MAT334H1-F – LEC0101 Complex Variables

WELCOME TO MAT334!



September 11th, 2020

Information about this section – 1



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Please start the subject with "MAT334:"

Lectures schedule:

- Monday, 10am to 11am
- Wednesday, 10am to 11am
- Friday, 10am to 11am

Office hours:

- Monday, 11am to 12pm (Online via Zoom)
- Friday, 11am to 12pm (Online via Zoom)

Website for this section:



http://uoft.me/MAT334-LEC0101

The course will take place online via Zoom. I will send you a message through Quercus if the credentials change.

I am not going to record my lectures, nonetheless I will post my slides and my notes on my webpage. Lectures from Section LEC5101 (Victor Ivrii) will be recorded.

I will probably need a few lectures in order to become comfortable with online lectures, so I apologize in advance if the first lectures are not smooth and for the technical issues we will probably face at the beginning...

Information about the course - 1

Coordinator: Victor Ivrii

ivrii@math.toronto.edu



 \bowtie

Textbook: *Complex Variables*, 2nd Edition (Dover Books on Mathematics) by Stephen D. Fisher. Chapters 1, 2 and 3.



Quercus is the main source of information for the course: Annoucements, Discussions, Syllabus/Outline (read it)...

A Make sure that you are enrolled in a tutorial!

- There will be 4 *short* tests: Oct 15, Oct 29, Nov 26, Dec 3.
- There will be 7 quizzes of 20min each: they will take place during the last 20minutes of a lecture, the planning is available on Quercus.
- You will find more details about the marking scheme on Quercus.

This course is about functions of a complex variable, i.e. of the form

$$f: \begin{array}{ccc} D & \to & \mathbb{C} \\ z & \mapsto & f(z) \end{array}$$

where $D \subset \mathbb{C}$.

More precisely, we will focus about C-differentiability of such a function.

The definition is going to be quite similar to the one you are used to from *calculus*, but \mathbb{C} -differentiable functions will behave quite differently from \mathbb{R} -differentiable functions.

Before going further, let's have a look at some examples from *calculus over the reals* to highlight these differences.

• Let
$$f : \mathbb{R} \to \mathbb{R}$$
 be defined by $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0\\ 0 & \text{otherwise} \end{cases}$

f is differentiable but not C^1 .

~

1...

• $\sin : \mathbb{R} \to \mathbb{R}$ is differentiable, even analytic, it is non-constant and admits local maxima.

h is differentiable, C^{∞} , it vanishes on the open set $(-\infty, 0)$ but is not everywhere 0.

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•
$$\sin(1) \neq \frac{1}{2r} \int_{1-r}^{1+r} \sin(t) dt$$
 for $r > 0$.

At the end of the term, you will know that all these phenomena from the previous slide concerning \mathbb{R} -differentiable functions of a real variable are not possible for \mathbb{C} -differentiable functions of a complex variable.

To summarize: \mathbb{C} -differentiability admits a definition similar to the one you are used to, but it gives a more rigid notion. For this reason, we introduce the name *holomorphic*.

Actually, *holomorphic* functions may be seen as the complex analog of *harmonic* functions that you may have met in *multivariable calculus*, for instance in my MAT237 section last year.

Roadmap

There are several equivalent viewpoints concerning holomorphic functions:

1 C-differentiability: $f'(z_0) \coloneqq \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

2 \mathbb{R} -differentiability + Cauchy-Riemann equations, which admit different equivalent flavours: • $\frac{\partial f}{\partial y}(z_0) = i \frac{\partial f}{\partial x}(z_0)$, or • $\overline{\partial} f(z_0) = 0$, or • $\frac{\partial \Re(f)}{\partial x} = \frac{\partial \Im(f)}{\partial y}$ and $\frac{\partial \Re(f)}{\partial y} = -\frac{\partial \Im(f)}{\partial x}$

3 Analyticity, i.e. f can be expressed with a convergent power series around each z_0 :

$$f(z) = \sum_{n \ge 0} a_n (z - z_0)^n$$

• *f* is continuous and $\int_{\gamma} f = 0$ for continuous piecewise C^1 closed curves which *can be continuously deformed* to a point.

 $\mathbf{5}$ f is continuous and admits local antiderivatives.

We will go through most of these viewpoints during the term (and their consequences).

According to the previous slide, it is a good idea to review:

- Power series/Analytic functions (definition, properties) from your first year calculus class¹.
- Second year multivariable calculus class² (for functions ℝ² → ℝ²), especially the following topics: topology/continuity, differentiability, line integrals, Green's theorem.

¹See for instance: http://www.math.toronto.edu/campesat/ens/1819/lec47-0401.pdf ²See for instance: http://www.math.toronto.edu/campesat/mat237.html

Why study complex calculus?

•

- Several applications in physics: waves (Fourier analysis), quantum mechanics, quantum field theory (regularization), aerodynamics (the Joukowsky transform)...
- Computation of integrals (of real functions) using Cauchy's residue theorem.
- Applications to ODEs, PDEs (e.g. Laplace transform).
- Applications to number theory (e.g. proofs of the prime number theorem).
- Conformal geometry (assuming that f is \mathbb{R} -differentiable and $\operatorname{Jac}_{x,y}(\mathfrak{R}(f),\mathfrak{T}(f)) \neq \mathbf{0}$, then f is holomorphic if and only if it preserves angles).

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• Last but not least, that's a (maybe too³) beautiful theory!

³According to René Thom in *Stabilité structurelle et morphogénèse*, see the hidden slide.

Welcome to the complex world!

Before studying complex functions, we first need to introduce \mathbb{C} (definition, geometric properties, topology...).

Let's start now with the definition!

- C := {x + iy : x, y ∈ ℝ}
 We define the following two operations:
 - Addition: $(x_1 + iy_1) + (x_2 + iy_2) := (x_1 + x_2) + i(y_1 + y_2)$
 - *Multiplication:* $(x_1 + iy_1) \cdot (x_2 + iy_2) \coloneqq (x_1x_2 y_1y_2) + i(x_1y_2 + y_1x_2)$

⁴There are several rigorous constructions of \mathbb{C} , but this informal presentation is enough for our purpose.

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- We see R as a subset of C (i.e. R ⊂ C): for x ∈ R, x = x + i0 ∈ C.
 Note that the addition and multiplication of C extend the ones of R:
 (x₁ + i0) + (x₂ + i0) = (x₁ + x₂) + i0 and (x₁ + i0) · (x₂ + i0) = x₁x₂ + i0

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- Note that $i^2 = (0 + i1)(0 + i1) = -1 + i0 = -1$. Then the above defined operations are compatible with the usual distributive laws: $(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

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We will often denote an element of C by the letter *z*.
 When writing "*z* = *x* + *iy* ∈ C" it is implicit that *x*, *y* ∈ R.

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 - How to recover this formula: $\frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$

(reduction to the canonical form by taking the conjugate, we will come back later to that)

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- The order on R does NOT extend to a total order on C compatible with the addition and the multiplication⁵: otherwise we would get that *i*² > 0, i.e. −1 > 0 which is a contradiction. Hence,
 - You should **NOT** write that $z_1 < z_2$ for complex numbers.
 - You should **NOT** write that a complex number is positive (or negative).

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compatible with the operations, e.g. the lexicographic order) or orders compatible with the operations (but not total, e.g. $x_1 + iy_1 \leq x_2 + iy_2 \Leftrightarrow x_1 \leq x_2$ and $y_1 = y_2$).

Proposition

 \mathbb{C} is a 2-dimensional vector space over \mathbb{R} spanned by < 1, i >.

Proposition

 $\ensuremath{\mathbb{C}}$ is a field, meaning that

- $\forall z_1, z_2, z_3 \in \mathbb{C}, \ z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
- $\forall z \in \mathbb{C}, \ z + 0 = 0 + z = z$
- $\forall z \in \mathbb{C}, \ z + (-z) = (-z) + z = 0$ where -(x + iy) = (-x) + i(-y)
- $\forall z_1, z_2 \in \mathbb{C}, \ z_1 + z_2 = z_2 + z_1$
- $\forall z_1, z_2, z_3 \in \mathbb{C}, \ z_1(z_2z_3) = (z_1z_2)z_3$
- $\forall z_1, z_2, z_2 \in \mathbb{C}, \ z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \text{ and } (z_1 + z_2)z_3 = z_1z_3 + z_2z_3$
- $\forall z \in \mathbb{C}, \ 1 \cdot z = z \cdot 1 = z$
- $\forall z \in \mathbb{C} \setminus \{0\}, \ z \cdot z^{-1} = z^{-1} \cdot z = 1 \text{ where } (x + iy)^{-1} = \frac{x}{x^2 + y^2} i \frac{y}{x^2 + y^2}$
- $\forall z_1, z_2 \in \mathbb{C}, \ z_1 z_2 = z_2 z_1$

Conclusion: you should remember that $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}\$ where $i^2 = -1$ with addition and multiplication which behave as you expect them to.

The complex plane

Definition

Given $z = x + iy \in \mathbb{C}$, we define

- The real part of z by $\Re(z) \coloneqq x$ (or $\operatorname{Re}(z) \coloneqq x$).
- The *imaginary part of* z by $\Im(z) \coloneqq y$ (or $\operatorname{Im}(z) \coloneqq y$).

Note that $\Re(z), \Im(z) \in \mathbb{R}$.

It may be convenient to identify \mathbb{C} with the Euclidean plane \mathbb{R}^2 :



Definition

Given $z = x + iy \in \mathbb{C}$, we define the *(complex)* conjugate of z by $\overline{z} := x - iy$.

Geometrically, it is the reflection with respect to the real axis:



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Proposition

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$$\forall z_1, z_2 \in \mathbb{C}, \ \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \text{ and } \overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$$

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• Let
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 then $z\overline{z} = x^2 + y^2$

Note that $z\overline{z} \in \mathbb{R}$, it is useful to write a fraction in its canonical form, for instance:

$$\frac{3+4i}{1+i} = \frac{(3+4i)(1-i)}{(1+i)(1-i)} = \frac{7+i}{2} = \frac{7}{2} + i\frac{1}{2}$$

Euler's formulae

$$\forall z \in \mathbb{C}, \ \Re(z) = \frac{z + \overline{z}}{2}$$
$$\forall z \in \mathbb{C}, \ \Im(z) = \frac{z - \overline{z}}{2i}$$

Don't forget the *i* in the denominator for $\Im(z)$.

Proposition

$$z \in \mathbb{R} \Leftrightarrow z = \Re(z) \Leftrightarrow \Im(z) = 0 \Leftrightarrow z = \overline{z}$$

A too beautiful theory?

« On peut se demander si l'importance attribuée par l'Analyse du siècle passé au corps complexe, et à la théorie des fonctions analytiques n'a pas joué un rôle néfaste sur l'orientation des mathématiques. En permettant l'édification d'une doctrine très belle, trop belle, qui s'accordait d'ailleurs parfaitement à la conception alors triomphante du caractère quantitatif des lois physiques, elle a amené à négliger l'aspect réel et qualitatif des choses. Il a fallu l'essor de la Topologie, au milieu du XXème siècle, pour que les mathématiciens reviennent à l'étude directe des objets géométriques, étude qui n'est d'ailleurs qu'à peine abordée actuellement; qu'on compare l'état d'abandon où se trouve maintenant la Géométrie algébrique réelle, avec le degré de sophistication et de perfection formelle atteint par la Géométrie algébrique complexe! Pour tout phénomène naturel dont l'évolution est régie par une équation algébrique, il est de première importance de savoir si cette équation a des solutions, des racines réelles. En avoir ou pas, telle est la question, la question que supprime précisément le recours aux nombres complexes. Comme exemple de situations où la notion de réalité joue un rôle qualitatif essentiel, on citera la réalité des valeurs propres d'un système différentiel, l'index d'un point critique d'une fonction, le caractère elliptique ou hyperbolique d'un opérateur différentiel linéaire. »

René Thom, Stabilité structurelle et morphogénèse.