Topology of the complex plane

UNIVERSITY OF TORONTO

September 18^{th}, 2020

Today's topic: topology of $\mathbb{C}$
After having tried to convince you that a line is a circle, today I will divide by 0…

LEC0101 website: `http://uoft.me/MAT334-LEC0101`

How to practice for MAT334:

- Immediate practice questions from the slides.
- "Problems_to_…" from Quercus.
- Tutorial problems.

Make sure that you have read the Outline/Syllabus and *readme* pages posted on Quercus, they contain valuable information such as:
Each Quiz is drawn from the problems for a week (or weeks) in the Quiz description.
(Problems_to_…).

Beware if you look online for *inversion*: it can mean the complex inversion $z \mapsto z^{-1}$ (we use this one) but also the geometric inversion, which in terms of complex numbers is given by $z \mapsto \overline{z^{-1}}$, do NOT confuse them!

Disks, neighborhoods and bounded sets

Open disk

The *open disk centered at* $z_0 \in \mathbb{C}$ *and of radius* $r \in \mathbb{R}_{>0}$ is $D_r(z_0) := \left\{ z \in \mathbb{C} \;:\; |z - z_0| < r \right\}$.
(You may also see the name r*-vicinity of* z_0*.)*

Closed disk

The *closed disk centered at* $z_0 \in \mathbb{C}$ *and of radius* $r \in \mathbb{R}_{>0}$ is $\overline{D_r}(z_0) := \left\{ z \in \mathbb{C} \;:\; |z - z_0| \leq r \right\}$.

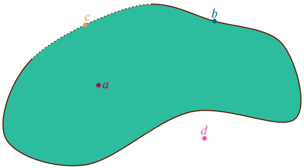
Neighborhood

We say that $S \subset \mathbb{C}$ is a *neighborhood* of $z_0 \in \mathbb{C}$ if there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $D_\varepsilon(z_0) \subset S$.

Bounded sets

We say that $S \subset \mathbb{C}$ is *bounded* if there exists $r \in \mathbb{R}_{>0}$ such that $S \subset D_r(0)$.

Interior, closure and boundary



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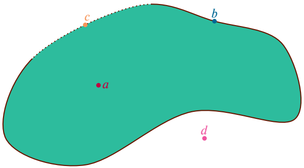

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Definition: interior

The *interior* of $S \subset \mathbb{C}$ is $\mathring{S} \coloneqq \left\{ z \in \mathbb{C} \ : \ \exists \varepsilon > 0,\ D_\varepsilon(z) \subset S \right\}$ (also denoted S^{int}).

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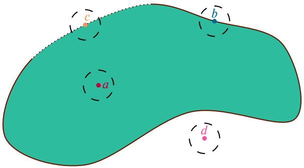

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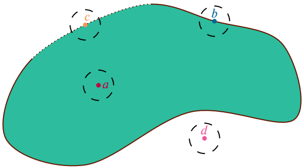

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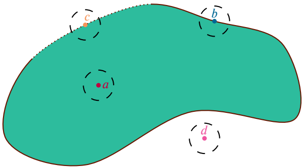

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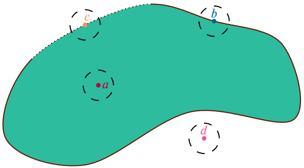

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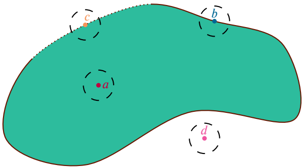

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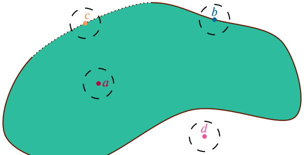

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Properties

1. $\mathring{S} \subset S \subset \overline{S}$
2. $\overline{S} = S \cup \partial S$
3. $\mathring{S} \cap \partial S = \varnothing$
4. $\partial S = \left\{ z \in \mathbb{C} \ : \ \forall \varepsilon > 0,\ D_\varepsilon(z) \cap S \neq \varnothing \text{ and } D_\varepsilon(z) \cap S^c \neq \varnothing \right\}$
5. $\partial \left(S^c \right) = \partial S$

Open and closed sets

Definition: Open sets

We say that $S \subset \mathbb{C}$ is *open* if $\mathring{S} = S$.

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Examples

- $\{z \in \mathbb{C} \ : \ |z| < 1\}$ is open not closed.
- $\{z \in \mathbb{C} \ : \ |z| \leq 1\}$ is closed not open.
- $\mathbb{C}$ is both open and closed.
- $\{z \in \mathbb{C} \ : \ \Re(z) = 0, \ \Im(z) > 0\}$ is neither open nor closed.

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Properties

1. S is open if and only if $S \cap \partial S = \varnothing$
2. S is open if and only if it is a neighborhood of each of its elements,
   i.e. $\forall z \in S,\ \exists \varepsilon > 0,\ D_\varepsilon(z) \subset S$
3. S is closed if and only if $\partial S \subset S$
4. S is closed if and only if S^c is open
5. Open sets are stable by unions and finite intersections.
6. Closed sets are stable by intersections and finite unions.

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Homework

Are the following sets open? closed?

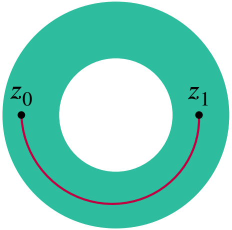
1. $\{z \in \mathbb{C} \ : \ \Re(z) \geq 0\}$
2. $\{z \in \mathbb{C} \ : \ \Re(z) > 0\}$
3. $\varnothing$
4. $\{z \in \mathbb{C} \ : \ |z| = 1\}$

Connectedness – 1

Definition: Path-connectedness

A subset $S \subset \mathbb{C}$ is *path-connected* if for any $z_0, z_1 \in S$ there exists $\gamma : [0,1] \to \mathbb{C}$ continuous such that 1 $\forall t \in [0,1],\ \gamma(t) \in S$ 2 $\gamma(0) = z_0$ 3 $\gamma(1) = z_1$.

Intuitively, it means that S is made of only one piece.



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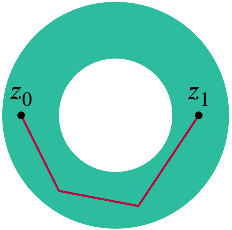
Theorem

An **open** subset $S \subset \mathbb{C}$ is path-connected if and only if for any $z_0, z_1 \in S$ there exists a polygonal curve from z_0 to z_1 which is included in S,
i.e. there exists $w_0, \dots, w_k$ such that $[w_i, w_{i+1}] \subset S$, $w_0 = z_0$ and $w_k = z_1$.



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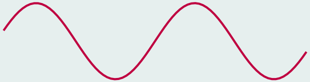
Beware

The openness assumption is very important in the previous theorem.
Indeed, the following set is path-connected but two point on it can't be joined by a polygonal curve staying in the curve.



Connectedness – 2

Definition: Connectedness

We say that an **open** subset $S \subset \mathbb{C}$ is *connected* if it is path-connected.

Remark

We defined connectedness only for open sets: there exists a more general notion of connectedness but it coincides with path-connectedness for open sets.

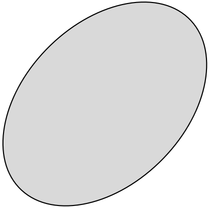
Definition: Domain

We say that a subset $D \subset \mathbb{C}$ is a *domain* if it is open and connected.

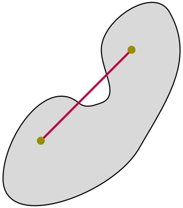
Convex sets and star-shaped sets

Definition: Convex sets

We say that $S \subset \mathbb{C}$ is *convex* if $\forall z_0, z_1 \in S$, $\forall t \in [0,1]$, $(1-t)z_0 + tz_1 \in S$.



(a) Convex set



(b) Non-convex set

Convex sets and star-shaped sets

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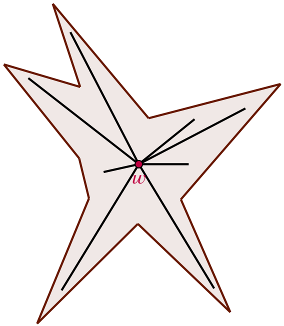
Definition: Star-shaped sets

We say that $S \subset \mathbb{C}$ is *star-shaped* if $\exists w \in S$, $\forall z \in S$, $\forall t \in [0,1]$, $(1-t)w + tz \in S$.



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Proposition

Convex and non-empty $\implies$ star-shaped $\implies$ path-connected.

Beware: $\varnothing$ is convex but not star-shaped.

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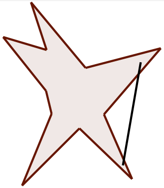
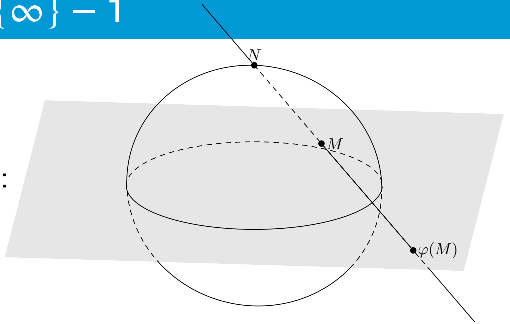


We set $S^2 := \left\{(r,s,t) \in \mathbb{R}^3 \ : \ r^2+s^2+t^2=1\right\}$
and $N=(0,0,1)$ (the *north pole* of S^2).
We identify $\mathbb{C}$ with the equatorial plane $P=\{t=0\}$.
We define the stereographic projection with respect to N:

$$\varphi : \begin{cases} S^2 \setminus \{N\} & \to & P \\ M & \mapsto & (NM) \cap P \end{cases}$$



The extended complex plane: $\hat{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$ – 1

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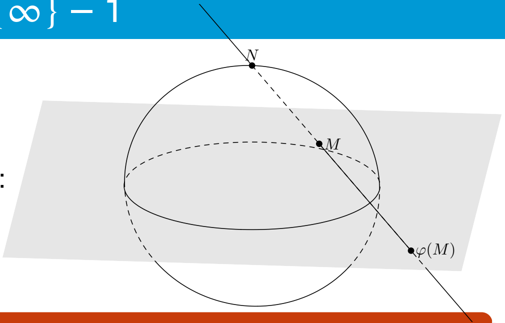
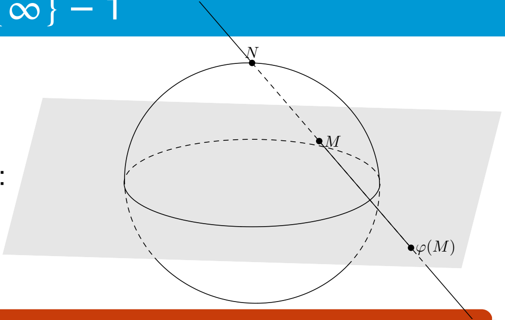


Theorem

φ is a bijection.



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φ is a bijection.

It allows to identify $\mathbb{C}$ with $S^2 \setminus \{N\}$ and then to see N as *the point at infinity*, i.e. to identify S^2 with $\widehat{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$.

There are several models for $\widehat{\mathbb{C}}$, this one is called the ***Riemann sphere****.*

The extended complex plane: $\hat{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$ – 2

Definition: Neighborhood of the ∞

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Remember that a set is open if and only if it is a neighborhood of each of its points. Hence, we defined a topology on $\hat{\mathbb{C}}$. It makes $\varphi : S^2 \to \hat{\mathbb{C}}$ a homeomorphism.

Definition: Open sets of $\hat{\mathbb{C}}$

A subset $S \subset \hat{\mathbb{C}}$ is open if

- $S \subset \mathbb{C}$ is open or
- $S = \{\infty\} \cup U$ where $U = K^c \subset \mathbb{C}$ is the complement of a compact $K \subset \mathbb{C}$ (closed and bounded).

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*The Riemann sphere is a special case of a general topological construction called the **one-point compactification** or the **Alexandrov compactification**.*

The extended complex plane: $\hat{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$ – 3

We may extend the inversion[a] to $\hat{\mathbb{C}}$ by $\operatorname{inv} : \begin{cases} \hat{\mathbb{C}} & \to & \hat{\mathbb{C}} \\ z & \mapsto & z^{-1} & \text{if } z \in \mathbb{C} \setminus \{0\} \\ 0 & \mapsto & \infty \\ \infty & \mapsto & 0 \end{cases}$

[a] Actually, it is possible to define division by 0, what is **not** possible is to define a multiplicative inverse of 0.

Remember from last time the generalized circle (or line-circle) equation

$$a\bar{z}z - \bar{\eta}z - \eta\bar{z} + k = 0 \tag{1}$$

where $a, k \in \mathbb{R}$ and $\eta \in \mathbb{C}$ satisfy $|\eta|^2 - ak > 0$.

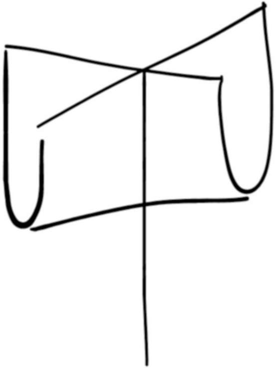
Set $w = z^{-1}$ (so that we swap 0 and ∞ in $\hat{\mathbb{C}}$).
Then (1) becomes

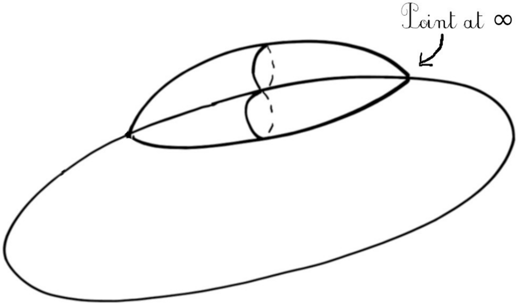
$$k\bar{w}w - \bar{\alpha}w - \alpha\bar{w} + a = 0$$

where $\alpha = \bar{\eta}$.

Not part of MAT334, just a gift 🎂:
another example of one-point/Alexandrov compactification

$x^2 - zy^2 = 0 \subset \mathbb{R}^3$





The drawing on the left is probably my favourite real algebraic set (Whitney umbrella) and the one on the right is its one-point/Alexandrov compactification.