{n=1}^{+\infty} \frac{z^{n-1}}{(n-1)!} = S(z)$.

3. For all $z \in \mathbb{C}$, $e^z = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$.

   Indeed, $\frac{\partial}{\partial z}\left(e^{-z}S(z)\right) = -e^{-z}S(z) + e^{-z}S'(z) = -e^{-z}S(z) + e^{-z}S(z) = 0$.
   Thence $e^{-z}S(z) = \lambda$ is constant since $\mathbb{C}$ is connected.
   We evaluate at $z = 0$ to obtain that $\lambda = 1$, so that $S(z) = e^z$.

Example: power series expansion of the complex cosine

For $z \in \mathbb{C}$, we have

$$\begin{aligned}\cos(z) &= \frac{e^{iz}+e^{-iz}}{2}\\ &= \frac{1}{2}\left(\sum_{n=0}^{+\infty}\frac{(iz)^n}{n!}+\sum_{n=0}^{+\infty}\frac{(iz)^n}{n!}\right)\\ &= \sum_{n=0}^{+\infty}(-1)^n\frac{z^{2n}}{(2n)!}\end{aligned}$$