Complex Variables

Functions, sequences and series

UNIVERSITY OF TORONTO

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Functions: generalities – 1

Definition (informal): function

A ***function*** (or ***map***) is the data of two sets A and B together with a ”process” that associates to each $x \in A$ a unique $f(x) \in B$:

$$f : \begin{cases} A & \rightarrow & B \\ x & \mapsto & f(x) \end{cases}$$

Here: f is the name of the function, A is the ***domain***, and B is the ***codomain***.

Beware

The domain and codomain are part of the definition of a function:
$\sin : \mathbb{R} \rightarrow \mathbb{R}$ and $\sin : [-\pi, \pi] \rightarrow \mathbb{R}$ are not the same function.

A function is not simply a ”formula”.
Nonetheless, if you have a question like ”what's the domain of $f(z) = \frac{1}{1+z}$?”, it is implied that you are asked to give the greatest domain such that this formula makes sense (here: $\mathbb{C} \setminus \{-1\}$).

Functions: generalities – 2

We fix a function $f : A \to B$.

- The ***image of*** $E \subset A$ ***by*** f is $f(E) \coloneqq \{f(x) : x \in E\} \subset B$.
- The ***image of f*** (or ***range of*** f) is $\mathrm{Range}(f) \coloneqq f(A)$.
- The ***preimage of*** $F \subset B$ ***by*** f is $f^{-1}(F) \coloneqq \{x \in A : f(x) \in F\}$.
- The ***graph of*** f is the set $\Gamma_f \coloneqq \{(x, y) \in A \times B : y = f(x)\}$.

The graph of a function $f : \mathbb{C} \to \mathbb{C}$ is quite difficult to visualize: it is a 2-dimensional object lying in a 4-dimensional space.
Instead, it can be useful to visualize separately the graphs of the real and imaginary parts. Or to study how the function transforms some subsets.

Functions: generalities – 3

We fix a function $f : A \to B$.

- We say that f is ***injective*** (or ***one-to-one***) if $\forall x_1, x_2 \in A,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$ or equivalently by taking the contrapositive $\forall x_1, x_2 \in A,\ f(x_1) = f(x_2) \implies x_1 = x_2$

Figure: Injective

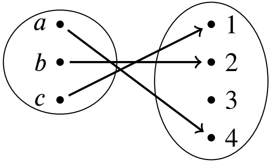
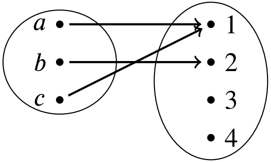


Figure: Not injective



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- We say that f is ***surjective*** (or ***onto***) if $\forall y \in B, \exists x \in A,\ y = f(x)$

Figure: Surjective

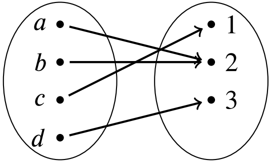
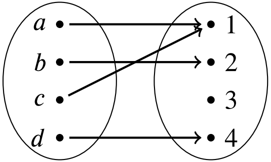


Figure: Not surjective



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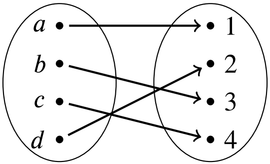
- We say that f is ***surjective*** (or ***onto***) if $\forall y \in B, \exists x \in A,\ y = f(x)$
- We say that f is ***bijective*** if it is injective and surjective, i.e. $\forall y \in B, \exists! x \in A,\ y = f(x)$

Figure: Bijective



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Proposition

$f : A \to B$ is bijective if and only if there exists $g : B \to A$ such that $\begin{cases} \forall x \in A,\ g(f(x)) = x \\ \forall y \in B,\ f(g(y)) = y \end{cases}$.
Then g is unique, it is called the ***inverse of*** f and denoted by $f^{-1} : B \to A$.

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Figure: Bijective function

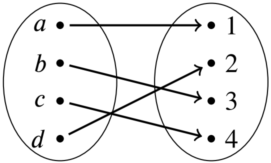
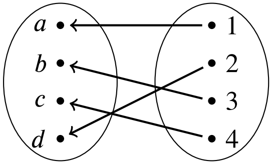


Figure: Its inverse



Limits of complex functions – 1

Definition: limit point

Let $S \subset \mathbb{C}$. We say that $z_0 \in \mathbb{C}$ is a ***limit point of S*** if

$$\forall \delta > 0,\ \exists z \in S,\ 0 < |z - z_0| < \delta$$

or equivalently

$$\forall \delta > 0,\ \left(D_\delta(z_0) \cap S\right) \setminus \{z_0\} \neq \varnothing$$

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Definition: limit at z_0

Let $S \subset \mathbb{C}$, $f : S \to \mathbb{C}$, z_0 be a limit point of S and $\ell \in \mathbb{C}$.
We say that f ***tends to ℓ as z tends to z_0***, denoted by $\lim_{z \to z_0} f(z) = \ell$, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall z \in S, 0 < |z - z_0| < \delta \implies |f(z) - \ell| < \varepsilon.$$

Limits of complex functions – 2

Definition: limit at ∞

Let $S \subset \mathbb{C}$ be unbounded, $f : S \to \mathbb{C}$ and $\ell \in \mathbb{C}$. We say that f ***tends to*** ℓ ***as*** z ***tends to*** ∞, denoted by $\lim_{z\to\infty} f(z) = \ell$, if

$$\forall \varepsilon > 0, \exists M > 0, \forall z \in S, |z| > M \implies |f(z) - \ell| < \varepsilon$$

Examples

$$\lim_{z\to\infty} \frac{1}{z} = 0$$

$$\lim_{z\to 0} \frac{z}{\bar{z}} \text{ DNE}$$

$$\lim_{z\to\infty} \frac{z}{\bar{z}} \text{ DNE}$$

Limits of complex functions – 3

Proposition (here $z_0 \in \mathbb{C} \sqcup \{\infty\}$)

$$\lim_{z\to z_0} f(z) = L \Leftrightarrow \begin{cases} \lim\limits_{z\to z_0} \Re(f) = \Re(L) \\ \lim\limits_{z\to z_0} \Im(f) = \Im(L) \end{cases}$$

Proposition

Assume that $\lim\limits_{z\to z_0} f(z) = L$, $\lim\limits_{z\to z_0} g(z) = M$, $\alpha, \beta \in \mathbb{C}$ (here $z_0 \in \mathbb{C} \sqcup \{\infty\}$). Then

- $\lim\limits_{z\to z_0} (\alpha f + \beta g) = \alpha L + \beta M$
- $\lim\limits_{z\to z_0} fg = LM$
- $\lim\limits_{z\to z_0} \overline{f} = \overline{L}$
- $\lim\limits_{z\to z_0} |f| = |L|$
- If $M \neq 0$, $\lim\limits_{z\to z_0} \dfrac{f}{g} = \dfrac{L}{M}$

Continuity – 1

Definition

Let $S \subset \mathbb{C}$, $f : S \to \mathbb{C}$ and $z_0 \in S$.
We say that f ***is continuous at*** z_0 if
$$\forall \varepsilon > 0, \exists \delta > 0, \forall z \in S, |z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon.$$
We say that f is ***continuous*** if f is continuous at every $z_0 \in S$.

Continuity – 1

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We say that f is ***continuous*** if f is continuous at every $z_0 \in S$.

If $z_0 \in S$ is a limit point then f is continuous at z_0 if and only if $\lim\limits_{z \to z_0} f(z) = f(z_0)$

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If $z_0 \in S$ is a limit point then f is continuous at z_0 if and only if $\lim\limits_{z \to z_0} f(z) = f(z_0)$

Theorem

Let $S \subset \mathbb{C}$, $f : S \to \mathbb{C}$. The following are equivalent:

1. f is continuous,
2. $\forall \mathscr{U} \subset \mathbb{C}$ open, $\exists \mathscr{V} \subset \mathbb{C}$ open, $f^{-1}(\mathscr{U}) = \mathscr{V} \cap S$,
3. $\forall \mathscr{C} \subset \mathbb{C}$ closed, $\exists \mathscr{D} \subset \mathbb{C}$ closed, $f^{-1}(\mathscr{C}) = \mathscr{D} \cap S$.

Continuity – 2

Proposition

f is continuous at z_0 if and only if $\Re(f)$ and $\Im(f)$ are.

Proposition

If f and g are continuous at z_0, so are:

- $\alpha f + \beta g$ (here $\alpha, \beta \in \mathbb{C}$)
- fg
- $\overline{f}$
- $|f|$
- $\dfrac{f}{g}$ (assuming that $g(z_0) \neq 0$).

Sequences – 1

Definition

We say that a sequence of complex numbers $(z_k)_{k\in\mathbb{N}}$ ***tends to*** $\ell \in \mathbb{C}$ ***as*** n ***tends to*** $+\infty$, denoted by $\lim\limits_{k\to+\infty} z_k = \ell$, if

$$\forall \varepsilon > 0,\ \exists K \in \mathbb{N},\ \forall k \in \mathbb{N},\ k \geq K \implies |z_k - \ell| < \varepsilon$$

Proposition

$$\lim_{k\to+\infty} z_k = \ell \Leftrightarrow \lim_{k\to+\infty} |z_k - \ell| = 0$$

Proposition

$$\lim_{k\to+\infty} z_k = \ell \Leftrightarrow \begin{cases} \lim\limits_{k\to+\infty} \Re(z_k) = \Re(\ell) \\ \lim\limits_{k\to+\infty} \Im(z_k) = \Im(\ell) \end{cases}$$

Sequences – 2

Proposition

Assume that $\lim\limits_{k\to+\infty} z_k = L$ and $\lim\limits_{k\to+\infty} w_k = M$ then

- $\lim\limits_{k\to+\infty} (z_k + w_k) = L + M$
- $\lim\limits_{k\to+\infty} z_k w_k = LM$
- $\lim\limits_{k\to+\infty} \frac{z_k}{w_k} = \frac{L}{M}$ (assuming $M \neq 0$)
- $\lim\limits_{k\to+\infty} \overline{z_k} = \overline{L}$
- $\lim\limits_{k\to+\infty} |z_k| = |L|$

Series – 1

Definitions

Given a sequence $(z_k)_{k\in\mathbb{N}}$:

- We define the ***n-th partial sum associated to*** $(z_k)_{k\in\mathbb{N}}$ by $S_n = \sum_{k=0}^{n} z_k$,
- We say that the series $\sum_{k=0}^{+\infty} z_k$ is ***convergent*** if $\lim_{n\to+\infty} S_n$ exists then we set $\sum_{k=0}^{+\infty} z_k := \lim_{n\to+\infty} \sum_{k=0}^{n} z_k$,
- Otherwise, we say that the series $\sum_{k=0}^{+\infty} z_k$ is ***divergent***.

Convergence doesn't depend on the starting index: $\sum_{k=0}^{+\infty} z_k$ is convergent $\Leftrightarrow \sum_{k=k_0}^{+\infty} z_k$ is convergent.

But the value of the series DOES depend on the starting index.

Series – 2

Proposition

If $\sum_{k=0}^{+\infty} z_k$ is convergent then $\lim_{k\to+\infty} z_k = 0$.

- The contrapositive can be very useful: if $\lim_{k\to+\infty} z_k \neq 0$ then $\sum_{k=0}^{+\infty} z_k$ is divergent.
- The converse is false: $\sum_{k=1}^{+\infty} \frac{1}{k}$ is divergent.

Homework: geometric series

Study the series $\sum_{k=k_0}^{+\infty} z^k$.

Series – 3

Definition: absolute convergence

We say that the series $\sum_{k=0}^{+\infty} z_k$ is ***absolutely convergent*** if $\sum_{k=0}^{+\infty} |z_k|$ is convergent.

Theorem

If $\sum_{k=0}^{+\infty} z_k$ is ***absolutely convergent*** then $\sum_{k=0}^{+\infty} z_k$ is ***convergent***.

Beware

The converse is false: $\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$ is convergent but not absolutely convergent.

Theorem: d'Alembert's ratio test

We consider a series $\sum_{k=0}^{+\infty} z_k$. We assume that $\ell := \lim_{k\to+\infty} \left|\frac{z_{k+1}}{z_k}\right|$ exists, then

- If $\ell < 1$ then $\sum_{k=0}^{+\infty} z_k$ is absolutely convergent.
- If $\ell > 1$ then $\sum_{k=0}^{+\infty} z_k$ is divergent.

We can't conclude when $\ell = 1$:

- $\sum_{k=0}^{+\infty} 1$ is divergent.
- $\sum_{k=1}^{+\infty} \frac{(-1)^k}{k^2}$ is absolutely convergent.
- $\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$ is convergent but not absolutely convergent.