Geometry in the complex plane

September 16th, 2020

Today's topic: Geometry in the complex plane

I'll try to convince you that a line is a circle…

How to practice for MAT334:

- Immediate practice questions from the slides.
- "Problems_to_…" from Quercus.
- Tutorial problems.

Remember that **practice makes perfect.**[1]

Make sure that you have read the Outline/Syllabus and *readme* pages posted on Quercus, they contain valuable information such as:
Each Quiz is drawn from the problems for a week (or weeks) in the Quiz description. (Problems_to_…).

[1] Or in French: *C'est en forgeant qu'on devient forgeron.*

Dot product and norm

Dot product and norm

Dot product

For $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we have

$$\Re\left(\overline{z_1}z_2\right) = \Re\left(z_1\overline{z_2}\right) = x_1x_2 + y_1y_2 = \left(x_1, y_1\right) \cdot \left(x_2, y_2\right)$$

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Norm

For $z = x + iy$, we have $|z| = \sqrt{x^2 + y^2} = \|(x, y)\|$.

Straight lines

Equation of a straight line – 1

Lines admit equations in z of the form

$$\Re\left(\overline{w}z\right) = k$$

where $w \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{R}$.

Straight lines

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$$\mathfrak{R}\left(\overline{w}z\right) = k$$

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Equation of a straight line – 2

Lines admit equations in z of the form

$$\overline{w}z + w\overline{z} = r$$

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In both cases $\omega = a + ib$ corresponds to the coordinates of a vector (a, b) normal to the line.
Note that $r = 2k$ if you pass from one equation to the other one with the same w.

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Homework

Give a complex equation of the line through 1 and i.

Circles

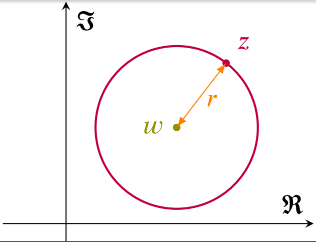
Equation of a circle

The circle centered at $w \in \mathbb{C}$ and of radius $r \in \mathbb{R}_{>0}$ is described by

$$|z - w| = r$$

or equivalently

$$z\bar{z} - \overline{w}z - w\bar{z} = r^2 - |w|^2$$



Homework

Give a complex equation of the circle centered at $1 + 2i$ and passing through i.

Generalized circles (or the *circle-line equation*)

Circle-line equation

The equation

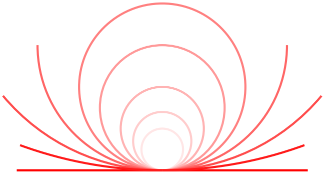
$$a\bar{z}z - \bar{\eta}z - \eta\bar{z} + k = 0$$

where $a, k \in \mathbb{R}$ and $\eta \in \mathbb{C}$ satisfy $|\eta|^2 - ak > 0$ describes

- A line if $a = 0$,
- A circle if $a \neq 0$.

Conversely, any circle or line admits an equation of this form.

Intuitively, a line is a *degenerate circle*, see for instance the following family of circles centered at $0 + ir$ and radius r when r goes to $+\infty$.



This statement will be made precise next lecture.

Apollonius' circles

Apollonius' circles associated to $w_1, w_2 \in \mathbb{C}$

The equation $\quad |z - w_1| = \rho|z - w_2| \quad$ where $w_1, w_2 \in \mathbb{C}$ and $\rho \in \mathbb{R}_{>0}$ describes

- A circle if $\rho \neq 1$,
- A line if $\rho = 1$ (the bisector of the segment line from w_1 to w_2).

Apollonius of Perga (c.240 BC – c.190 BC) proposed to define a circle as a set of points M in the plane for which the ratio MA/MB is constant (for A and B two given points).

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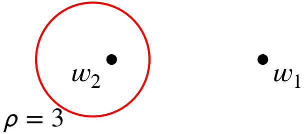

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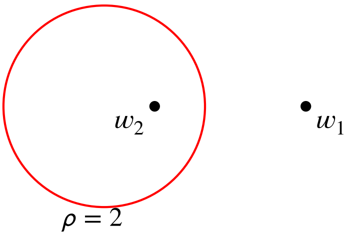

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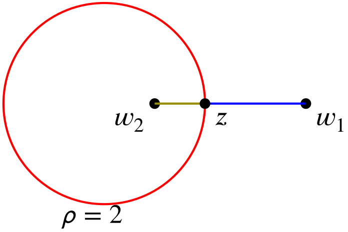

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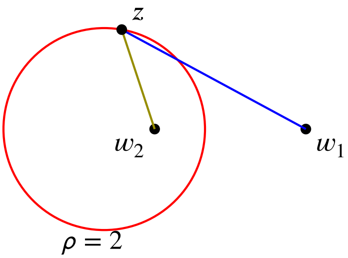

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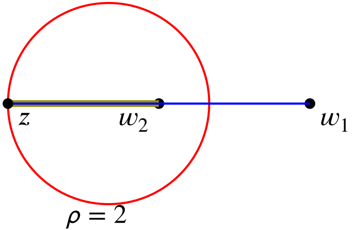

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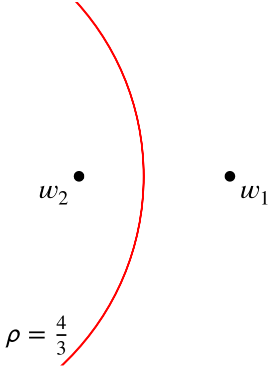

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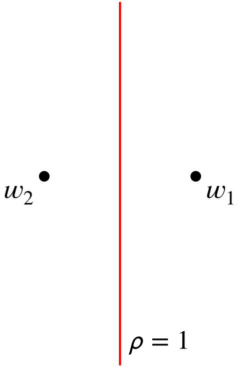

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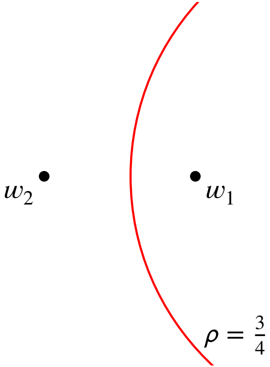

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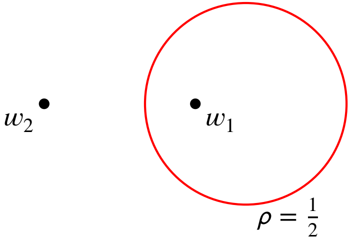

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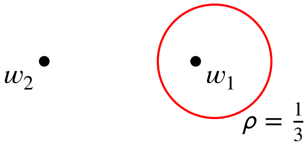

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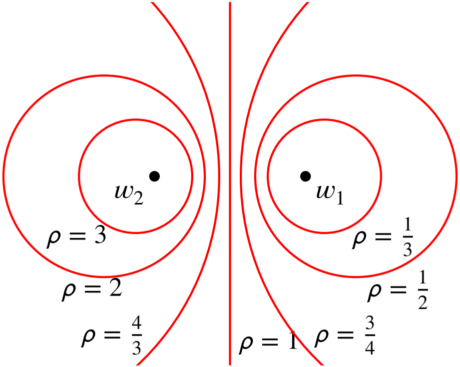

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Homework

Study the set of points z such that:

1. $|z - i| = |z - 1|$
2. $|z - i| = 2|z - 1|$

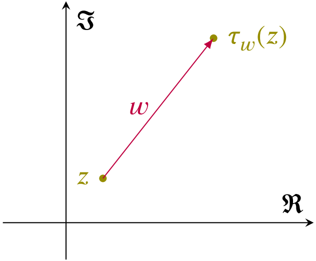
Homework

Read pp.18–20.

Transformations: translation

Translation by $w \in \mathbb{C}$

$$\tau_w : \left\{ \begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ z & \mapsto & z + w \end{array} \right.$$

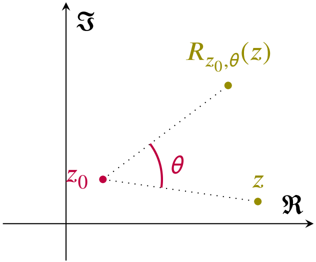


A translation maps lines to lines and circles to circles.

Transformations: rotations

Rotations centered at $z_0 \in \mathbb{C}$ with angle $\theta \in \mathbb{R}$

$$Rz_0, \theta : \begin{cases} \mathbb{C} & \to & \mathbb{C} \\ z & \mapsto & (z - z_0)e^{i\theta} + z_0 \end{cases}$$

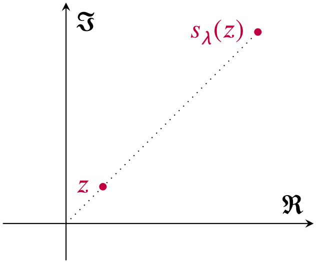


A rotation maps lines to lines and circles to circles.

Transformations: scaling

Scaling by $\lambda \in \mathbb{R}_{>0}$

$$s_\lambda : \left\{ \begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ z & \mapsto & \lambda z \end{array} \right.$$

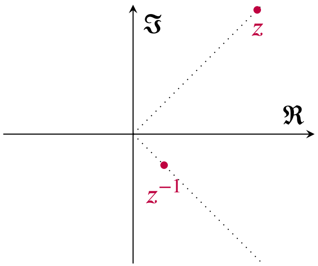


A scaling maps lines to lines and circles to circles.

Transformations: inversion[2].

Inversion

$$\mathrm{inv} : \begin{cases} \mathbb{C} \setminus \{0\} & \to & \mathbb{C} \setminus \{0\} \\ z & \mapsto & z^{-1} \end{cases}$$



$$\left|z^{-1}\right| = |z|^{-1}$$
$$\arg\left(z^{-1}\right) \equiv -\arg(z) \mod 2\pi$$

The inversion maps {lines, circles} to {lines, circles}
Beware: it may maps a line to a circle and vice-versa.

[2] It shouldn't be confused with the *geometric inversion* $z \mapsto (\bar{z})^{-1}$

Transformations: inversion[2].

Inversion

$$\mathrm{inv}:\left\{\begin{array}{ccc}\mathbb{C}\setminus\{0\} & \to & \mathbb{C}\setminus\{0\}\\ z & \mapsto & z^{-1}\end{array}\right.$$

Homework

1. What's the image of a line passing through the origin?
2. What's the image of a circle centered at the origin?
3. What's the image of a circle passing through the origin?
4. What's the image of a line not passing through the origin?

[2] It shouldn't be confused with the *geometric inversion* $z \mapsto (\overline{z})^{-1}$