MAT237 - LEC5201 - 2019–2020
2020 Winter Term Notes

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Contents

The Implicit Function Theorem

Recollection:

$\triangle$ $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_p)$, $F = (F_1, \dots, F_p)$

Theorem: $U \subset \mathbb{R}^m$ open, $V \subset \mathbb{R}^p$ open

$$\begin{aligned} F : U \times V &\longrightarrow \mathbb{R}^p \\ (x, y) &\longmapsto F(x, y) \end{aligned} \quad \text{of class } C^1$$

$(x_0, y_0) \in U \times V$

If $D_y F(x_0, y_0)$ is invertible then there exist $r, s > 0$ satisfying $B(x_0, r) \subset U$, $B(y_0, s) \subset V$ and $\varphi : B(x_0, r) \longrightarrow B(y_0, s)$ of class C^1 such that

(*) $$\forall (x, y) \in B(x_0, r) \times B(y_0, s), \quad F(x, y) = F(x_0, y_0) \iff y = \varphi(x)$$

Remember that $D_y F(x_0, y_0)$ is the Jacobian matrix of $y \mapsto F(x_0, y)$ at y_0 and that $D_x F(x_0, y_0)$ is the Jacobian matrix of $x \mapsto F(x, y_0)$ at x_0

$$\underset{M_{p, m+p}(\mathbb{R})}{\underset{\cap}{DF(x_0, y_0)}} = \left(\underbrace{\begin{matrix} \frac{\partial F_1}{\partial x_1}(x_0, y_0) & \cdots & \frac{\partial F_1}{\partial x_m}(x_0, y_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_p}{\partial x_1}(x_0, y_0) & \cdots & \frac{\partial F_p}{\partial x_m}(x_0, y_0) \end{matrix}}_{\substack{\| \\ D_x F(x_0, y_0) \\ \cap \\ M_{p,m}(\mathbb{R})}} \quad \underbrace{\begin{matrix} \frac{\partial F_1}{\partial y_1}(x_0, y_0) & \cdots & \frac{\partial F_1}{\partial y_p}(x_0, y_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_p}{\partial y_1}(x_0, y_0) & \cdots & \frac{\partial F_p}{\partial y_p}(x_0, y_0) \end{matrix}}_{\substack{\| \\ D_y F(x_0, y_0) \\ \cap \\ M_{p,p}(\mathbb{R})}} \right)$$

Remark: $F(x_0, y_0) = F(x_0, y_0)$ so $y_0 = \varphi(x_0)$ by $(*)$

Remark: $F(x, \varphi(x)) = F(x_0, y_0) \quad \forall x \in B(x_0, r)$

$$\Rightarrow DF(x_0, \underset{y_0}{\underset{\shortparallel}{\varphi(x_0)}}) \begin{pmatrix} I_{m,m} \\ D\varphi(x_0) \end{pmatrix} = 0$$

↳ the RHS is constant

↳ by the chain rule applied to $F \circ G(x)$ where $G(x) = (x, \varphi(x))$

$$\Rightarrow \begin{pmatrix} D_x F(x_0, y_0) & D_y F(x_0, y_0) \end{pmatrix} \begin{pmatrix} I_{m,m} \\ D\varphi(x_0) \end{pmatrix} = 0$$

$$\Rightarrow D_x F(x_0, y_0) + D_y F(x_0, y_0) D\varphi(x_0) = 0$$

$$\Rightarrow D\varphi(x_0) = -\left[D_y F(x_0, y_0)\right]^{-1} D_x F(x_0, y_0)$$

recall that $D_y F(x_0, y_0)$ is invertible

Ccl: We know how to compute $D\varphi(x_0)$ in terms of F

$$\boxed{D\varphi(x_0) = -\left[D_y F(x_0, y_0)\right]^{-1} D_x F(x_0, y_0)}$$

You should know this formula

(or better: be able to quickly recover it)

Special case of the IFT when $p = 1$

Theorem: $U \subset \mathbb{R}^m$ open, $I = (a, b)$, $F : \begin{array}{l} U \times I \longrightarrow \mathbb{R} \\ (x_1, \dots, x_m, y) \mapsto F(x_1, \dots, x_m, y) \end{array}$ C^1

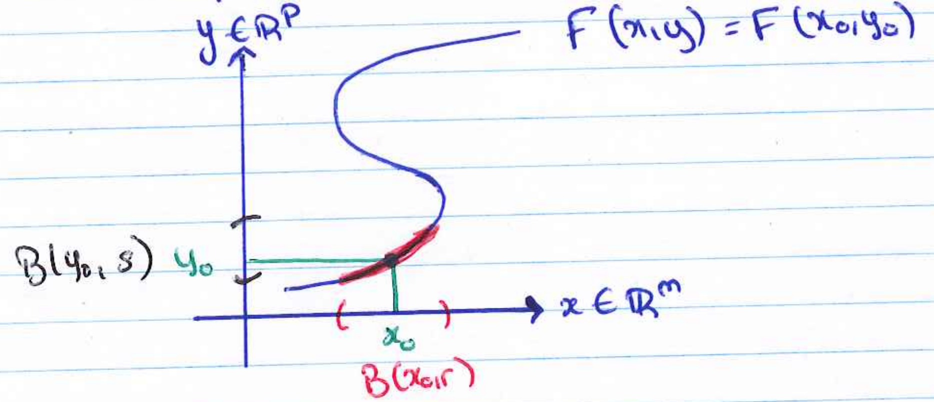
If $\dfrac{\partial F}{\partial y}(x_0, y_0) \neq 0$ then there exist $r, s > 0$ with $B(x_0, r) \subset U$ and $(y_0 - s, y_0 + s) \subset I$ and $\varphi : B(x_0, r) \to (y_0 - s, y_0 + s)$ C^1 s.t. $\forall (x, y) \in B(x_0, r) \times (y_0 - s, y_0 + s)$, $F(x, y) = F(x_0, y_0) \Leftrightarrow y = \varphi(x)$

Remark: by computing the $\dfrac{\partial}{\partial x_i}$'s derivative at x_0 of $F(x, \varphi(x)) = F(x_0, y_0)$

we get:

$$\boxed{\frac{\partial \varphi}{\partial x_i}(x_0, y_0) = -\frac{\dfrac{\partial F}{\partial x_i}(x_0, y_0)}{\dfrac{\partial F}{\partial y}(x_0, y_0)}}$$

Geometric interpretation:

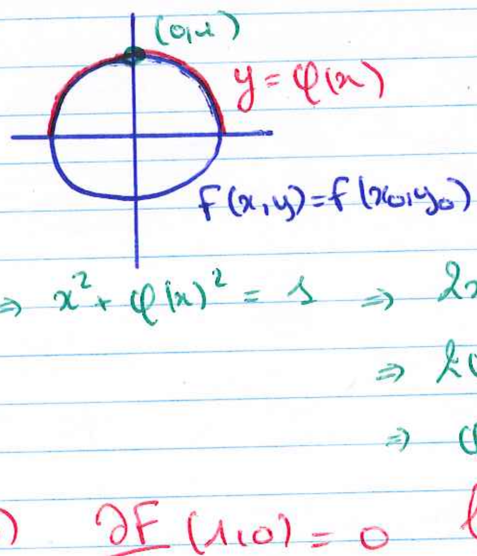


Under the assumptions of the IFT, the level set $F(x,y) = F(x_0,y_0)$ defines locally around (x_0,y_0) a function $y = \varphi(x)$ of class C^1

Example:

$F(x,y) = x^2+y^2$, $(x_0,y_0) = (0,1)$, $\frac{\partial F}{\partial y}(0,1) = 2 \neq 0$

$F(x,y) = F(x_0,y_0) \Leftrightarrow x^2+y^2 = 1$



$\varphi : \begin{array}{ccc} (-1,1) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \sqrt{1-x^2} \end{array}$

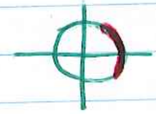
$F(x,\varphi(x)) = 1 \Rightarrow x^2 + \varphi(x)^2 = 1 \Rightarrow 2x + 2\varphi(x)\varphi'(x) = 0$

$\Rightarrow 2\varphi(0)\varphi'(0) = 0$

$\Rightarrow \varphi'(0) = 0$

Remark: at $(1,0)$ $\frac{\partial F}{\partial y}(1,0) = 0$ but $\frac{\partial F}{\partial x}(1,0) = 2 \neq 0$

so we can express $F(x,y) = 1$ as a function $x = \varphi(y)$



Homework: Questions from 3.1

The Inverse Function Theorem

⚠ Notice that the domain and codomain have same dimension m

<u>Theorem</u>: $\mathcal{U} \subset \mathbb{R}^m$ open, $f: \mathcal{U} \to \mathbb{R}^m$ C^1, $a \in \mathcal{U}$

If $Df(a)$ is invertible then $\exists \mathcal{V}, \mathcal{W} \subset \mathbb{R}^m$ open sets satisfying $a \in \mathcal{V} \subset \mathcal{U}$, $f(a) \in \mathcal{W}$ s.t. $f: \mathcal{V} \to \mathcal{W}$ is a C^1-diffeomorphism.

(ie: $f: \mathcal{V} \to \mathcal{W}$ is C^1, bijective and $f^{-1}: \mathcal{W} \to \mathcal{V}$ is also C^1)

⚠ If $Df(a)$ is invertible then <u>locally</u> around a and $f(a)$ f is a C^1-diffeomorphism : we shrunk the domain from $\mathcal{U}$ to a smaller $\mathcal{V}$

<u>Remark</u>: the implicit function theorem and the inverse function theorem are equivalent

Δ <u>ImpFT $\Rightarrow$ InvFT</u>:

Let $f: \mathcal{U} \xrightarrow{C^1} \mathbb{R}^m$, $\mathcal{U} \subset \mathbb{R}^m$ open, $a \in \mathcal{U}$ s.t. $Df(a)$ is invertible.

Define $F: \mathcal{U} \times \mathbb{R}^m \to \mathbb{R}^m$ by $F(x,y) = f(x) - y$

then $D_x F(a, f(a)) = Df(a)$ is invertible. { $D_x F$ is invertible so the ImpFT gives x in terms of y }

Hence, by the ImpFT, $\exists \varphi: B(f(a), r) \to B(a,s)$ C^1

s.t. $\forall (x,y) \in B(a,s) \times B(f(a), r)$, $F(x,y) = \overbrace{F(a, f(a))}^{0} \Leftrightarrow x = \varphi(y)$

<u>ie</u> $y = f(x) \Leftrightarrow x = \varphi(y)$

so $\varphi = f^{-1}$ for $f: B(a,s) \cap f^{-1}(B(f(a), r)) \longrightarrow B(f(a), r)$

InvFT $\Rightarrow$ ImpFT:

Let $F : \mathcal{U} \times \mathcal{V} \to \mathbb{R}^p$ C^1, $\mathcal{U} \subset \mathbb{R}^m$ open, $\mathcal{V} \subset \mathbb{R}^p$ open, assume $D_y F(x_0, y_0)$ invertible

Notice that $\mathcal{U} \times \mathcal{V}$ is an open subset of $\mathcal{U} \times \mathcal{V}$ and define

$f : \mathcal{U} \times \mathcal{V} \to \mathbb{R}^{m+p}$ by $f(x,y) = (x, F(x,y))$

$$Df(x_0, y_0) = \begin{pmatrix} I_{m,m} & 0 \\ D_x F(x_0, y_0) & D_y F(x_0, y_0) \end{pmatrix} \text{ is invertible}$$

so, by the InvFT, we can find $M, N \subset \mathbb{R}^{m+p}$ open s.t.

$(x_0, y_0) \in M \subset \mathcal{U} \times \mathcal{V}$, $(x_0, F(x_0, y_0)) \in N$, $f : M \to N$ C^1-difféo

notice that, by definition of f, $f^{-1}(x, F(x_0, y_0)) = (x, \varphi(x))$

for some φ C^1 defined in a neighborhood of x_0 ↳ we should be a little bit more careful here

but $\underset{\overset{\shortparallel}{(x, F(x, \varphi(x)))}}{f(x, \varphi(x))} = (x, F(x_0, y_0))$, ie $F(x, \varphi(x)) = F(x_0, y_0)$

$\square$

Rem: from $f^{-1} \circ f = id$ we obtain: $Df^{-1}(f(a)) \cdot Df(a) = I_{m,m}$

$$\Rightarrow Df^{-1}(f(a)) = [Df(a)]^{-1}$$

Definitions:
- $f : A \to B$ homeomorphism means f bij $+ f$ $C^0 + f^{-1}$ C^0
- $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^m$ open, $f : \mathcal{U} \to \mathcal{V}$ C^k-diffeomorphism means: f bijective $+ f$ $C^k + f^{-1}$ C^k

Singularity points:

Case 1: Curves in $\mathbb{R}^2$

Observation: A curve may be described in 3 natural ways

① As a graph $S = \{(x,y) \in \mathbb{R}^2 : x \in I,\ y = f(x)\}$

or $S = \{(x,y) \in \mathbb{R}^2 : y \in I,\ x = f(y)\}$

where $f: I \to \mathbb{R}$, $I \subset \mathbb{R}$ open interval

② As a level set: $S = \{(x,y) \in \mathcal{U} : F(x,y) = c\}$

where $\mathcal{U} \subset \mathbb{R}^2$ is open and $F: \mathcal{U} \to \mathbb{R}$ and $c \in \mathbb{R}$

⚠ In this case we can get something too big

$S = \{(x,y) \in \mathbb{R}^2 : F(x,y) = 0\}$ where $F(x,y) = 0$

$\mathbb{R}^2$

or too small

$S = \{(x,y) \in \mathbb{R}^2 : \sin^2(x) + \cos^2(y) = 4\} = \emptyset$

$S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 0\} = \{(0,0)\}$

③ Parametrically, $S = \{\gamma(t) : t \in I\}$ where $I \subset \mathbb{R}$ open interval

$\gamma: I \to \mathbb{R}^2$

⚠ Again we can get something too small:

$\{\gamma(t) : t \in \mathbb{R}\} = \{(0,0)\}$ for $\gamma(t) = (0,0)$ constant

or too big: type "Peano curve" on google

↳ this last phenomenon is not possible for γ C^1.

Remark 1: A curve represented by a graph may be represented by a level set

Indeed : $\{(x,y) \in \mathbb{R}^2 : x \in I \ , \ y = f(x)\}$

$= \{(x,y) \in I \times \mathbb{R} \ : \ y - f(x) = 0\}$

$= \{(x,y) \in I \times \mathbb{R} \ : \ F(x,y) = 0\}$ where $F(x,y) = y - f(x)$

But the converse is false: the lemniscate

$x^4 - x^2 + y^2 = 0$ $\qquad \infty$

may not be described as a graph around $(0,0)$

Remark 2: A curve represented by a graph admits a parametrization

Indeed : $\{(x,y) \in \mathbb{R}^2 : x \in I \ , \ y = f(x)\}$

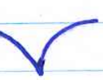
$= \{(t, f(t)) : t \in I\}$

$= \{\gamma(t) : t \in I\}$ where $\gamma(t) = (t, f(t))$

But the converse is false : $\gamma(t) = \left(\frac{t^2-1}{t^2+1} \ , \ \frac{2t(t^2-1)}{(t^2+1)^2} \right)$

gives again the lemniscate ∞ which may not be described as a graph around $(0,0)$

another parametrization is $\gamma(t) = (\sin(t) \ , \sin(t)\cos(t))$

Remark 3: $y^3 - x^2 = 0$.  may be described as the graph of $f(x) = |x|^{2/3}$ but may not be described as the graph of a C^1 function

The above observations lead us to the following definitions:

Def. Let $C \subset \mathbb{R}^2$ be a curve. We say that C is *regular* at $a \in C$ if there exists $\varepsilon > 0$ s.t. $C \cap B(a,\varepsilon)$ is the graph of a C^1 function.
We say that C is *singular* at a if for every $\varepsilon > 0$, $C \cap B(a,\varepsilon)$ is not the graph of a C^1 function.

Ex. $x^2+y^2=1$ is regular at all its points

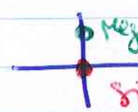
is the graph of some $x=g(y)$ C^1

is the graph of some $y=f(x)$ C^1

Ex: $y^3=x^2$

Ex: $x^4-x^2+y^2=0$

Ex: $xy=0$



Theorem. $F: U \to \mathbb{R}$ C^1, $U \subset \mathbb{R}^2$ open, $a = (x_0,y_0) \in U$.
If $\nabla F(a) \neq \vec{0}$ then $C = \{(x,y) \in U : F(x,y) = F(x_0,y_0)\}$ is regular at a.

Δ We know that $\nabla F(a) = \left(\frac{\partial F}{\partial x}(a), \frac{\partial F}{\partial y}(a)\right) \neq (0,0)$

Case 1: $\frac{\partial F}{\partial y}(a) \neq 0$ then by the Implicit Function Theorem, then C is locally the graph of a function $y=\varphi(x)$ around a.

Case 2. $\frac{\partial F}{\partial x}(a) \neq 0$ same but $x=\varphi(y)$

$\square$

Theorem: $I \subset \mathbb{R}$ open interval, $\gamma : I \to \mathbb{R}^2$ C^1, $t_0 \in I$

If $\gamma'(t_0) \neq \vec{0}$ then there exists $r > 0$ s.t. $(t_0 - r, t_0 + r) \subset I$ and

$C = \{\gamma(t) : t \in (t_0 - r, t_0 + r)\}$ is regular.

Δ $\gamma'(t_0) = (\gamma_1'(t_0), \gamma_2'(t_0))$ so WLOG we may assume that

$\gamma_1'(t_0) \neq 0$.

By the Inverse Function Theorem, $\exists r > 0$ s.t.

$J = (t_0 - r, t_0 + r) \subset I$, $K = \gamma_1(J)$ open interval, $\gamma_1 : J \to K$ bijection

and $\gamma_1^{-1} : K \to J$ is C^1.

Define $f : K \to \mathbb{R}$ by $f(x) = \gamma_2(\gamma_1^{-1}(x))$

then $\{(x,y) \in \mathbb{R}^2 : x \in K, \; y = f(x)\}$

$$= \{(x,y) \in \mathbb{R}^2 : t \in J, \; x = \gamma_1(t), \; y = \gamma_2(t)\}$$

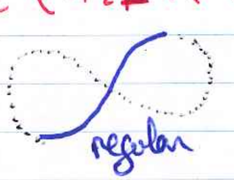
$$= \{\gamma(t) : t \in J\}$$

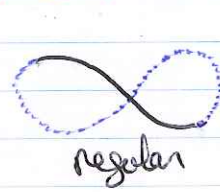
The idea is the following: around to "$x \underset{bij}{\overset{\gamma_1}{\longleftrightarrow}} t \xrightarrow{\gamma_2} y$" $\square$

⚠ **Beware**: the domain of f is shrunk, we don't conclude that $\{f(t) : t \in I\}$ is regular only $\{f(t) : \text{for } t \text{ close to } t_0\}$

Ex: $\gamma(t) = (\sin(t), \sin(t)\cos(t))$

$t \in \mathbb{R}$: $t \in (\pi/2 - r, \pi/2 + r)$ $t \in (3\pi/2 - r, 3\pi/2 + r)$



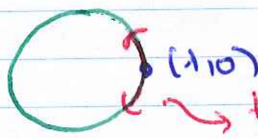


Case 2: The general case

Def: Let $M \subset \mathbb{R}^N$, $a \in M$. We say that M is regular of dimension d at a if there exists $\varepsilon > 0$ s.t., up to permuting the variables, $B(a,\varepsilon) \cap M$ is the graph of a C^1-function $f: \mathcal{U} \to \mathbb{R}^{N-d}$ where $\mathcal{U} \subset \mathbb{R}^d$ is open

⚠ "Up to permuting the variables" means that we express $N-d$ variables in terms of d variables but not necessarily the $N-d$ last in terms of the d first.

Ex: $M = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$ is regular of dim 1 at $(1,0)$



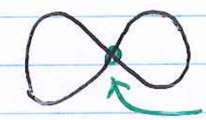
this part may not be described as a graph $y = \varphi(x)$ but it can be described as a graph $x = \varphi(y)$

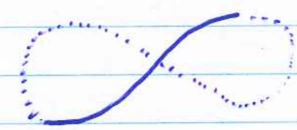
Theorem: $\mathcal{U} \subset \mathbb{R}^N$ open, $F: \mathcal{U} \to \mathbb{R}^{N-d}$ C^1, $a \in \mathcal{U}$.
If $\operatorname{rank}(DF(a)) = N-d$ then $M = \{x \in \mathcal{U} : F(x) = F(a)\}$ is regular of dimension d at a.

Δ Let's denote $DF(a) = (v_1\ v_2 \ \cdots\ v_N) \in M_{N-d,N}(\mathbb{R})$, by assumption we may find $i_1, \dots, i_{N-d}$ s.t $v_{i_1}, \dots, v_{i_{N-d}}$ are linearly independant then $D_{(x_{i_1}, \dots, x_{i_{N-d}})} F(a)$ is invertible.
By the Imp FT, we may locally describe M around a as a C^1 graph $(x_{i_1}, \dots, x_{i_{N-d}}) = \varphi$(of the d remaining variables) $\square$

Theorem: $\mathcal{U} \subset \mathbb{R}^d$ open, $\sigma : \mathcal{U} \to \mathbb{R}^N$ C^1, $t_0 \in \mathcal{U}$.
If $\operatorname{rank}(D\sigma(t_0)) = d$ then $\exists r > 0$ s.t. $B(t_0, r) \subset \mathcal{U}$ and
$M = \{\sigma(t) : t \in B(t_0, r)\}$ is regular of dimension d

⚠ Again, as in the planar curve case, we shrunk the domain to avoid "self intersection"





Δ

$$D\sigma(t_0) = \begin{array}{c|c} & \overset{\longleftarrow d \longrightarrow}{} \\ N \updownarrow & \begin{pmatrix} \nabla\sigma_1(t_0) \\ \nabla\sigma_2(t_0) \\ \vdots \\ \nabla\sigma_N(t_0) \end{pmatrix} \end{array}$$

is of rank d, up to permuting the components we may assume that $\nabla\sigma_1(t_0), \dots, \nabla\sigma_d(t_0)$ are linearly independant

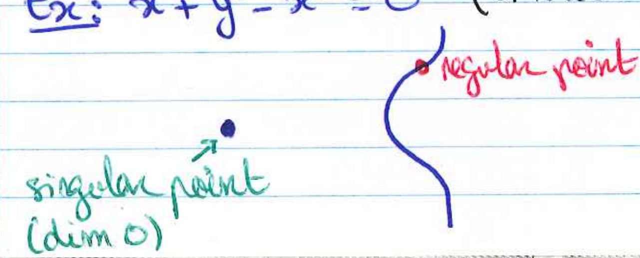
hence, by the Inv FT, $\varphi : t \mapsto (\sigma_1(t), \dots, \sigma_d(t))$ is a C^1-diffeo $\mathcal{V} \longrightarrow \mathcal{W}$ for $t_0 \in \mathcal{V}$ small enough

We define $f : \mathcal{W} \to \mathbb{R}^{N-d}$ by $f(\underbrace{x_1, \dots, x_d}_{x}) = (\sigma_{d+1}(\varphi^{-1}(x)), \dots, \sigma_N(\varphi^{-1}(x)))$

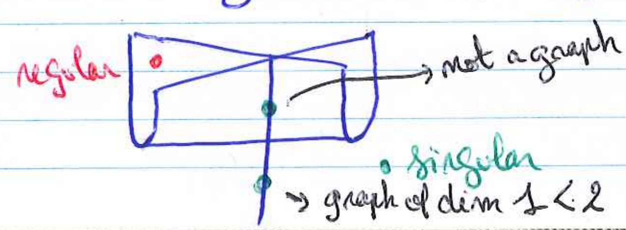
then $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^{m-d}, x \in \mathcal{W}, y = f(x)\} = \{\sigma(t) : t \in \mathcal{V}\}$ $\square$

Def. $M \subset \mathbb{R}^N$, $a \in M$. We say that a is a singular point of M if a is not a regular point of "maximal dimension"
Otherwise, we say that a is a regular point of M (with no precision about the dimension)

Ex: $x^2 + y^2 - x^3 = 0$ (it's a curve, dim 1)



Ex: Whitney's umbrella $x^2 - zy^2 = 0$



You may be tempted to use the following $\boxed{\text{false}}$ statement: DO NOT, it is $\boxed{\text{false}}$

FALSE STATEMENT

Let $\sigma : \mathcal{U} \xrightarrow{C^1} \mathbb{R}^N$ be a parametrization where $\mathcal{U} \subset \mathbb{R}^d$ open

If ① $\forall t \in \mathcal{U}$, $D\sigma(t)$ is of rank d

② $t \neq t' \Rightarrow \sigma(t) \neq \sigma(t')$

then $\underset{\mathbb{R}^N}{\underset{\cap}{C}} = \{\sigma(t) : t \in \mathcal{U}\}$ is regular of dimension d

BAD PROOF

Δ Indeed, by 1, $\forall t_0 \in \mathcal{U}$, $\exists \varepsilon > 0$, s.t. $\{\sigma(t) : t \in (t_0 - \varepsilon, t_0 + \varepsilon)\}$ is regular (ie there is no cusp)

and by 2 there is no self intersection, so we can't have something like ∞ $\{(\sin(t), \sin(2t)) : t \in \mathbb{R}\}$ $\square$

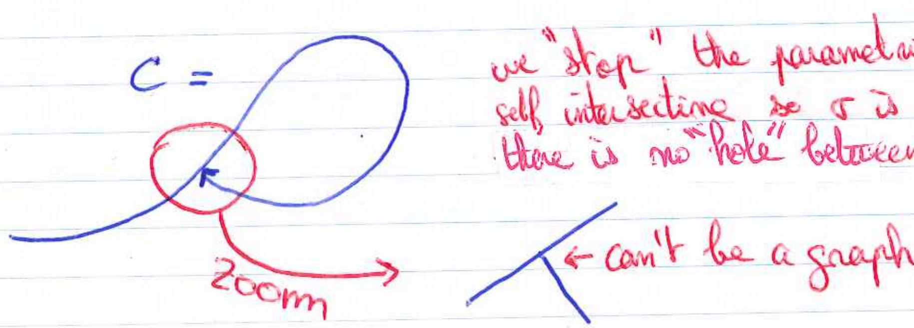
<u>Comment</u>: the last part of the proof is false:

take $\sigma(t) = (\sin(t), \sin(2t))$, $t \in (-1, \pi)$

then σ satisfies the assumptions ① and ② but

$C = \{\sigma(t) : t \in (-1, \pi)\}$ is not regular at $(0,0)$.

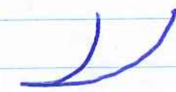
counter-example



we "stop" the parametrization just before self intersecting so σ is 1-to-1 but there is no "hole" between the two branches

$\underline{Ex}$: a curve in the 3-dimensional Euclidean space

- $t \mapsto (t^3, t^2, t^6)$ parametrization
- $\begin{cases} x^2 - z = 0 \\ y^3 - z = 0 \end{cases}$ level set / implicit equation



both define the same curve $C \subset \mathbb{R}^3$

Define $F(x,y,z) = (x^2 - z, y^3 - z)$ then $C = \{(x,y,z) : F(x,y,z) = (0,0)\}$

$DF(x,y,z) = \begin{pmatrix} 2x & 0 & -1 \\ 0 & 3y^2 & -1 \end{pmatrix}$ is of rank $2 \overset{3-1}{=}$ except at $(0,0,0) \in C$

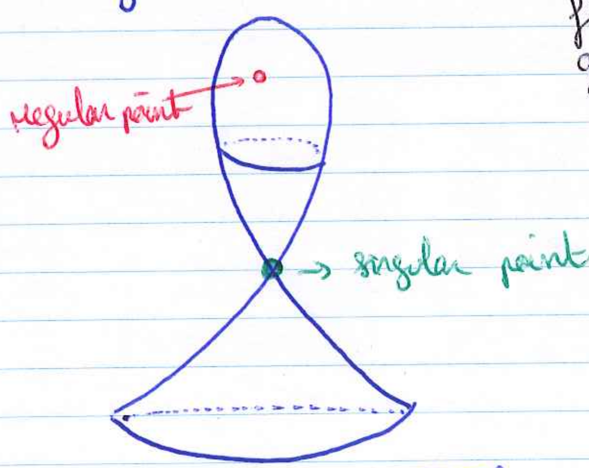
By the above theorem, C is regular for $(x,y,z) \neq (0,0,0)$

Graphically, we see that C is singular at $(0,0,0)$ → Prove it rigorously!

$\underline{Ex}$: $x^2 + y^2 - z^2(1-z) = 0$

Rem: this surface is easy to draw, for $z = z_0$ fixed, the distance ρ at the O_z axis is given by $\rho^2 = z_0^2(1-z_0)$

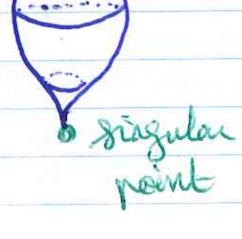
$z \mapsto \sqrt{z^2(1-z)}$ 



(Indeed, if we remove this point we have to path-connected components whereas if we remove a point of a graph $B(0,r) \subset \mathbb{R}^2 \to \mathbb{R}$, we have one)

$DF(x,y,z) = (2x \;\; 2y \;\; z(3z-2))$

$\underline{Ex}$: $x^2 + y^2 - z^3(1-z) = 0$



$DF(x,y,z) = (2x \;\; 2y \;\; z^2(-3+4z))$

Exercise : $C = \{ (x,y) \in \mathbb{R}^2 : x^3 + y^3 - 3xy = 0 \}$ "Folium of Descartes"

1. Study the singular points of C
2. Find a parametric description of C (Hint: study $C \cap \{ y = tx \}$ and use it to draw C

Exercise: Prove that the following sets are singular at the origine

1) $M = \{ (x,y) \in \mathbb{R}^2 : y = |x| \}$
2) $M = \{ (x,y) \in \mathbb{R}^2 : x^2 = y^2 \}$
3) $M = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = z^2 \}$
4) $M = \{ (x,y) \in \mathbb{R}^2 : x^2 = y^3 \}$

Exercise: Define $C \subset \mathbb{R}^3$ implicitely by

$$\begin{cases} x^2 + y^2 + z^2 = R^2 \\ x^2 + y^2 - 2x = 0 \end{cases}$$

where $R > 0$ is fixed

1. Prove that C is regular for $R \neq 2$
2. What do we get for $R = 2$

I wrote these solutions quickly after class
→ DOUBLE CHECK EVERYTHING

Solutions:

Exo 1. $C \cap \{y = tx\} \rightsquigarrow \begin{cases} y = tx \\ x^3 + y^3 - 3xy = 0 \end{cases}$

$\Rightarrow \begin{cases} y = tx \\ x^3 + t^3 x^3 - 3t\, x^2 = 0 \end{cases}$

$\Rightarrow \begin{cases} y = tx \\ x^2\left((1+t^3)x - 3t\right) = 0 \end{cases}$

$\Rightarrow x,y = (0,0)$ or $(x,y) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right)$

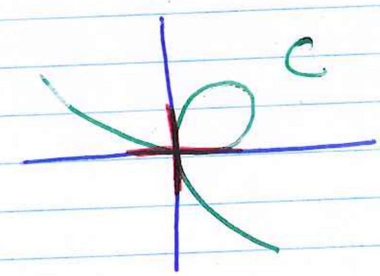
we study $x(t) = \frac{3t}{1+t^3}$ and $y(t) = \frac{3t^2}{1+t^3}$

t	$-\infty$		-1		0	$2^{-1/3}$	$2^{1/3}$	$+\infty$
$x(t)$	0 ↗	$+\infty$ ‖	$-\infty$ ↗		0 ↗	$2^{2/3}$ ↘	$2^{1/3}$ ↘	0
$y(t)$	0 ↘	$-\infty$ ‖	$+\infty$ ↘		0 ↗	$2^{1/3}$ ↗	$2^{2/3}$ ↘	0

$\frac{y}{x} = t \to -\infty$ vertical tangent at $(0,0)$

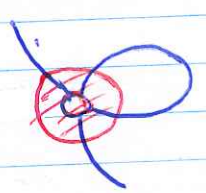
$\frac{y}{x} = t \to 0$ horizontal tangent at $(0,0)$

$\frac{y}{x} = t \to +\infty$ vertical tangent line at $(0,0)$



$f(x,y) = x^3 + y^3 - 3xy \qquad Df(x,y) = \begin{pmatrix} 3x^2 - 3y & 3y^2 - 3x \end{pmatrix}$

is of max rank on $C \setminus \{0\}$ so $C \setminus \{0\}$ is regular

At $(0,0)$:  $(C \setminus \{0\}) \cap B(0,\varepsilon)$ has 4 path-connected components : C is not regular of dim 1 at $(0,0)$

IDEM : Don't trust me !

Exo 2

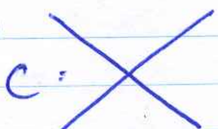
① • Assume $y = \varphi(x)$ in a $B(0,\varepsilon)$

then $\varphi(x) = |x|$ is not C^1

• Assume $x = \varphi(y)$ in a $B(0,\varepsilon)$

then $|\varphi(y)| = y \Rightarrow \varphi(y) = \pm y$: not a graph (2 possibles x-values for a y

②



C : $x^2 - y^2 = (x-y)(x+y)$

Method 1 : $y = \varphi(x) \Rightarrow (x - \varphi(x))(x + \varphi(x)) = 0$

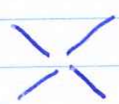
$\Rightarrow \varphi(x) = \pm x \rightsquigarrow$ not a graph

$x = \varphi(y) \Rightarrow \varphi(y) = \pm y \rightsquigarrow$ not a graph

Method 2 : if $\{(x, \varphi(x))\} = B(0,\varepsilon) \cap$ for $\varphi : I \to \mathbb{R}$ (small interval)

then $M \cap B(0,\varepsilon) = F(I)$ for $F(x) = (x, \varphi(x))$

$\Rightarrow (M \cap B(0,\varepsilon)) \setminus \{0\} = F(I \setminus \{0\})$



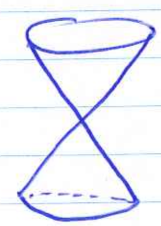
$= F((\longleftrightarrow) \cup (\longleftrightarrow))$ contradiction

↳ 4 components

↳ 2 components

↳ at most 2 components since the C^0 image of a p-c is p-c

③



Method 1 : $x = \varphi(y,z) \Rightarrow \varphi(y,z) = \pm\sqrt{z^2 - y^2}$ not a graph

$y = \varphi(x,z) \Rightarrow \varphi(x,z) = \pm\sqrt{z^2 - x^2}$ not a graph

$z = \varphi(x,y) \Rightarrow \varphi(x,y) = \pm\sqrt{x^2 + y^2}$

Method 2 : if M is regular then $(B(0,\varepsilon) \cap M) \setminus \{0\}$ ← 2 comp

IS

$B(0,\varepsilon) \setminus \{0\} \subset \mathbb{R}^2$

do as above ↗ 1 comp

④ $x = \varphi(y) \Rightarrow \varphi(y)^2 = y^3 \Rightarrow \varphi(y) = \pm y^{3/2}$ not a graph

$y = \varphi(x) \Rightarrow x^2 = \varphi(x)^3 \Rightarrow \varphi(x) = x^{2/3}$ not C^1

Why are we so interested by sets that are locally a graph?

<u>Extra</u>: (not part of MAT 237)

Assume that $M \subset \mathbb{R}^N$ is regular of dimension d at $a \in M$, then we may locally flatten M around a: around a M looks like $\mathbb{R}^d$.

Formally: $\exists \mathcal{U} \subset \mathbb{R}^N$ an open subset containing a
$\exists \mathcal{V} \subset \mathbb{R}^N$ an open subset containing $\vec{0}$
$\exists F : \mathcal{U} \to \mathcal{V}$ a C^1-diffeomorphism (φ is bijective, φ and φ^{-1} are C^1)

such that $F(\mathcal{U} \cap M) = \mathcal{V} \cap (\mathbb{R}^d \times \{0\}^{N-d})$

Δ By definition $\exists \varepsilon > 0$ s.t. $B(a, \varepsilon) \cap M = \{(x, \varphi(x)) : x \in \mathcal{W}\}$ where $\mathcal{W} \subset \mathbb{R}^d$ is open and $\varphi : \mathcal{W} \to \mathbb{R}^{N-d}$. (up to permuting the coordinates)

Define $F : \mathcal{W} \times \mathbb{R}^{N-d} \to \mathbb{R}^N$ by

$$F(x_1, \dots, x_d, y_1, \dots, y_{N-d}) = (x_1 - a_1, \dots, x_d - a_d, y_1 - \varphi_1(x), \dots, y_{N-d} - \varphi_{N-d}(x))$$

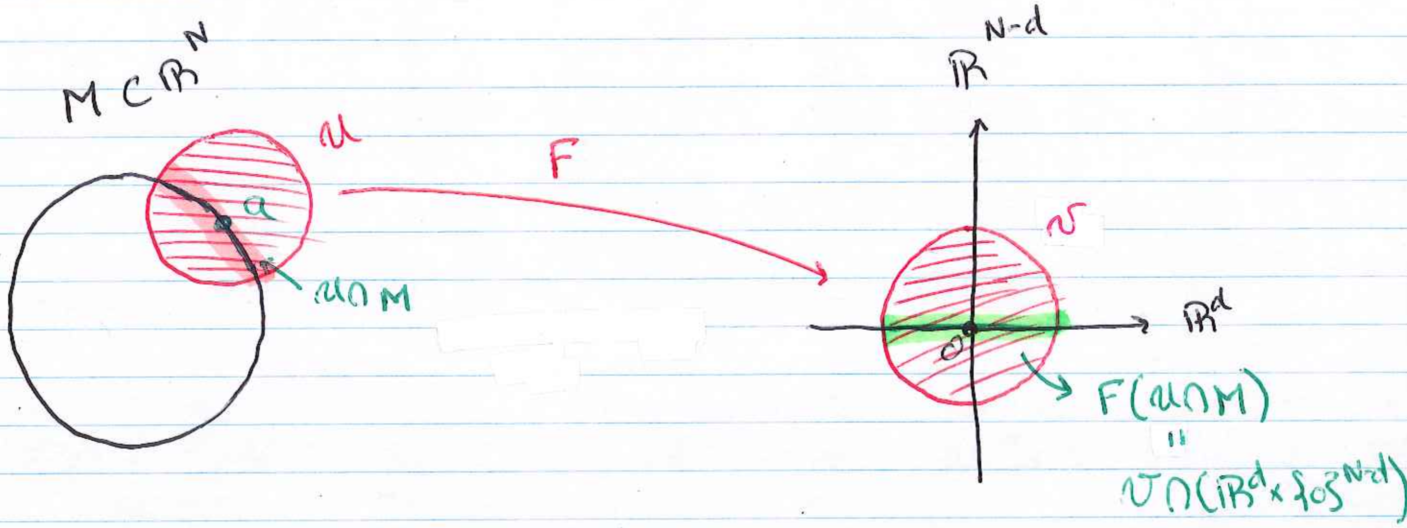
Then $DF(a) = \begin{pmatrix} I_{d,d} & 0 \\ * & I_{N-d,N-d} \end{pmatrix}$ is invertible, so by the inverse function theorem $\exists \mathcal{U} \subset \mathbb{R}^N$ open containing a, $\exists \mathcal{V} \subset \mathbb{R}^N$ open containing $F(a) = 0$ s.t. $F : \mathcal{U} \to \mathcal{V}$ is a C^1-diffeomorphism

Moreover, $F(\mathcal{U} \cap M) = \mathcal{V} \cap (\mathbb{R}^d \times \{0\}^{N-d})$ by definition of F $\square$

We say that M is a d-dim C^1-submanifold if it is everywhere regular of dim d.

Since "being C^1" is a local property, it allows to define C^1-functions defined on M

Ex :



So if we have a function $f : M \longrightarrow \mathbb{R}$, we may study

$$\tilde{f} = f \circ (F^{-1}) : F(\mathcal{U} \cap M) = \mathcal{V} \cap (\mathbb{R}^d \times \{0\}^{N-d}) \longrightarrow \mathbb{R}$$

then we may see $\tilde{f}$ as a function defined on $\mathbb{R}^d$ (locally) and do calculus ...

$\square$ End of extra.

$\triangle$ don't forget this assumption

Transformations / Change of coordinates

<u>Theorem</u>: $U \subset \mathbb{R}^m$ open, $f : U \to \mathbb{R}^m$ C^1 and injective

The following are equivalent:

(i) $\forall x_0 \in U$, $Df(x_0)$ is invertible

(ii) $f(U)$ is open and $f : U \to f(U)$ is a C^1-diffeomorphism

Δ <u>i ⇒ ii</u>: $f : U \to f(U)$ is obviously a bijection

Let $y_0 = f(x_0) \in f(U)$. Since $Df(x_0)$ is invertible, by the inverse function theorem $\exists M, N \subset \mathbb{R}^m$ open with $x_0 \in M$, $y_0 \in N$ s.t. $f : M \to N$ is a C^1-diffeo.

Then $y_0 \in N = f(M) \subset f(U)$ with N open so $\exists r > 0$ s.t. $B(y_0, r) \subset N \subset f(U)$

Since it is true for any $y_0 \in f(U)$, $f(U)$ is open

Moreover since $f^{-1} : N \to M$ is C^1, f^{-1} is C^1 at y_0

<u>ii ⇒ i</u>: we have $f^{-1} \circ f = id$ so $D(f^{-1})(f(x_0)) \cdot Df(x_0) = I_{m,m}$

and $Df(x_0)$ is invertible

□

$\triangle$ The injective assumption is important: $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (e^x \cos y, e^x \sin y)$ satisfies $\forall (x_0, y_0) \in \mathbb{R}^2$ $Df(x_0, y_0)$ is invertible but is not injective.

Exercise 1: $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (e^x \cos y, e^x \sin y)$

Set $\mathcal{U} = \{(x,y) \in \mathbb{R}^2 : y \in (0, 2\pi)\}$

① Compute $f(\mathcal{U})$

② Prove that $f(\mathcal{U})$ is open and $f: \mathcal{U} \to f(\mathcal{U})$ is a C^1-diffeomorphism

③ If $g = f^{-1}: f(\mathcal{U}) \to \mathcal{U}$, compute $Dg(0,1)$

Exercise 2: let $f: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x,y,z) = (e^{2y} + e^{2z}, e^{2x} - e^{2z}, x-y)$

Prove that $f(\mathbb{R}^3) \subsetneq \mathbb{R}^3$

and that $f(\mathbb{R}^3)$ is open

C^1-diffeomorphisms are important because they allow to study a C^1-function after a change of coordinates

Indeed, let $f : \mathcal{U} \to \mathbb{R}^p$ be C^1 with $\mathcal{U} \subset \mathbb{R}^n$ open and let $\varphi : \mathcal{V} \to \mathcal{U}$ be a C^1-diffeomorphism.
Then, if we set $\tilde{f} = f \circ \varphi : \mathcal{V} \to \mathbb{R}^p$ we have

$$\begin{cases} \tilde{f} = f \circ \varphi \\ f = \tilde{f} \circ \varphi^{-1} \end{cases}$$

so we may either study f or $\tilde{f}$.

<u>Ex</u>: polar coordinates

$$\mathcal{U} = \mathbb{R}^2 \setminus \{(x,0) : x \leq 0\}, \quad \mathcal{V} = \{(r,\theta) : r > 0, |\theta| < \pi\}$$

then $\varphi : \mathcal{V} \to \mathcal{U}$ defined by $\varphi(r,\theta) = (r\cos\theta, r\sin\theta)$ is a C^1-diffeomorphism.
Hence, instead of working with $f : \begin{matrix} \mathcal{U} \to \mathbb{R}^p \\ (x,y) \mapsto f(x,y) \end{matrix}$
we may work with $\tilde{f} : \mathcal{V} \to \mathbb{R}^p$ defined by
$\tilde{f}(r,\theta) = f(\varphi(r,\theta))$ (which may be useful if f is invariant w.r.t rotation centered at 0)
It's common to simply write $f(r,\theta)$ instead of $f(\varphi(r,\theta))$
but be careful, that's an abuse of notation

Uniform continuity:

In what follows: $S \subset \mathbb{R}^n$, $f : S \longrightarrow \mathbb{R}^p$

Definition: $x_0 \in S$.
We say that f is continuous at x_0 if:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in S, \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon$$

Definition: We say that f is continuous if it is everywhere, ie:

$$\forall x_0 \in S, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in S, \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon$$

Definition: We say that f is uniformly continuous if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in S, \|x_1 - x_2\| < \delta \Rightarrow \|f(x_1) - f(x_2)\| < \varepsilon$$

Let's compare these two definitions.

Continuity

1 $$\forall \varepsilon > 0, \boxed{\forall x_1 \in S, \exists \delta > 0}, \forall x_2 \in S, \|x_1 - x_2\| < \delta \Rightarrow \|f(x_1) - f(x_2)\| < \varepsilon$$

2 $$\forall \varepsilon > 0, \boxed{\exists \delta > 0, \forall x_1 \in S}, \forall x_2 \in S, \|x_1 - x_2\| < \delta \Rightarrow \|f(x_1) - f(x_2)\| < \varepsilon$$

Uniform continuity

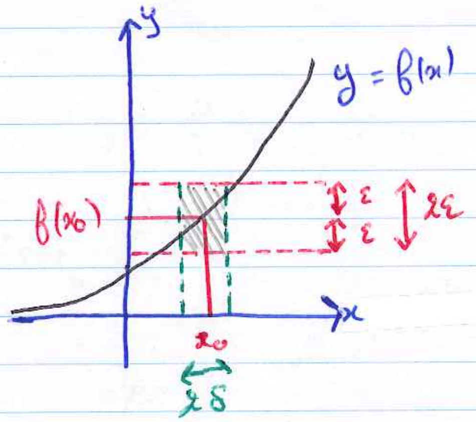
The only difference is that in 1 δ may depend on the choice of x_1
but in 2 δ is independant of x_1 or x_2:
it should be suitable everywhere

⚠ Continuity is a local notion (only depends on the behavior of f around x_0)

Uniform continuity is a global notion (depends on the domain)

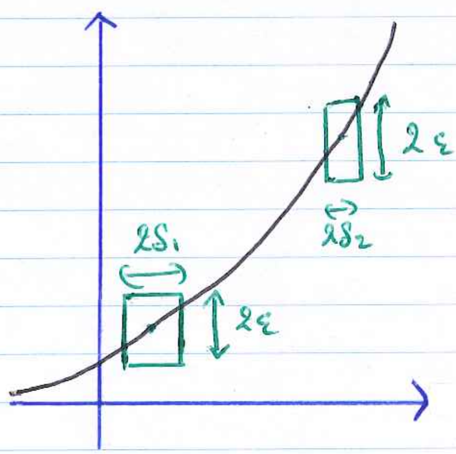
Remark: obviously: uniform continuity implies continuity.

Geometrically: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ is continuous since for any $x_0 \in \mathbb{R}$, and any $\varepsilon > 0$, you may find a $\delta > 0$ s.t. the graph of f stays in the $2\delta \times 2\varepsilon$ box around $(x_0, f(x_0))$ for $x \in (x_0 - \delta, x_0 + \delta)$

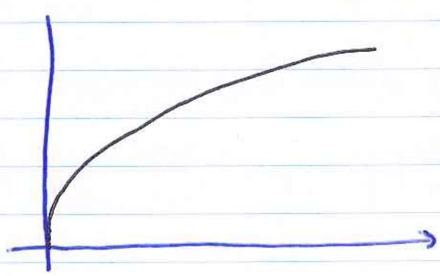


But f is not uniformly continuous: the more x_0 goes, the smaller must be δ (for a fixed $\varepsilon > 0$)

$\rightsquigarrow$ we can't find a δ suitable for everywhere



However $g: [0, +\infty) \to \mathbb{R}$, $g(x) = \sqrt{x}$ is uniformly continuous even if $\lim_{x \to +\infty} g(x) = +\infty$ and the graph becomes arbitrarily steeps at 0.



If you recall the discussion about Dedekind-Completeness
from Sep 24, the below stated theorem is another characterization
of the Dedekind completeness of $\mathbb{R}$ } You can safely skip this comment

The Heine-Cantor theorem:

$K \subset \mathbb{R}^n$ compact and $f : K \longrightarrow \mathbb{R}^p$

If f is continuous then f is uniformly continuous

Δ We prove the contrapositive:

f not uniformly continuous $\Rightarrow$ f not continuous.

Let's assume that f is not U.C.

$\exists \varepsilon > 0, \forall n \in \mathbb{N}_{>0}, \exists x^1_n, x^2_n \in K, \|x^1_n - x^2_n\| < \frac{1}{n}$ and $\|f(x^1_n) - f(x^2_n)\| \geq \varepsilon$

(x^1_n) is a sequence of terms in K compact so $\exists$ a subsequence $(x^1_{\varphi(n)})$ convergent to $\ell \in K$

$$\begin{aligned}\|x^2_{\varphi(n)} - \ell\| &= \|x^2_{\varphi(n)} - x^1_{\varphi(n)} + x^1_{\varphi(n)} - \ell\| \\ &\leq \|x^2_{\varphi(n)} - x^1_{\varphi(n)}\| + \|x^1_{\varphi(n)} - \ell\| \\ &\leq \frac{1}{\varphi(n)} + \|x^1_{\varphi(n)} - \ell\| \xrightarrow[n\to\infty]{} 0 + 0 = 0\end{aligned}$$

so $\lim\limits_{n\to+\infty} x^2_{\varphi(n)} = \ell$ too

Assume by contradiction that f is continuous at $\ell \in K$

then $\forall n$, $\|f(x^1_{\varphi(n)}) - f(x^2_{\varphi(n)})\| \geq \varepsilon$

$\Rightarrow \|f(\ell) - f(\ell)\| \geq \varepsilon$ by continuity of f and $\|\cdot\|$

ie $0 \geq \varepsilon > 0$

Contradiction.

Hence f is not continuous at ℓ $\square$

A few exercises to practice U.C.

Ex 1: I interval, $f: I \to \mathbb{R}$

Prove that: f Lipschitz $\Rightarrow$ f U.C.

Ex 2: $I = (a, b)$, $a \in \mathbb{R}$, $b = \mathbb{R} \cup \{+\infty\}$

$f: I \to \mathbb{R}$

① Prove that: f U.C $\Rightarrow \lim\limits_{x \to a^+} f(x)$ exists and is finite

② Deduce that: $\lim\limits_{x \to a^+} f(x)$ DNE $\Rightarrow f$ is not U.C.

Ex 3: $f: [0, +\infty) \to \mathbb{R}$

① Prove that f UC $\Rightarrow \exists a, b \in \mathbb{R}, \forall x \in [0, +\infty), f(x) \leq ax + b$

Remark: we have a similar result on $(-\infty, 0]$, but not if the domain is $\mathbb{R}$ entirely (eg: $f(x) = |x|$ is U.C. but not "upper bounded" by an affine function)

② Deduce that if $\lim\limits_{x \to +\infty} \frac{f(x)}{x} = +\infty$ then f is not U.C.

③ Then if $\lim\limits_{x \to +\infty} \frac{f(x)}{x} = -\infty$ then f is not U.C.

Ex 4: $f: [a, +\infty) \to \mathbb{R}$: If $\begin{cases} f \text{ is continuous} \\ \text{and} \\ \lim\limits_{x \to +\infty} f(x) \in \mathbb{R} \\ \quad = \ell \end{cases}$ then f is U.C.

Ex 5: Prove that

① $x^2: \mathbb{R} \to \mathbb{R}$ is not U.C.

② $\tan: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is not U.C.

③ $\frac{1}{x}: (0, +\infty) \to \mathbb{R}$ is not U.C

④ $\exp: \mathbb{R} \to \mathbb{R}$ is not U.C.

⑤ $\exp: [-\pi, 42] \to \mathbb{R}$ is U.C.

⑥ $\sqrt{\cdot}: [0, +\infty) \to \mathbb{R}$ is U.C.

⑦ $\sqrt[3]{\cdot}: \mathbb{R} \to \mathbb{R}$ is U.C.

⑧ $\sin: \mathbb{R} \to \mathbb{R}$ is U.C.

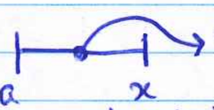
⑨ $\sin(x^2): \mathbb{R} \to \mathbb{R}$ is not U.C

⑩ $\sin(\frac{1}{x}): (0, 1) \to \mathbb{R}$ is not U.C.

Ex2 ① Check that if (x_n) is a sequence of I converging to a then $(f(x_n))$ is Cauchy

Hence $\ell = \lim f(x_n)$ exists

Then prove that $\lim\limits_{x\to a^+} f(x) = \ell$

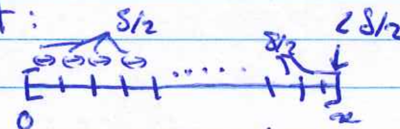
Hint:  $\to x_n$ for n big enough: $|a-x_n| < |a-x| < \delta$

So $|f(x)-\ell| \leq |f(x)-f(x_n)| + |f(x_n)-\ell| < \underbrace{\frac{\varepsilon}{2}}_{\text{UC}} + \underbrace{\varepsilon/2}_{\lim}$

② Contrapositive

Ex3 ① We know that $\exists \delta > 0, \forall x,y \in [0,+\infty), |x-y| < \delta \Rightarrow |f(x)-f(y)| < 1$ (*)

Let $x \in [0,+\infty)$. Divide $[0,x]$ in a partition such that:

$\forall k \in [0,n-2], \ x_{k+1}-x_k = \delta/2, \ x_n - x_{n-1} < \delta/2$: 

$$|f(x)-f(0)| = \left|\sum_{k=0}^{n-1} f(x_{k+1}) - f(x_k)\right| \leq \sum_{k=0}^{n-1} |f(x_{k+1})-f(x_k)|$$

$$(n-1)\delta/2 < x \quad (*) \quad \to \ \leq n < \frac{2}{\delta}x + 1$$

Hence $f(x) \leq \frac{2}{\delta}x + 1 + f(0)$

② By contrapositive: f UC $\Rightarrow f(x) \leq ax+b \Rightarrow \frac{f(x)}{x} \leq a + b/x$

③ Replace f by $-f$

Ex4: Let $\varepsilon > 0$, $\exists A > 0$ st. $\forall x \in [a,+\infty)$, $x \geq A \Rightarrow |f(x)-\ell| < \varepsilon/2$ (*)

Heine Cantor on $[a,A]$: $\exists \delta > 0, \forall x,y \in [a,A], |x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon/2$ (**)

Let $x,y \in [a,+\infty)$

Case1. $x,y \geq A$ then $|f(x)-f(y)| = |f(x)-\ell+\ell-f(y)| \leq |f(x)-\ell| + |f(y)-\ell| \overset{\text{by } (*)}{<} \frac{\varepsilon}{2} + \varepsilon/2$

Case2: $x,y \in [a,A]$, $|f(x)-f(y)| < \varepsilon/2 < \varepsilon$ by (**)

Case3: $x \in [a,A]$, $y \in [A,+\infty)$, $|f(x)-f(y)| = |f(x)-f(A)+f(A)-f(y)|$

$\leq |f(x)-f(A)| + |f(A)-f(y)|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$ by (**) and (*)

Ex 5 $\to$ some hints only

⑥ • $\sqrt{\cdot}$ is U.C. on $[0,1]$ by Heine-Cantor

• if $x, y \geq 1$ then $|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \frac{|x-y|}{2}$

so $\sqrt{\cdot}$ is U.C. on $[1,+\infty)$

then "patch" as in Ex 4 (Case 1: $x, y \in [0,1]$ Case 2: $x, y \geq 1$ Case 3: $x \in [0,1], y \geq 1$)

⑨ $x_n = \sqrt{n\pi + \frac{\pi}{2}}$ $y_n = \sqrt{n\pi}$

then or $x_n - y_n = \sqrt{n\pi + \frac{\pi}{2}} - \sqrt{n\pi} = \frac{n\pi + \frac{\pi}{2} - \pi n}{\sqrt{n\pi + \frac{\pi}{2}} + \sqrt{n\pi}} = \frac{\pi/2}{\sqrt{n\pi + \pi/2} + \sqrt{n\pi}} \xrightarrow[n\infty]{} 0$

but $\sin(x_n^2) = \pm 1$ & $\sin(y_n^2) = 0$

⑦ Similar to ⑥

② Method 1: $x_n = \frac{\pi}{2} - \frac{1}{n}$, $y_n = \frac{\pi}{2} - \frac{1}{2n}$, $|y_n - x_n| \xrightarrow[n\infty]{} 0$

but $|f(x_n) - f(y_n)| = \frac{1}{\sin(1/n)} \geq 1$

Method 2: Use Exo 2

For the others: use the previous exercises + Heine-Cantor

Infimum and Supremum (Recollection from MAT137)

Def: $A \subset \mathbb{R}$, $L, U \in \mathbb{R}$.
We say that:

- L is a lower bound of A if $\forall x \in A$, $L \leq x$
- U is an upper bound of A if $\forall x \in A$, $x \leq U$

Def: $A \subset \mathbb{R}$, $S \in \mathbb{R}$
We say that S is the supremum (or least upper bound) of A if

1. S is an upper bound of A
   ie: $\forall x \in A$, $x \leq S$
2. it is the least one
   ie: T is an upper bound of $A \Rightarrow S \leq T$

Def: $A \subset \mathbb{R}$, $I \in \mathbb{R}$
We say that I is the infimum (or greatest lower bound) of A if

1. I is a lower bound of A
   ie: $\forall x \in A$, $I \leq x$
2. it is the greatest one
   ie: J is a lower bound of $A \Rightarrow J \leq I$

Fundamental property of $\mathbb{R}$: "Dedekind completeness"

LUB principle: a non-empty subset of $\mathbb{R}$ which is bounded from above admits a supremum

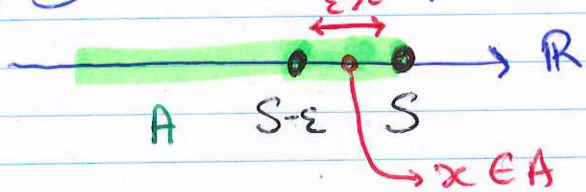
GLB principle: a non-empty subset of $\mathbb{R}$ which is bounded from below admits an infimum

<u>Theorem</u>: $A \subset \mathbb{R}$, $S, I \in \mathbb{R}$

$$S = \sup(A) \Leftrightarrow \begin{cases} \forall x \in A, & x \leq S \\ \forall \varepsilon > 0, \exists x \in A, & S - \varepsilon < x \end{cases}$$

$$I = \inf(A) \Leftrightarrow \begin{cases} \forall x \in A, & I \leq x \\ \forall \varepsilon > 0, \exists x \in A, & x < I + \varepsilon \end{cases}$$

Δ I am only going to prove the first one:



$\Rightarrow$: By definition, $\forall x \in A, x \leq S$

Let $\varepsilon > 0$, then $S - \varepsilon < S$ and hence $S - \varepsilon$ is not an upper bound of A since S is the least one.

<u>ie</u>: $\exists x \in A, S - \varepsilon < x$

$\Leftarrow$: $\forall x \in A, x \leq S$ means that S is an upper bound of A

Let's prove it is the least one.
We will prove the contrapositive: $T < S \Rightarrow T$ is not an upper bound
Let $T \in \mathbb{R}$. Assume that $T < S$.
Set $\varepsilon = S - T$, then $\varepsilon > 0$.
By assumption, $\exists x \in A$ s.t. $S - \varepsilon < x$ <u>ie</u> $T < x$
So T is not an upperbound

$\square$

Integration

Comment 1: Be sure that you remember the one-variable construction from last year

(There is a recap on my webpage)

Comment 2: We are going to talk about the integral of Darboux (but with several variables)

Historical comment, you can skip it

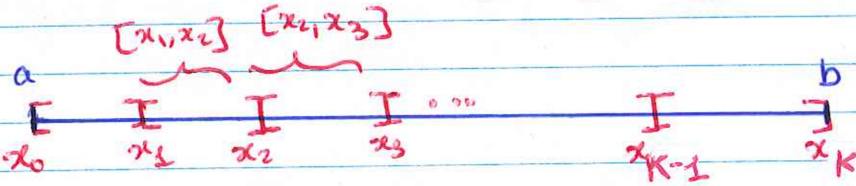
Since over $\mathbb{R}$ this construction gives the formerly defined Riemann's integral, it is common to simply call the result of both definitions "Riemann's integral"

Darboux integrable ⇒ Riemann integrable
Archimedean property
} is another characterization of the Dedekind-completeness of $\mathbb{R}$

Definition: A partition of the segment line $[a,b]$ is a finite subset of $[a,b]$ containing a and b.

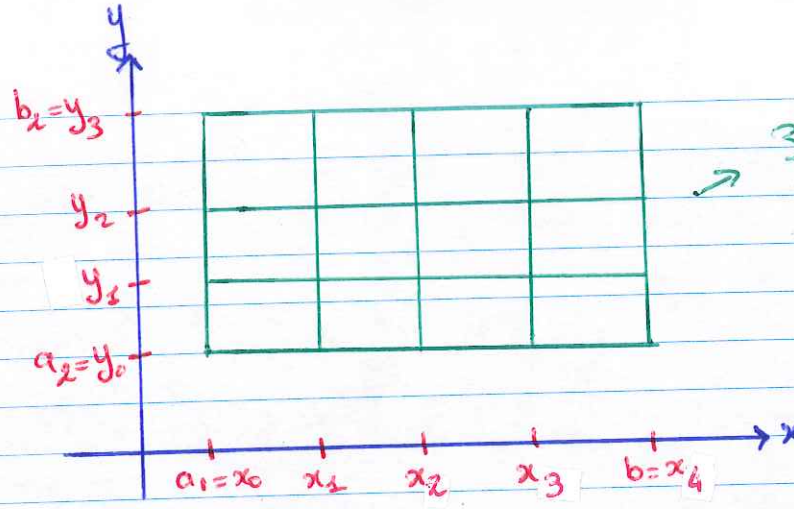
i.e. $P = \{a = x_0 < x_1 < \dots < x_K = b\}$

Intuitively, we break $[a,b]$ into finitely many closed subintervals.



Definition: A partition of the rectangle $[a_1,b_1] \times [a_2,b_2] \times \dots \times [a_m,b_m]$ is a collection $P = (P_1, \dots, P_m)$ where P_i is a partition of $[a_i,b_i]$

Intuitively, we break the rectangle into finitely many closed subrectangles



$\nearrow$ $3\times 4=12$ subrectangles

<u>Remark</u>: • a partition $P=\{a=x_0<x_1<\dots<x_K=b\}$ contains K subintervals

• a partition $P=(P_1,\dots,P_m)$ of $[a_1,b_1]\times\dots\times[a_m,b_m]$ contains $K_1\cdot K_2\cdots K_m$ subrectangles where

$$P_i=\{a_i=x_{i0}<x_{i1}<\dots<x_{iK_i}=b_i\}$$

<u>Definition</u>: the length of an interval $[a,b]$ is $v([a,b]):=b-a$

<u>Definition</u>: the volume of a rectangle $S=[a_1,b_1]\times\dots\times[a_m,b_m]$ is

$$v(S):=(b_1-a_1)\cdot(b_2-a_2)\cdots(b_m-a_m)$$
$$=v([a_1,b_1])\,v([a_2,b_2])\cdots v([a_m,b_m])$$

<u>Definition</u>: Let P,Q be two partitions of $[a,b]$. We say that Q is finer than P if $P\subset Q$

{ I recall that $P\subset Q$ means $P\subseteq Q$ (subset or equal)

<u>Definition</u>: Let $P=(P_1,\dots,P_m)$, $Q=(Q_1,\dots,Q_m)$ be 2 partitions of $[a_1,b_1]\times[a_2,b_2]\times\dots\times[a_m,b_m]$.

We say that Q is finer than P if $\forall i$, $P_i\subset Q_i$

Remark: Given two partitions P and Q, it is always possible to find a third partition O which is finer than both P and Q

Let $O = P \cup Q$ (or $O_i = P_i \cup Q_i$ for a rectangle)

Definition: Let $P = \{a = x_0 < x_1 < \dots < x_m = b\}$ be a partition of $[a,b]$ and $f : [a,b] \to \mathbb{R}$ be a bounded function

The upper-Darboux-sum of f with respect to P is

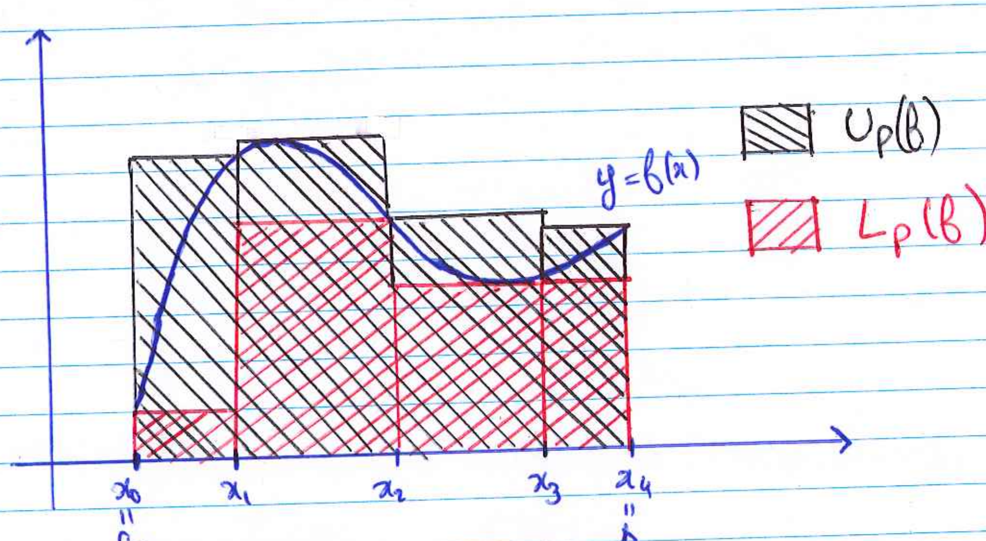
$$U_P(f) := \sum_{k=1}^{m} (x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} (f)$$

$$= \sum_I \partial(I) \sup_I (f) \quad \text{where } I \text{ varies through the subintervals of } P$$

The lower-Darboux-sum of f w.r.t. P is

$$L_P(f) := \sum_{k=1}^{m} (x_k - x_{k-1}) \inf_{[x_{k-1}, x_k]} (f)$$

$$= \sum_I \partial(I) \inf_I (f)$$



$$P = \{ \underset{a}{\overset{\shortparallel}{x_0}} < x_1 < \dots < \underset{b}{\overset{\shortparallel}{x_4}} \}$$

Definition: Let $P=(P_1,\dots,P_m)$ be a partition of $[a_1,b_1]\times\dots\times[a_m,b_m]$ and $f:[a_1,b_1]\times\dots\times[a_m,b_m]\longrightarrow\mathbb{R}$ be a bounded function

We define the upper-Darboux-sum of f w.r.t P by

$$U_P(f)=\sum_S{}' \,\partial(S)\,\sup_S(f)$$

and the lower-Darboux-sum of f w.r.t. P by

$$L_P(f)=\sum_S{}' \,\partial(S)\,\inf_S(f)$$

where S varies through the subrectangles of P.

Rem: The assumption "f bounded" ensures that $\sup_S f$ and $\inf_S f$ are well defined as the sup/inf of a non-empty bounded subset of $\mathbb{R}$

(Recall: $\sup_S f=\sup\{f(x): x\in S\}$, $\inf_S f=\inf\{f(x): x\in S\}$)

Proposition: R = rectangle or segment, $f:R\to\mathbb{R}$ bounded, P partition of R

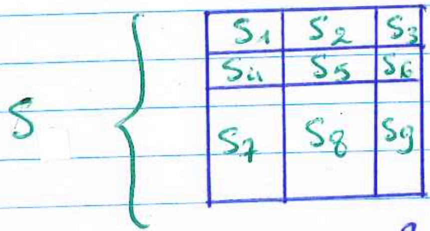
then $L_P(f)\leqslant U_P(f)$

Δ $L_P(f)=\sum_S{}' \,\partial(S)\inf_S(f)\leqslant\sum_S\partial(S)\sup_S(f)=U_P(f)$ $\square$

Proposition: R = rectangle or segment, $f:R\to\mathbb{R}$ bounded, P,Q are 2 partitions of R

If Q is finer than P then $\begin{cases}U_Q(f)\leqslant U_P(f)\\ L_P(f)\leqslant L_Q(f)\end{cases}$

Δ Assume that S is divided into subrectangles $S_1, \dots, S_9$



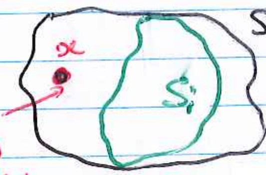
then $\displaystyle\sum_{i=1}^{9} \partial(S_i) \sup_{S_i}(f) \leq \sum_{i=1}^{9} \partial(S_i) \sup_{S}(f)$

$$= \left(\sum_{i=1}^{9} \partial(S_i)\right) \sup_S (f)$$

$$= \partial(S) \sup_S (f)$$

For this inequality, I used that $S_i \subset S \Rightarrow \sup_{S_i} \leq \sup_S$

i.e.: when you shrink the domain, you may decrease the supremum



max of f for $f(x)$, $x \in S \setminus S_i$

$\square$

<u>Corollary</u>: $f : R \to \mathbb{R}$ bounded, R = rectangle or segment, P, Q 2 partitions of R

$$L_P(f) \leq U_Q(f)$$

Δ Take Θ a partition of R which is finer than P and than Q

$$L_P(f) \leq L_\Theta(f) \leq U_\Theta(f) \leq U_Q(f)$$

$\square$

<u>Definition</u>: R = a rectangle or segment line, $f : R \to \mathbb{R}$ bounded

We define the Darboux lower integral of f by

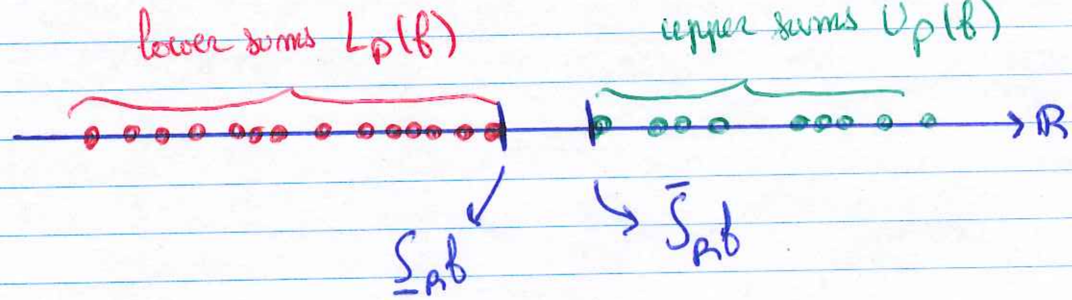
$$\underline{\int_R} f := \sup\{L_P(f) : P \text{ partition of } R\}$$

and the Darboux upper integral of f by

$$\overline{\int_R} f := \inf\{U_P(f) : P \text{ partition of } R\}$$

<u>Remark</u>: they are always well-defined (as soon as f is bounded)

indeed $\{U_P(f) : P \text{ partition of } R\}$ is not empty since it contains $U_{\{a<b\}}(f)$ for instance and it is bounded from below by $L_{\{a<b\}}(f)$ so $\inf\{U_P(f)\} =: \overline{\int}_R f$ exists and similarly for $\underline{\int}_R f$.



<u>Def</u>: $R :=$ rectangle or segment line, $f : R \longrightarrow \mathbb{R}$ bounded.

We say that f is (Darboux) integrable if $\underline{\int}_R f = \overline{\int}_R f$ and then we define

$$\int_R f := \underline{\int}_R f = \overline{\int}_R f$$

<u>Theorem</u> (ε-criterion for integrability)

R = rectangle or segment line, $f : R \longrightarrow \mathbb{R}$ bounded

f is integrable $\iff \forall \varepsilon > 0, \exists P$ partition of R s.t. $U_P(f) - L_P(f) < \varepsilon$

Δ $\Rightarrow$: By assumption

$$\sup\{L_P(f)\} =: \underline{\int_R} f = \overline{\int_R} f := \inf\{U_P(f)\}$$

Let $\varepsilon > 0$.

Since $\overline{\int_R} f + \frac{\varepsilon}{2} > \overline{\int_R} f$ and $\overline{\int_R} f$ is the GLB of $\{U_P(f)\}$

$\overline{\int_R} f + \frac{\varepsilon}{2}$ is not a lower bound of $\{U_P(f)\}$,

ie $\exists P_1$ partition s.t. $U_{P_1}(f) < \overline{\int_R} f + \frac{\varepsilon}{2}$

Similarly $\exists P_2$ partition s.t. $L_{P_2}(f) > \overline{\int_R} f - \frac{\varepsilon}{2}$

Take P a common refinement of P_1 and P_2 then

$$\begin{cases} U_P(f) \leq U_{P_1}(f) < \overline{\int_R} f + \frac{\varepsilon}{2} \\ L_P(f) \geq L_{P_2}(f) > \underline{\int_R} f - \frac{\varepsilon}{2} \end{cases}$$

$$\Rightarrow U_P(f) - L_P(f) < \cancel{\overline{\int_R} f} + \frac{\varepsilon}{2} - \cancel{\underline{\int_R} f} + \frac{\varepsilon}{2} = \varepsilon$$

are equal since f is integrable

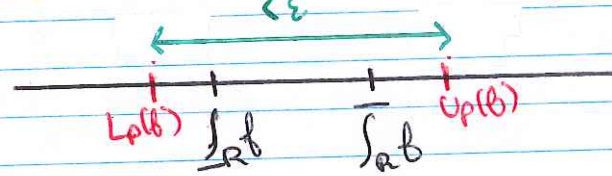
$\Leftarrow$: Let $\varepsilon > 0$, then $\exists$ a partition s.t. $U_P(f) - L_P(f) < \varepsilon$

But $L_P(f) \leq \underline{\int_R} f \leq \overline{\int_R} f \leq U_P(f)$

Hence $0 \leq \overline{\int_R} f - \underline{\int_R} f \leq U_P(f) - L_P(f) < \varepsilon$

ie: $\forall \varepsilon > 0$, $0 \leq \overline{\int_R} f - \underline{\int_R} f < \varepsilon$

$\Rightarrow \overline{\int_R} f = \underline{\int_R} f$



$\square$

__Theorem__: $f, g : R \to \mathbb{R}$ integrable, $c \in \mathbb{R}$ then

① $(f+g) : R \to \mathbb{R}$ is integrable and

$$\int_R (f+g) = \int_R f + \int_R g$$

② $(cf) : R \to \mathbb{R}$ is integrable and

$$\int_R (cf) = c \int_R f$$

③ $(fg) : R \to \mathbb{R}$ is integrable and

$$\left[\int_R (fg)\right]^2 \leqslant \int_R f^2 \int_R g^2$$ "Cauchy-Schwarz inequality"

④ If $\forall x \in R$, $f(x) \leqslant g(x)$

then $\int_R f \leqslant \int_R g$

⑤ $|f| : R \to \mathbb{R}$ is integrable and

$$\left| \int_R f \right| \leqslant \int_R |f|$$

⚠

- $\int (fg) \neq \int f \cdot \int g$; eg: $f(x) = \begin{cases} 1 & \text{on } [0, 1/2) \\ 0 & \text{on } [1/2, 1] \end{cases}$ $g(x) = \begin{cases} 1 & \text{on } (1/2, 1] \\ 0 & \text{on } [0, 1/2] \end{cases}$
- $|f|$ integrable $\not\Rightarrow$ f integrable ; eg $f(x) = \begin{cases} 1 & \text{on } \mathbb{Q} \cap [0,1] \\ -1 & \text{on } (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \end{cases}$
- $\int |f| \neq \left| \int f \right|$: eg $f(x) = x$ on $[-1, 1]$

The proofs may be difficult/technical if you directly start computing the upper/lower integrals. Below are some strategies to simplify the proofs.

Δ Sketch of proof:

① $\int f + \int g = \underline{\int} f + \underline{\int} g$ ← since f, g are integrable

$\leq \underline{\int}(f+g)$ ← Prove it from the definition of $L_P(f)$

$\leq \overline{\int}(f+g)$

$\leq \overline{\int} f + \overline{\int} g$

$= \int f + \int g$ ← since f, g are integrable

$\Rightarrow \underline{\int}(f+g) = \overline{\int}(f+g)$

since f is integrable

② If $c \geq 0$: $\underline{\int}(cf) = c\underline{\int} f = c\overline{\int} f = \overline{\int}(cf)$

To get the result for $c<0$: prove it using the Darboux sums

$\underline{\int}(-f) = -\overline{\int} f = -\underline{\int} f = \overline{\int}(-f)$ → since $\sup(-f) = -\inf(f)$

since f is integrable

since $\inf(-f) = -\sup(f)$

③ We first prove that f^2 is integrable:

$$|f(x)^2 - f(y)^2| = |f(x)+f(y)|\,|f(x)-f(y)|$$
$$\leq (|f(x)|+|f(y)|)\,|f(x)-f(y)|$$
$$\leq 2M\,|f(x)-f(y)|$$ since f is bounded: $\exists M>0, \forall x, |f(x)| \leq M$

$\Rightarrow \sup_S(f^2) - \inf_S(f^2) \leq 2M(\sup_S f - \inf_S f)$ where S is a rectangle

$\Rightarrow U_P(f^2) - L_P(f^2) \leq 2M(U_P(f) - L_P(f))$

and we conclude with the ε-criterion.

then notice that

$$(fg) = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$$ which is integrable by ①, ② and the above

We may adapt the proof of the usual CS inequality (Sep 5)

④ $\int g - \int f = \int (g-f) = \overline{\int}(\underbrace{g-f}_{\geq 0}) \geq 0$

prove it using the definition. $\forall p,\ U_p \geq 0 \Rightarrow \overline{\int} \geq 0$

⑤ From the reverse triangle inequality:

$$\forall x,y,\quad |f(x)| - |f(y)| \leq |f(x) - f(y)|$$

$$\Rightarrow \sup_S |f| - \inf_S |f| \leq \sup f - \inf f$$

$$\Rightarrow U_p(|f|) - L_p(|f|) \leq U_p(f) - L_p(f)$$

and we conclude with the ε-criterion

For the remaining inequality:

$$|f| \geq f \Rightarrow \int |f| \geq \int f$$

$$|f| \geq -f \Rightarrow \int |f| \geq \int (-f) = -\int f$$

hence $\int |f| \geq \left| \int f \right|$

□

<u>Theorem</u> : R = rectangle or segment line

If $f : R \to \mathbb{R}$ is continuous then f is integrable

Δ. First notice that f is bounded as a continuous function defined on a compact set : $\begin{cases} R \text{ compact} \\ f \ C^0 \end{cases} \Rightarrow f(R)$ compact $\Rightarrow f(R)$ bounded

• Next, since f is continuous on R compact, by Heine-Cantor theorem f is uniformly continuous :

Let $\varepsilon > 0$,

$$\exists \delta > 0, \ \forall x_1, x_2 \in R, \quad \|x_1 - x_2\| < \delta \Rightarrow |f(x_1) - f(x_2)| < \frac{\varepsilon}{2 v(R)}$$

Let P be a partition of R such that for any subrectangle S,

$$x_1, x_2 \in S \Rightarrow \|x_1 - x_2\| < \delta$$

Then

$$U_P(f) - L_P(f) = \sum_S v(S) \left(\sup_S f - \inf_S f \right)$$

$$\leqslant \sum_S v(S) \cdot \frac{\varepsilon}{2 v(R)}$$

$$= \frac{\varepsilon}{2 v(R)} \cdot \sum_S v(S)$$

$$= \frac{\varepsilon}{2} \frac{v(R)}{v(R)}$$

$$= \frac{\varepsilon}{2} < \varepsilon$$

So f is integrable by the ε-criterion $\square$

The FTC - Recollection from MAT137

Let $a<c<b$ and $f:[a,b]\to\mathbb{R}$ integrable then $f:[a,c]\to\mathbb{R}$ and $f:[c,b]\to\mathbb{R}$ are too and $\int_a^b f = \int_a^c f + \int_c^b f$

Hence it is natural to define $\int_b^a f := -\int_a^b f$ when $a>b$

Theorem: If $f:[a,b]\to\mathbb{R}$ continuous then $\exists c\in[a,b]$ s.t. $\int_a^b f = (b-a)f(c)$

Δ f is continuous on a compact hence $\exists s,S\in[a,b]$ s.t. $\forall x\in[a,b], f(s)\leq f(x)\leq f(S)$

$$\Rightarrow \underbrace{\int_a^b f(s)\,dx}_{=f(s)(b-a)} \leq \int_a^b f(x)\,dx \leq \underbrace{\int_a^b f(S)\,dx}_{=f(S)(b-a)}$$

$$\Rightarrow f(s) \leq \frac{\int_a^b f}{b-a} \leq f(S)$$

We conclude with the IVT $\square$

Theorem (FTC - Part 1)

Let I interval, $f:I\to\mathbb{R}$ continuous, and $a\in I$.

Define $F:I\to\mathbb{R}$ by $F(x)=\int_a^x f(t)\,dt$

Then F is differentiable and $F'=f$

Δ • f is continuous on $[a,x]$ if $x\geq a$ or $[x,a]$ otherwise so F is well-defined

• Let $x_0\in I$ then

$$F(x)-F(x_0) = \int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt = \int_{x}^{x_0} f(t)\,dt = (x-x_0)f(\xi)$$

for some $\xi\in[x,x_0]$ or $[x_0,x]$ by the above theorem

$$\Rightarrow \frac{F(x)-F(x_0)}{x-x_0} = f(\xi) \xrightarrow[\substack{x\to x_0\\ \Rightarrow \xi\to x_0}]{} f(x_0) \text{ since } f \text{ is } C^0$$

$\Rightarrow F'(x_0)=f(x_0)$ $\square$

Cor: Let $f: I \to \mathbb{R}$ be a C^0 function defined on an interval, $a \in I$.

If $F: I \to \mathbb{R}$ is an antiderivative of f

then $\exists\, C \in \mathbb{R}$ s.t. $F(x) = \int_a^x f(t)\,dt + C$

Δ F and $x \mapsto \int_a^x f$ are two antiderivatives of f on an interval

hence they differ by a constant by the MVT $\square$

Remark: If the domain is not an interval, it's possible to find two antiderivatives which don't differ by a const

eg: $F_1, F_2 : \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $F_1(x) = \ln|x|$, $F_2(x) = \begin{cases} \ln|x| + 42 & \text{for } x > 0 \\ \ln|x| - \pi & \text{for } x < 0 \end{cases}$

then $F_1' = F_2' = f$ for $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$ $f(x) = \frac{1}{x}$

but $F_1 - F_2 \neq$ constant

Theorem: (FTC – part 2) $f: [a,b] \to \mathbb{R}$ C^0, $F: [a,b] \to \mathbb{R}$ an antiderivative of f

then $\int_a^b f(t)\,dt = F(b) - F(a)$

Δ By the above $F(x) = \int_a^x f(t)\,dt + C$ for some C

hence $F(b) - F(a) = \int_a^b f(t)\,dt + C - \int_a^a f(t)\,dt - C = \int_a^b f(t)\,dt$ $\square$

Remark: we may replace f C^0 by f integrable in the above but the proof becomes a little bit more technical

⚠ If f is integrable but not C^0, $F(x) = \int_a^x f(t)\,dt$ may not be differentiable (but it is uniformly C^0)

⚠ It is not enough to have an antiderivative to be integrable

Eg: $F(x) = \begin{cases} x^2 \sin(\pi/x^2) & \text{on } (0,1] \\ 0 & \text{at } 0 \end{cases}$: F is differentiable but F' is not integrable (F' is not bounded)

Zero content sets

Def. We say that $S \subset \mathbb{R}^n$ has *zero content* if for every $\varepsilon > 0$ there exist finitely many rectangles (or segment lines if $n=1$) $R_1, \dots, R_q$ such that

① $\sum_{i=1}^{q} \mathcal{V}(R_i) < \varepsilon$

② $S \subset \bigcup_{i=1}^{q} R_i$

Exercises
- S has content zero $\Rightarrow S$ is bounded
- $[0,1]$ doesn't have zero content

Proposition.

① $\left.\begin{array}{l}\tilde{S} \subset S \\ S \text{ has zero content}\end{array}\right\} \Rightarrow \tilde{S}$ has zero content

② S has zero content $\Leftrightarrow \overline{S}$ has zero content

③ $S_1, \dots, S_r$ have zero content $\Rightarrow S = \bigcup_{i=1}^{r} S_i$ has zero content

④ S finite $\Rightarrow S$ has zero content

⑤ Let $R \subset \mathbb{R}^n$ be a rectangle (or segment line for $n=1$)
If $f : R \to \mathbb{R}$ is integrable then the graph of f
$$\Gamma_f = \{(x, f(x)) : x \in R\} \subset \mathbb{R}^{n+1}$$
has zero content

Δ ① Let $\varepsilon > 0$, since S has zero content, there exist finitely many rectangles $R_1, \dots, R_q$ s.t.

① $\sum_{i=1}^{q} \mathcal{V}(R_i) < \varepsilon$ ② $\tilde{S} \subset S \subset \bigcup_{i=1}^{q} R_i$

hence $\tilde{S}$ has zero content

② $\Leftarrow$: if $\overline{S}$ has zero content then $S \subset \overline{S}$ has too by ①
$\Rightarrow$: assume that S has zero content and let $\varepsilon > 0$. Then $\exists$ rectangles $R_1, \dots, R_q$ s.t. $\sum \mathcal{V}(R_i) < \varepsilon$ and $S \subset \bigcup R_i \Rightarrow \overline{S} \subset \overline{\bigcup R_i} = \bigcup R_i$ since $\bigcup R_i$ is closed as a finite union of closed sets. Hence $\overline{S}$ has zero content

③ Let $\varepsilon > 0$. Since S_i has zero content, $\exists R_1^i, \dots, R_{q_i}^i$ rectangles s.t.

$$\sum_{j=1}^{q_i} \partial(R_j^i) < \frac{\varepsilon}{r} \quad \text{and} \quad S_i \subset \bigcup_{j=1}^{q_i} R_j^i$$

then $S = \bigcup_{i=1}^{r} S_i \subset \bigcup_{i=1}^{r} \bigcup_{j=1}^{q_i} R_j^i$ and $\sum_{i=1}^{r} \sum_{j=1}^{q_i} \partial(R_j^i) < \sum_{i=1}^{r} \frac{\varepsilon}{r} = \varepsilon$

hence S has zero content

④ First assume that $S = \{p\}$ has only one element $p = (x_1, \dots, x_m)$

Let $\varepsilon > 0$, set $\delta = \sqrt[m]{\varepsilon/2}$

Set $R = [x_1 - \delta/2, x_1 + \delta/2] \times \dots \times [x_m - \delta/2, x_m + \delta/2]$.

then $S \subset R$ and $\partial(R) = \delta^m = \varepsilon/2 < \varepsilon$

hence S has zero content

Finally $S = \{P_1, \dots, P_r\} = \bigcup_{i=1}^{r} \{P_i\}$ has zero content by ③

⑤ Let $\varepsilon > 0$. Since f is integrable there exists a partition P s.t.

$\sum_S (\sup_S f - \inf_S f)\, \partial(S) < \varepsilon$ where S goes through the subrectangles of P

finitely many

Notice that $\Gamma_f \subset \bigcup_S \big(\underbrace{[\inf_S f, \sup_S f] \times S}_{\text{rectangle}}\big)$

and $\sum_S \partial([\inf_S f, \sup_S f] \times S) = \sum_S (\sup_S f - \inf_S f)\, \partial(S) < \varepsilon$ $\square$

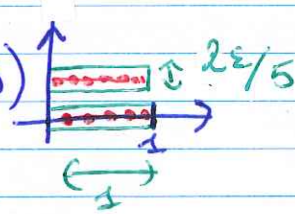
<u>Remark</u>: the converse of ⑤ is $\boxed{\text{false}}$. ⚠

Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

then f is not integrable.

Let $\varepsilon > 0$,

then $\Gamma_f \subset \big(\overbrace{[0,1] \times [1-\varepsilon/5, 1+\varepsilon/5]}^{R_1}\big) \cup \big(\overbrace{[0,1] \times [-\varepsilon/5, \varepsilon/5]}^{R_2}\big)$



and $\partial(R_1) + \partial(R_2) = \frac{2\varepsilon}{5} + \frac{2\varepsilon}{5} < \varepsilon$

hence Γ_f has zero content.

1

Theorem : R = rectangle or segment line, $f : R \to \mathbb{R}$ bounded

If $\{x \in R : f \text{ is not continuous at } x\}$ has zero content

then f is integrable

Δ Let $\varepsilon > 0$.

By assumption $\{x \in R : f \text{ is not continuous at } x\}$ has zero content

hence it may be covered by rectangles $S_1, \dots, S_q$ s.t.

(**) $$\sum_1 \mathcal{V}(S_i) < \frac{\varepsilon}{2(\sup_R f - \inf_R f)}$$ Here I assume that f is not constant otherwise it is obvious.

Without loss of generality, we may assume that they are subrectangles of a partition P of R and that the other subrectangles don't contain discontinuity points

Now, since the union K of the remaining subrectangles is compact, f is uniformly continuous on K.

Hence we may refine the partition s.t. $\displaystyle\sum_{S \text{ has no disc point}} (\sup_S f - \inf_S f) \cdot \mathcal{V}(S) < \frac{\varepsilon}{2}$ (*)
(as in the proof "$\mathcal{C} \Rightarrow$ integrable")

then $$\sum_S (\sup_S f - \inf_S f) \cdot \mathcal{V}(S)$$
$$= \sum_{S \text{ has no disc}} (\sup_S f - \inf_S f)\mathcal{V}(S) + \sum_{S \text{ has disc}} (\sup_S f - \inf_S f) \cdot \mathcal{V}(S)$$
$$= \qquad '' \qquad + \sum (\sup_R f - \inf_R f) \cdot \mathcal{V}(S)$$
$$< \quad \underbrace{\varepsilon/2}_{\text{by } (*)} \quad + \underbrace{\varepsilon/2}_{\text{by } (**)}$$

Hence f is integrable $\square$

Remark: the converse is false!

Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{1}{q} & \text{if } x = p/q,\ p \in \mathbb{Z}\setminus\{0\},\ q \in \mathbb{N}_{>0},\ \gcd(p,q) = 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

Then • f is integrable

• The discontinuity set of f is $[0,1] \cap \mathbb{Q}$

But • $[0,1] \cap \mathbb{Q}$ doesn't have zero content

indeed, assume by contradiction that $[0,1] \cap \mathbb{Q}$ has zero content

then $\overline{[0,1] \cap \mathbb{Q}} = [0,1]$ has zero content: contradiction. Δ

x — x

Remark (NOT PART OF MAT237)

If we replace "finitely many" by "countably many" in the definition of zero content set then we obtain the definition of "set of measure 0"

Then we have Lebesgue Criterion for Riemann's integrability:

Theorem: $f : \mathbb{R} \to \mathbb{R}$ bounded on a rectangle/segment line

f is integrable $\iff \{x \in \mathbb{R} : f \text{ is discontinuous}\}$ has measure 0

↳ both way now!

How to integrate on a set which is not a rectangle?

Let $S \subset \mathbb{R}^n$ be a bounded subset

We define the characteristic function of S by

$$\chi_S : \mathbb{R}^n \to \mathbb{R}, \quad \chi_S(x) = \begin{cases} 1 \text{ if } x \in S \\ 0 \text{ otherwise} \end{cases}$$

Let $f : S \to \mathbb{R}$ be a function

We say that f is integrable on S if there exists a rectangle $R \subset \mathbb{R}^n$ such that $S \subset R$ and $f\chi_S : R \to \mathbb{R}$ is integrable

We write $\int_S f = \int_R f\chi_S$

Remarks: ① $f\chi_S$ is an abuse of notation for the function

$$\begin{aligned} R &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} f(x) \text{ if } x \in S \\ 0 \quad \text{if } x \in R \setminus S \end{cases} \end{aligned}$$

② One may check that this definition is independent of the choice of the rectangle R: if R' is another suitable rectangle then $\int_R f\chi_S = \int_{R'} f\chi_S$

③ The basic properties ($f+g$, cf, $|f|$) remain true

④ $\int_{S_1 \cup S_2} f = \int_{S_1} f + \int_{S_2} f - \int_{S_1 \cap S_2} f$

⑤ Even if f is C^0, there is no reason for $\chi_S f$ to be (look at points on ∂S)

Exercise 1 : χ_S is discontinuous at $x \Leftrightarrow x \in \partial S$

Exercise 2 : $S \subset \mathbb{R}^n$ bounded, $f : S \to \mathbb{R}$ bounded

If $\begin{cases} \partial S \text{ has zero content} \\ \{x \in S : f \text{ is not } C^0 \text{ at } x\} \text{ has zero content} \end{cases}$

then f is integrable on S, ie $\int_S f$ is well-defined

Exercise 3 : If S has zero content and $f : S \to \mathbb{R}$ is bounded

then f is integrable on S and $\int_S f = 0$

Exercise 4 : $f, g : S \to \mathbb{R}$ are integrable

If $\{x \in S : f(x) \neq g(x)\}$ has zero content then $\int_S f = \int_S g$

Exercise 5 : If f is integrable on S and on T and $S \cap T$ has zero content then f is integrable on $S \cup T$ and

$$\int_{S \cup T} f = \int_S f + \int_T f$$

⚠ ∂S bounded $\not\Rightarrow$ S bounded

Ex : $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$ is not bounded

but $\partial S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is bounded

Solutions

Ex 1 : • If $x_0 \in \mathring{S}$, $\exists \varepsilon > 0$, $B(x_0, \varepsilon) \subset \mathring{S} \subset S$

and f is constant equal 1 on this ball

• If $x_0 \in (\overline{S})^c = (S^c)^\circ$ then $\exists \varepsilon > 0$, $B(x_0, \varepsilon) \subset (S^c)^\circ \subset S^c$

and f is constant equal 0 on this ball

• If $x_0 \in \partial S$ then for any $\delta > 0$,

$\exists y_1 \in B(x_0, \delta)$ s.t. $f(y_1) = 0$
$\exists y_2 \in B(x_0, \delta)$ s.t. $f(y_2) = 1$
} prove it! using the definition of ∂S

Ex 2 : Hint : the set of discontinuity of $\chi_S f : R \to \mathbb{R}$

is a subset of $\underbrace{\partial S}_{\text{zero content}} \cup \underbrace{\{x \in S : f \text{ not } C^0 \text{ at } x\}}_{\text{zero content}}$

where R is a rectangle containing S

Ex 3 : Let $\varepsilon > 0$, $M = \sup |f|$.
$\exists R_1, \dots, R_q$ rectangle s.t. $S \subset \bigcup_{i=1}^{q} R_i$ and $\sum_{i=1}^{q} \partial(R_i) < \varepsilon / 2M$

We may assume $R_1, \dots, R_q$ are subrectangles of a partition P of a rectangle R containing S.

Then, for $\chi_S f : R \to \mathbb{R}$

$$-\frac{\varepsilon}{2} = -\frac{\varepsilon M}{2M} < \sum_{i=1}^{q} \partial(R_i)(-M) \leq L_P(\chi_S f) \leq U_P(\chi_S f) \leq \sum_{i=1}^{q} \partial(R_i) M \leq \frac{\varepsilon M}{2M} = \varepsilon/2$$

$$\Rightarrow U_P(\chi_S f) - L_P(\chi_S f) < \varepsilon$$

So $\chi_S f$ is integrable on R and f is integrable on S

Ex 4: $h = f - g$ is zero except on a set T which has zero content

So h is integrable and $\int_S h = 0$ by the previous exo

$$\int_S h = \int_S (f-g) = \int_S f - \int_S g$$

since they are both integrable

$$\Rightarrow \int_S f = \int_S g$$

Ex 5: $\chi_{S\cup T} f = \chi_S f + \chi_T f - \chi_{S\cap T} f$

integrable by assumption

integrable by Exo 3 since $S \cap T$ has zero content

So $\chi_{S\cup T} f$ is integrable

and $\int_{S\cup T} f = \int \chi_{S\cup T} f = \int \chi_S f + \int \chi_T f - \int \chi_{S\cap T} f$

$= \int_S f + \int_T f - 0$

Iterated integrals

Theorem (Fubini's theorem)

Let $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^p$ be two rectangles

Let $f : A \times B \to \mathbb{R}$ be integrable ⚠

Define $l, u : A \longrightarrow \mathbb{R}$ by

$$\left.\begin{aligned} l(x) &:= \underline{\int_B} f(x,y)\,dy \\ u(x) &:= \overline{\int_B} f(x,y)\,dy \end{aligned}\right\}$$

For $x \in A$ fixed, $f_x : \overset{B \longrightarrow \mathbb{R}}{y \longmapsto f(x,y)}$ is bounded.
Hence $\underline{\int_B} f_x$ and $\overline{\int_B} f$ are well defined
However there is no reason for f_x to be integrable
ie: it is possible that $l(x) \neq u(x)$

Then l and u are integrable and

$$\int_{A\times B} f = \int_A l(x)\,dx = \int_A \left(\underline{\int_B} f(x,y)\,dy \right) dx$$

$$\int_{A\times B} f = \int_A u(x)\,dx = \int_A \left(\overline{\int_B} f(x,y)\,dy \right) dx$$

⚠ Remark: it is possible for $f_x : \overset{B \longrightarrow \mathbb{R}}{y \longmapsto f(x,y)}$ to not be integrable

ie the lower/upper integral is important, we can't omit it!

Ex: $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$, $f(x,y) = \begin{cases} 1 & \text{if } x = 1/2 \text{ and } y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

then $\underline{\int_B} f(1/2, y)\,dy = 0 \neq \overline{\int_B} f(1/2,y)\,dy$ so $f_{1/2}$ is not integrable

Nevertheless f is integrable and: $0 = \int_{A\times B} f = \int_A \left(\underline{\int_B} f(x,y)\,dy \right) dx$
$= \int_A \left(\overline{\int_B} f(x,y)\,dy \right) dx$

Ex: even worse: $f: [0,1]\times[0,1] \longrightarrow \mathbb{R}$

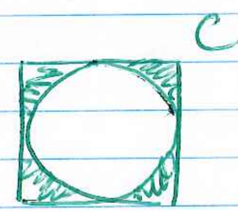
$$(x, y) \longmapsto \begin{cases} 1 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q},\ y \notin \mathbb{Q} \\ 1 - 1/q & \text{if } x = p/q,\ \gcd(p,q)=1,\ p,q \in \mathbb{N}_{\geq 0},\ q \neq 0,\ y \in \mathbb{Q} \end{cases}$$

$$\int_{[0,1]\times[0,1]} f = 1$$

$$\int_0^1 f(x,y)\,dy = \begin{cases} 1 & \text{if } x \notin \mathbb{Q} \\ \text{DNE} & \text{if } x \in \mathbb{Q} \end{cases}$$

Ex: how to use Fubini's theorem to compute an integral

$$C = \left\{ (x,y) \in [-1,1]^2 : \|(x,y)\| \geq 1 \right\}$$



$$\int_C f = \int_{[-1,1]\times[-1,1]} \chi_C f$$

$$= \int_{-1}^{1} \left(\int_{-1}^{1} f(x,y)\, \chi_C(x,y)\, dy \right) dx$$

$$= \int_{-1}^{1} \left(\int_{-1}^{-\sqrt{1-x^2}} f(x,y)\,dy + \int_{\sqrt{1-x^2}}^{1} f(x,y)\,dy \right) dx$$

since $\chi_C(x,y) = \begin{cases} 1 & \text{if } y \geq \sqrt{1-x^2} \text{ or } y \leq -\sqrt{1-x^2} \\ 0 & \text{otherwise} \end{cases}$

I do the proof for ℓ only, the proof for u is similar

1 Let $\varepsilon > 0$

Then there exists a partition P of $A \times B$ s.t. $U_P(f) - L_P(f) < \varepsilon$

P induces a partition P_A of A and a partition P_B of B

(a subrectangle S of P is of the form $S_A \times S_B$)

$$L_P(f) = \sum_S{}' \partial(S) \inf_S (f)$$

$$= \sum_{S_A, S_B}{}' \partial(S_A \times S_B) \inf_{S_A \times S_B} (f)$$

$$(*) \quad L_P(f) = \sum_{S_A}{}' \left(\sum_{S_B}{}' \inf_{S_A \times S_B} (f) \, \partial(S_B) \right) \partial(S_A)$$

If S_A is fixed and $x \in S_A$, then:

$$\sum_{S_B}{}' \inf_{S_A \times S_B} (f) \, \partial(S_B) \leqslant \sum_{S_B}{}' \inf_{y \in S_B} (f(x,y)) \, \partial(S_B)$$

$$\leqslant \underline{\int}_B f(x,y)\, dy = \ell(x)$$

(We shrink the domain: $\{x\} \times S_B \subset S_A \times S_B$
$\Rightarrow \inf_{S_A \times S_B} f \leqslant \inf_{\{x\} \times S_B} f$)

So $\sum_{S_B}{}' \inf_{S_A \times S_B} (f) \, \partial(S_B) \leqslant \inf_{S_A} \ell(x)$

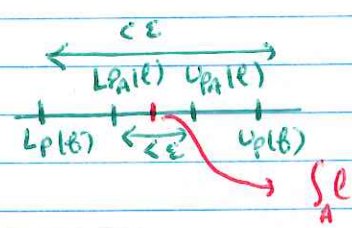
Hence $(*)$ gives:

$$L_P(f) = \sum_{S_A}{}' \left(\sum_{S_B}{}' \inf_{S_A \times S_B} (f) \, \partial(S_B) \right) \partial(S_A) \leqslant \sum_{S_A}{}' \left(\inf_{S_A} \ell(x) \right) \partial(S_A) = L_{P_A}(\ell)$$

ie $L_P(f) \leqslant L_{P_A}(\ell) \quad (**)$

and similarly, we could prove $U_{P_A}(u) \leqslant U_P(f) \quad (***)$

So: $L_P(f) \underset{(**)}{\leqslant} L_{P_A}(\ell) \underset{\text{always true}}{\leqslant} U_{P_A}(\ell) \underset{\text{since } \ell \leqslant u}{\leqslant} U_{P_A}(u) \underset{(***)}{\leqslant} U_P(f)$



then $U_{P_A}(\ell) - L_{P_A}(\ell) \leqslant U_P(f) - L_P(f) < \varepsilon$

So ℓ is integrable and moreover: $\int_{A \times B} f = \int_A \ell$

$\square$

<u>Corollary</u>: Let $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^p$ be two rectangles
Let $f: A\times B \to \mathbb{R}$

If (i) $f: A\times B \longrightarrow \mathbb{R}$ is integrable

(ii) $\forall x \in A$, $f_x: B \to \mathbb{R}$ defined by $f_x(y) = f(x,y)$ is integrable

Then

(1) $g: A \longrightarrow \mathbb{R}$ defined by $g(x) = \int_B f_x(y)\,dy = \int_B f(x,y)\,dy$ is integrable.

(2) $\int_{A\times B} f = \int_A g = \int_A \left(\int_B f(x,y)\,dy\right) dx$

Δ (ii) ensures that $g = U = L$ in Fubini's theorem $\square$

<u>Corollary</u>. $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^p$ rectangles, $f: A\times B \longrightarrow \mathbb{R}$ continuous

then $\int_{A\times B} f = \int_A \left(\int_B f(x,y)\,dy\right) dx$

Δ f and f_x are continuous and hence integrable and we may apply the above corollary $\square$

⚠ We can NOT weaken the assumption to assume that the discontinuity set has ZC

$f: [0,1]\times[0,1] \longrightarrow \mathbb{R}$, $f(x,y) = \begin{cases} 1 & \text{if } x = 1/2,\ y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

The discontinuity set of f has ZC but the discontinuity set of $f_{1/2}: y \mapsto f(1/2, y)$ doesn't have ZC and $\int_{[0,1]} f(1/2, y)\,dy$ is not defined.

Change of variables formulae

Heuristic

Let $\varphi : J \to I$ be a C^1-diffeomorphism between 2 bounded intervals.

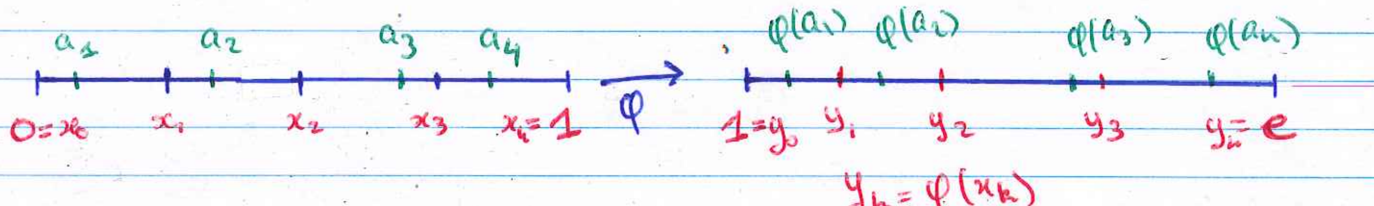
We want to compute $\int_I f$ for $f : I \to \mathbb{R}$ continuous

Since $\varphi : J \to I$ is a bijection between J and I preserving regularity it looks like that we can write $\int_I f = \int_J f \circ \varphi$.

Let's try on an example: $f(x) = x$, $\varphi : [0,1] \to [1,e]$, $\varphi(x) = e^x$

$$\int_{[1,e]} f = \frac{e^2}{2} - \frac{1}{2} \neq \int_{[0,1]} f \circ \varphi = e - 1$$

What went wrong?



A Riemann sum for $f \circ \varphi$ is $f(\varphi(a_1))(x_1 - x_0) + f(\varphi(a_2))(x_2 - x_1) + \cdots$

Notice that it involves $(x_{k+1} - x_k)$ and not $(y_{k+1} - y_k)$ the partition induces by φ on $[1,e]$ from the partition on $[0,1]$.

It doesn't take into account the speed of φ...

We would like to find F on $[0,1]$ s.t.

$$\int_{[0,1]} F = \int_{[1,e]} f$$

At the level of Riemann sums we would like

$$F(a_1)(x_1-x_0)+F(a_2)(x_2-x_1)+F(a_3)(x_3-x_2)+\cdots$$

to be equal to (at least when the step $x_{k+1}-x_k$ goes to 0)

$$f(\varphi(a_1))\,|y_1-y_0|+f(\varphi(a_2))\,|y_2-y_1|+f(\varphi(a_3))\,|y_3-y_2|+\cdots$$

Comment: absolute values because φ could be decreasing

We can set for example

$$F(a_k)(x_{k+1}-x_k)=f(\varphi(a_k))\,|\underbrace{y_{k+1}}_{=\varphi(x_{k+1})}-\underbrace{y_k}_{=\varphi(x_k)}|$$

ie $F(a_k)=f(\varphi(a_k))\left|\dfrac{\varphi(x_{k+1})-\varphi(x_k)}{x_{k+1}-x_k}\right| \xrightarrow[x_{k+1}-x_k\to 0]{} f(\varphi(x))\cdot|\varphi'(x)|$

ie $\displaystyle\int_I f=\int_J f\circ\varphi\cdot|\varphi'|$

Remark: in the MAT137 the absolute value was hidden in the fact that $\int_\alpha^\beta=-\int_\beta^\alpha$

So if φ is decreasing ie $\varphi'<0$ $\displaystyle\int_{\varphi(\alpha)}^{\varphi(\beta)} f\circ\varphi\cdot\varphi'=\int_{\varphi(\beta)}^{\varphi(\alpha)} f\circ\varphi\cdot|\varphi'|$ with $\varphi(\beta)<\varphi(\alpha)$

We won't be able to use this trick in the multivariable case ie the absolute values are going to be important

Conclusion: $\displaystyle\int_I f\neq\int_J f\circ\varphi$ if $\varphi:J\to I$ is a C^1-diffeomorphism

but $\displaystyle\int_I f=\int_J f\circ\varphi\cdot|\varphi'|$

⚠ The above discussion is informal, that's not a proof but it explains well the situation that we are going to clarify now

The one variable case (Recollection from MAT137/MAT157)

Theorem: $I \subset \mathbb{R}$ interval, $\varphi : [a,b] \longrightarrow I$ C^1, $f : I \longrightarrow \mathbb{R}$ C^0

then $\int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx$

Remarks: ① the functions involved are integrable since C^0 on segment lines

that will be a necessary assumption in the multivariable case

② we don't assume φ to be injective

eg: we can compute $\int_{-\sqrt{\pi/2}}^{\sqrt{\pi/2}} 2t\cos(t^2)\,dt$ using $\varphi(t) = t^2$
even if φ is not injective

③ We may rewrite: $\int_{[a,b]} f(\varphi(t))|\varphi'(t)|\,dt = \int_{\varphi([a,b])} f(x)\,dx$

when φ is monotonic (ie nondecreasing or nonincreasing)

Proof: Since f is continuous, it admits an antiderivative $F : I \to \mathbb{R}$

Notice that $(F\circ\varphi)' = F'\circ\varphi \cdot \varphi' = f\circ\varphi \cdot \varphi'$

Hence
$$\int_a^b f\circ\varphi(t)\,\varphi'(t)\,dt = \int_a^b (F\circ\varphi)'(t)\,dt$$
$$= [F\circ\varphi]_a^b$$
$$= [F]_{\varphi(a)}^{\varphi(b)}$$
$$= \int_{\varphi(a)}^{\varphi(b)} F'(x)\,dx = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx$$
$\square$

Remarks: It is possible to weaken the assumptions, but the proof becomes more complicated (cf Thm 39 in darboux.pdf)

• Ex (among others): If φ is monotonic, we may simply assume that f is integrable:

ie: If $f : [a,b] \to \mathbb{R}$ is integrable, $\varphi : [c,d] \to \mathbb{R}$ monotonic, $\varphi([c,d]) \subset [a,b]$ and φ' integrable then $\int_{\varphi(c)}^{\varphi(d)} f(x)\,dx = \int_c^d f(\varphi(t))\,\varphi'(t)\,dt$

Mnemonic device

$$\int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(u)\,du \qquad (*)$$

So if you write $u = \varphi(t)$

and then $\dfrac{du}{dt} = \varphi'(t)$

you recover $du = \varphi'(t)\,dt$

That's just a mnemonic device to remember the proved formula $(*)$

That's not a correct mathematical proof or reasoning

Do NOT forget to change the bounds:

$$\int_{a}^{b} f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(u)\,du$$

The multivariable case

Theorem: • $\mathcal{U} \subset \mathbb{R}^n$ open, $\Phi : \mathcal{U} \longrightarrow \mathbb{R}^n$ C^1 injective and $\forall x \in \mathcal{U}$, $\det D\Phi(x) \neq 0$

• Let $T \subset \mathcal{U}$ be a compact Jordan measurable set (ie ∂T has ZC)

• If f is integrable on $\Phi(T)$ then $f \circ \Phi \cdot |\det D\Phi|$ is integrable on T and

$$\int_{\Phi(T)} f(x)\, dx = \int_T f(\Phi(u))\, |\det D\Phi(u)|\, du$$

Remark: ① We have already seen that for $\Phi : \mathcal{U} \to \mathbb{R}^n$ injective (January 16, Transformations)

$$\forall x \in \mathcal{U},\ D\Phi(x) \text{ invertible} \iff \begin{cases} \Phi(\mathcal{U}) \text{ is open} \\ \Phi : \mathcal{U} \longrightarrow \Phi(\mathcal{U}) \text{ is a } C^1 \text{ diffeo} \end{cases}$$

So the condition on Φ is simply that $\Phi : \mathcal{U} \longrightarrow \mathcal{V}$ is a C^1-diffeomorphism where $\mathcal{V} = \Phi(\mathcal{U})$

② $\Phi(T)$ is compact as the continuous image of a compact and $\Phi(T)$ is Jordan measurable (ie $\partial(\Phi(T))$ has ZC

Δ Idea of proof for ②
- ∂T is closed and bounded, hence compact
- Since Φ is C^1 on ∂T compact, $\exists C > 0$, $\forall x, y \in \partial T$, $\|\Phi(x) - \Phi(y)\| \leq C\|x - y\|$
  [See the file "A MVT like inequality" for details]
- Hence $\Phi(\partial T)$ has ZC
- Since Φ is a homeomorphism $\partial(\Phi(T)) = \Phi(\partial T)$
- Ccl: $\partial(\Phi(T))$ has ZC, ie $\Phi(T)$ is Jordan measurable

□

ie we present an idea describing the geometric intuition, it's not a proof!!! ⚠

It is possible to generalize the heuristic idea from before:

$$\int_{\Phi(T)} f(x)\,dx \simeq \sum_S{}' f(\Phi(a_S)) \cdot Vol(\Phi(S))$$

← (where S is a subrectangle of a partition of a rectangle containing T)

$$\simeq \sum_S{}' f(\Phi(a_S)) \cdot \frac{Vol(\Phi(S))}{Vol(S)} \cdot Vol(S)$$

when $Vol(S) \to 0$ ⟶

$$\simeq \sum_S{}' f(\Phi(a_S)) \, |\det D\Phi(a_S)| \, Vol(S)$$

$$\simeq \int_T f \circ \Phi \cdot |\det D\Phi|$$

↳ that's actually the geometric interpretation of the Jacobian determinant:

DIFFICULT, NOT MANDATORY:

After a translation, we may assume that $0 \in U$ and that $\Phi(0) = 0$, and then use the following lemma:

NOT PART OF MAT351 BUT USEFUL

Lemma: $U, V \subset \mathbb{R}^n$ two open sets containing 0, $f : U \to V$ C^1-diffeo, $0 \mapsto 0$

$\forall \varepsilon > 0, \exists R > 0, \forall r \in [0, R], \; (1-\varepsilon) Df(0)(B_r) \subset f(B_r) \subset (1+\varepsilon) Df(0)(B_r)$

DIFFICULT, DON'T READ IT ⚠

△ $f(x) - Df(0)x = \eta(x)$ where $\frac{\eta(x)}{\|x\|} \xrightarrow[x \to 0]{} 0$ by differentiability

so $\|f(x) - Df(0)x\| \leq C \varepsilon' \|x\|$ for x close to 0

$\Rightarrow \|Df(0)^{-1} f(x) - x\| = \|Df(0)^{-1}(f(x) - Df(0)x)\| \leq \underbrace{\|Df(0)^{-1}\| \cdot C \cdot \varepsilon' \|x\|}_{\text{take } \varepsilon' \text{ s.t. it is } = \varepsilon}$ for x close to 0 ie in a small ball B_R

$\Rightarrow \|Df(0)^{-1} f(x)\| \leq (1+\varepsilon) \|x\|$

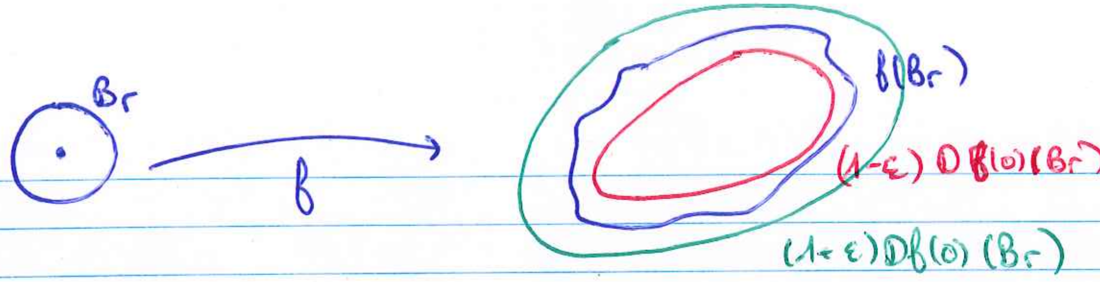
hence the second inclusion

For the first one: $y \xmapsto{\varphi} (1-\varepsilon) Df(0) y$ is C^0 so $\exists K$ s.t. $\varphi(B(0,K)) \subset \varphi(B_R)$

then if $(1-\varepsilon) Df(0) y = f(x)$, $\|x\| - (1-\varepsilon)\|y\| \leq \|(1-\varepsilon) y - x\| = \|Df(0)^{-1}(f(x)) - x\| \leq \varepsilon \|x\|$

from which we deduce the first inclusion

∎



and
$$\mathrm{Vol}\big((1+\varepsilon)\,Df(0)\,(B_r)\big) = \big|\det\big((1+\varepsilon)\,Df(0)\big)\big|\ \mathrm{Vol}(B_r)$$
$$= (1+\varepsilon)^m\,|\det Df(0)|\ \mathrm{Vol}(B_r)$$

So
$$(1-\varepsilon)^m\,|\det Df(0)| \leq \frac{\mathrm{Vol}(f(B_r))}{\mathrm{Vol}(B_r)} \leq (1+\varepsilon)^m\,|\det Df(0)|$$

END OF THE HEURISTIC IDEA
(which can be fixed to become a formal difficult proof)

<u>Historical comment:</u> the above idea may be fixed and made correct, see for instance Folland

• There exists another proof, by induction on m, which is more computational, easier, but hides a little bit the geometric idea. See for instance Spivak or Munkres

University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus

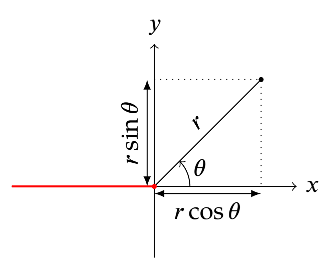
Change of variables: usual coordinate systems

Jean-Baptiste Campesato

February 13th, 2020

1 Polar coordinates

$$\Phi : \begin{array}{ccc} (0,+\infty)\times(-\pi,\pi) & \to & \mathbb{R}^2 \setminus \{(x,0)\in\mathbb{R}^2 \,:\, x\le 0\} \\ (r,\theta) & \mapsto & (r\cos\theta, r\sin\theta) \end{array}$$



- Φ is C^1.
- Φ is bijective.
- $\det D\Phi(r,\theta) = \det\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix} = r\left(\cos^2\theta+\sin^2\theta\right) = r > 0.$
- Hence Φ is a C^1-diffeomorphism.
- And $|\det D\Phi(r,\theta)| = r$.

Example 1. Let $\Delta = \left\{(x,y)\in\mathbb{R}^2 \,:\, 1\le x^2+y^2\le 9,\ x\ge 0\right\}$.
We want to compute $\displaystyle\int_\Delta e^{x^2+y^2}$.

First, notice that $\Delta = \Phi\left([1,3]\times[-\pi/2,\pi/2]\right)$.
Hence,

$$\begin{aligned}\int_\Delta e^{x^2+y^2}dxdy &= \int_{[1,3]\times[-\pi/2,\pi/2]} e^{r^2} r dr d\theta && \text{by the CoV formula} \\ &= \int_{-\pi/2}^{\pi/2}\int_1^3 re^{r^2} dr d\theta && \text{by the iterated integrals theorem} \\ &= \int_{-\pi/2}^{\pi/2} \frac{e^9-e}{2} d\theta \\ &= \frac{\pi}{2}\left(e^9-e\right)\end{aligned}$$

Example 2. We want to compute $\displaystyle\int_{\overline{B}((1,1),1)} x^2+y^2-2y\,dx\,dy$.

First notice that $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\Psi(x,y)=(x+1,y+1)$ is a C^1-diffeomorphism and that $|\det D\Psi(x,y)|=1$.
Moreover $\overline{B}((1,1),1)=\Psi\left(\overline{B}(\mathbf{0},1)\right)$. Hence

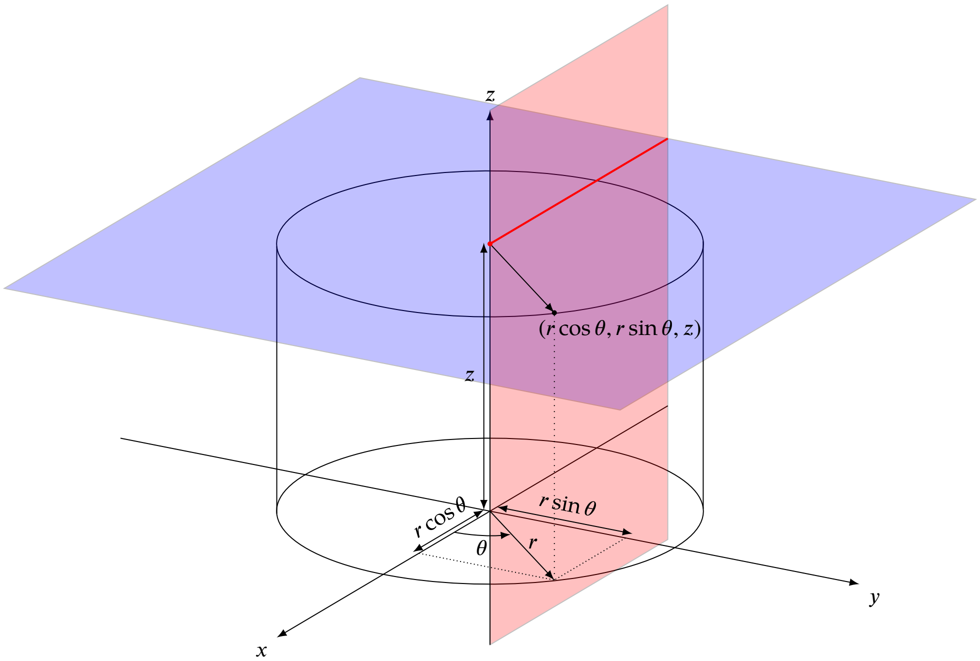
$$\begin{aligned}\int_{\overline{B}((1,1),1)} x^2+y^2-2y\,dx\,dy &= \int_{\overline{B}(\mathbf{0},1)} (x+1)^2+(y+1)^2-2(y+1)\,dx\,dy \qquad \text{by the CoV formula}\\ &= \int_{\overline{B}(\mathbf{0},1)} x^2+y^2-2x\,dx\,dy\end{aligned}$$

Next, we have $\overline{B}(\mathbf{0},1)=\Phi([0,1]\times[-\pi,\pi])$.
Notice that there is an issue for $r=0$ or $\theta=\pm\pi$ (i.e. $\{(x,0)\ :\ x\in[-1,0]\}$) but these sets have zero content.
Hence

$$\begin{aligned}\int_{\overline{B}(\mathbf{0},1)} x^2+y^2-2x\,dx\,dy &= \int_{[0,1]\times[-\pi,\pi]} (r^2-2r\cos\theta)r\,dr\,d\theta \qquad \text{by the CoV formula}\\ &= \int_{-\pi}^{\pi}\int_0^1 r^3-2r^2\cos\theta\,dr\,d\theta \qquad \text{by the iterated integrals theorem}\\ &= \int_{-\pi}^{\pi} \frac{1}{4}-\frac{2}{3}\cos\theta\,d\theta\\ &= \frac{\pi}{2}\end{aligned}$$

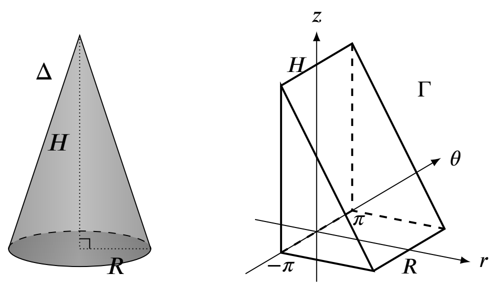
2 Cylindrical coordinates

$$\Phi : \begin{array}{ccc} (0,+\infty)\times(-\pi,\pi)\times\mathbb{R} & \to & \mathbb{R}^3\setminus((-\infty,0]\times\{0\}\times\mathbb{R}) \\ (r,\theta,z) & \mapsto & (r\cos\theta, r\sin\theta, z) \end{array}$$



- Φ is C^1.
- Φ is bijective.
- $\det D\Phi(r,\theta,z) = \det\begin{pmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} = \det\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix} = r > 0.$
- Hence Φ is a C^1-diffeomorphism.
- And $|\det D\Phi(r,\theta,z)| = r$.

Example 3. We want to compute $\int_\Delta z dx dy dx$ where $\Delta = \{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 \le \frac{R^2}{H^2}(H - z)^2,\ 0 \le z \le H\}$.



Notice that $\Delta = \Phi(\Gamma)$ where $\Gamma = \left\{(r, \theta, z) \;:\; 0 \le r \le \frac{R}{H}(H - z),\ \theta \in [-\pi, \pi],\ z \in [0, H]\right\}$.
Again, Γ goes outside the domain of Φ but the involved sets have zero content.
Hence

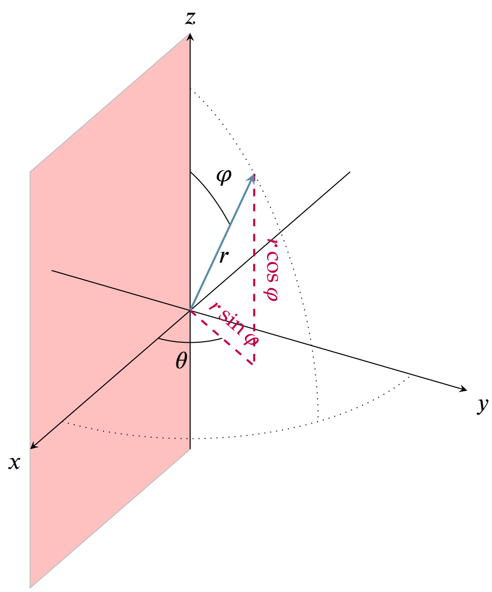
$$\begin{aligned}
\int_\Delta z dx dy dx &= \int_\Gamma z r dr d\theta dz \qquad \text{by the CoV formula} \\
&= \int_0^H \int_{-\pi}^{\pi} \int_0^{\frac{R}{H}(H-z)} r z dr d\theta dz \\
&= \int_0^H \int_{-\pi}^{\pi} \frac{R^2}{2H^2}(H - z)^2 z d\theta dz \\
&= \frac{\pi R^2}{H^2} \int_0^H (H - z)^2 z dz \\
&= \frac{\pi R^2 H^2}{12}
\end{aligned}$$

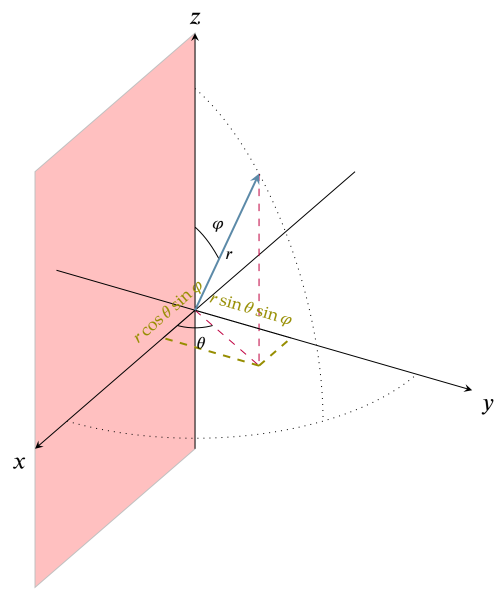
3 Spherical coordinates

$$\Phi : \begin{array}{ccc} (0,+\infty)\times(0,2\pi)\times(0,\pi) & \to & \mathbb{R}^3 \setminus ([0,+\infty)\times\{0\}\times\mathbb{R}) \\ (r,\theta,\varphi) & \mapsto & (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi) \end{array}$$

In this course, we use the following convention ⋆ :

$$(r,\theta,\varphi) = (\text{radius/distance to the origin, longitude, colatitude})$$





- Φ is C^1.
- Φ is bijective.
- The Jacobian determinant is

$$\begin{aligned}
\det D\Phi(r,\theta,\varphi) &= \det\begin{pmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\varphi & 0 & -r\sin\varphi \end{pmatrix} \\
&= \cos\varphi\det\begin{pmatrix} -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{pmatrix} - r\sin\varphi\det\begin{pmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi \end{pmatrix} \\
&= r^2\cos^2\varphi\sin\varphi\det\begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} - r^2\sin^3\varphi\det\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
&= -r^2\cos^2\varphi\sin\varphi - r^2\sin^3\varphi \\
&= -r^2\sin\varphi < 0 \text{ since } \varphi\in(0,\pi)
\end{aligned}$$

- Hence Φ is a C^1-diffeomorphism.
- And $|\det D\Phi(r,\theta,\varphi)| = r^2\sin\varphi$.

⋆ This convention may differ from the one used in other courses in math or in physics (the meaning of θ and φ may be swapped, some people use the latitude and not the colatitude...).
I believe that the usual convention in physics is $(r,\theta,\varphi) =$ (radius, colatitude, longitude) as in ISO 80000-2, i.e. the meaning of θ and φ are swapped from our convention in MAT237.

Example 4. We want to compute $\int_\Delta z dx dy dz$ where $\Delta = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 \leq 1, \ z \geq 0\right\}$.
Notice that $\Delta = \Phi\left([0, 1] \times [0, 2\pi] \times [0, \pi/2]\right)$.
Again, there is an issue with the domain of Φ but the involved sets have zero content.
Hence

$$\begin{aligned}
\int_\Delta z dx dy dz &= \int_{[0,1]\times[0,2\pi]\times[0,\pi/2]} r^3 \cos\varphi \sin\varphi dr d\theta d\varphi && \text{by the CoV formula} \\
&= \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 r^3 \frac{\sin(2\varphi)}{2} dr d\theta d\varphi && \text{by the iterated integrals theorem} \\
&= 2\pi \frac{1}{4} \left(\frac{\cos 0}{4} - \frac{\cos \pi}{4} \right) \\
&= \frac{\pi}{4}
\end{aligned}$$

Functions of the form $F(x) = \int_S f(x,y)\,dy$

Example: Define $f: \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \begin{cases} \frac{x^2 y}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$

Let $x \in \mathbb{R} \setminus \{0\}$ then

$$F(x) := \int_0^1 f(x,y)\,dy = \int_0^1 \frac{x^2 y}{(x^2+y^2)^2}\,dy = \left[-\frac{x^2}{2(x^2+y^2)}\right]_0^1 = \frac{1}{2(1+x^2)}$$

Let $y \in [0,1]$, then $\lim_{x\to 0} f(x,y) = 0$ (Prove it)

Hence:

$$\underbrace{\lim_{x\to 0} \int_0^1 f(x,y)\,dy}_{=1/2} \neq \underbrace{\int_0^1 \lim_{x\to 0} f(x,y)\,dy}_{=0}$$

Conclusion: we can NOT simply swap $\int$ and lim

Nevertheless, notice that f is not C^0 at $(0,0)$ in the above example

since $f(x,x) = \frac{1}{4x} \xrightarrow[x\to 0^+]{} +\infty$

If we assume that f is continuous, we have the following result:

Theorem: Assume that:

- $S \subset \mathbb{R}^m$ is compact and Jordan measurable (ie ∂S has ZC)
- $T \subset \mathbb{R}^p$ satisfies: $\forall x_0 \in T, \exists r > 0, T \cap \bar{B}(x_0, r)$ is compact
- $f: T \times S \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$ is continuous

Define $F: T \to \mathbb{R}$ by $F(x) = \int_S f(x,y)\,dy$

Then F is well defined and continuous.

Remark: the condition on T is automatically satisfied if T is

- open: let $x_0 \in T$, since T is open $\exists \varepsilon > 0$ s.t. $B(x_0, \varepsilon) \subset T$
  set $r = \varepsilon/2$ then $\overline{B}(x_0, r) \cap T = \overline{B}(x_0, r)$ is compact

- closed: $\overline{B}(x_0, r) \cap T$ is closed and bounded hence compact

Δ. Let $x \in T$, then $\begin{matrix} S \to \mathbb{R} \\ y \mapsto f(x,y) \end{matrix}$ is C^0 on S compact and Jordan measurable so $F(x) = \int_S f(x,y)\,dy$ is well-defined

- If $\int_S 1 = 0$ then $F \equiv 0$ (F is constant equal to 0), so we may assume that $|S| := \int_S 1 > 0$
- Let's prove that F is C^0 at $x_0 \in T$.

Let $\varepsilon > 0$.
By assumption, $\exists r > 0$, $\overline{B}(x_0, r) \cap T$ is compact, hence f is continuous on $\overline{B}(x_0, r) \cap T$ compact and by Heine-Cantor $f : \overline{B}(x_0, r) \cap T \to \mathbb{R}$ is uniformly continuous

so $\exists \delta > 0$, $\forall (x,y), (x',y') \in \overline{B}(x_0, r) \times S$

$$\|(x,y) - (x',y')\| < \delta \Rightarrow |f(x,y) - f(x',y')| < \varepsilon/_{2|S|}$$

Let $x \in T$ satisfying $\|x - x_0\| < \delta' := \min(\delta, r)$

then $x \in \overline{B}(x_0, r)$ and

$\forall y \in S$, $|f(x,y) - f(x_0,y)| < \frac{\varepsilon}{2|S|}$ since $\|(x,y) - (x_0,y)\| = \|x - x_0\| < \delta$

Therefore: $|F(x) - F(x_0)| = \left|\int_S f(x,y) - f(x_0,y)\,dy\right|$
$\leq \int_S |f(x,y) - f(x_0,y)|\,dy \leq \int_S \frac{\varepsilon}{2|S|} = \frac{\varepsilon}{2} < \varepsilon$

We proved: $\forall \varepsilon > 0, \exists \delta' > 0, \forall x \in T, \|x - x_0\| < \delta' \Rightarrow |F(x) - F(x_0)| < \varepsilon$ $\square$

Theorem: Let $S \subset \mathbb{R}^n$ compact and Jordan measurable (ie ∂S has ZC)
$T \subset \mathbb{R}^p$ open
$f: \begin{array}{l} T \times S \longrightarrow \mathbb{R} \\ (x,y) \mapsto f(x,y) \end{array}$ such that

① f is continuous on $T \times S$

② $\forall i = 1, \dots, p$, $\frac{\partial f}{\partial x_i}$ exists and is continuous on $T \times S$

Then $F: T \to \mathbb{R}$ defined by $F(x) := \int_S f(x,y)\, dy$ is C^1

and moreover $\forall i = 1, \dots, p$,

$$\frac{\partial F}{\partial x_i}(x) = \int_S \frac{\partial f}{\partial x_i}(x,y)\, dy$$

Δ Notice that F is well-defined and continuous by the above theorem

Since $\frac{\partial f}{\partial x_i}$ is C^0 on S compact and Jordan measurable,
$\int_S \frac{\partial f}{\partial x_i}$ is well defined

Let $x_0 \in T$, since T is open, for t small enough, $x_0 + t e_i \in T$

We define

$$A(t) := \frac{F(x_0 + t e_i) - F(x_0)}{t} - \int_S \frac{\partial f}{\partial x_i}(x_0, y)\, dy \quad \text{for } t \neq 0 \text{ small enough}$$

$$= \int_S \frac{f(x_0 + t e_i, y) - f(x_0, y)}{t} - \frac{\partial f}{\partial x_i}(x_0, y)\, dy$$

$$= \int_S \frac{\partial f}{\partial x_i}(x_0 + \theta t, y) - \frac{\partial f}{\partial x_i}(x_0, y)\, dy$$

for some $\theta \in [0,1]$ depending on t and y, by the MVT (one-variable case)
$\theta(t,y)$

Let $\varepsilon > 0$.

Since T is open, $\exists r > 0$, $\overline{B}(x_0, r) \subset T$

hence $\frac{\partial f}{\partial x_i} : \overline{B}(x_0, r) \times S \to \mathbb{R}$ is uniformly continuous

as a continuous function on a compact set (Heine-Cantor)

ie $\exists \delta > 0$, $\forall (x,y), (x',y') \in \overline{B}(x_0, r) \times S$

$$\|(x,y) - (x',y')\| < \delta \Rightarrow \left| \frac{\partial f}{\partial x_i}(x,y) - \frac{\partial f}{\partial x_i}(x',y') \right| < \frac{\varepsilon}{2|S|}$$

(I assume $|S| > 0$ otherwise it's easy since $F \equiv 0$

Let $t \in \mathbb{R}$ satisfying $|t| < \min(\delta, r) =: \delta'$

then $\|(x_0, y) - (x_0 + t e_i, y)\| = |t| < \delta$

and $\|x_0 - (x_0 + t e_i)\| = |t| < r$ (ie $x_0 + t e_i \in \overline{B}(x_0, r)$)

so $|A(t)| \leqslant \int_S \frac{\varepsilon}{2|S|} = \frac{\varepsilon}{2} < \varepsilon$

ie $A(t) \xrightarrow[t \to 0]{} 0$

ie $\dfrac{F(x_0 + t e_i) - F(x_0)}{t} \xrightarrow[t \to 0]{} \int_S \frac{\partial f}{\partial x_i}(x,y)\, dy$

ie $\frac{\partial F}{\partial x_i}(x) = \int_S \frac{\partial f}{\partial x_i}(x,y)\, dy$ which is C^0 by the previous theorem

$\square$

Summary of this lecture

① Generally $\lim_{x\to x_0} \int_S f(x,y)\,dy \neq \int_S \lim_{x\to x_0} f(x,y)\,dy$

ie we can't swap $\int$ and lim!

② If $f: T\times S \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$ is C^0 (+ technical assumptions on S and T)

then $F: T \to \mathbb{R}$ defined by $F(x) = \int_S f(x,y)\,dy$

is C^0

ie we can swap $\int$ and lim

indeed:

$$\lim_{x\to x_0} \int_S f(x,y)\,dy = \lim_{x\to x_0} F(x) \overset{F\ C^0}{=} F(x_0) = \int_S f(x_0,y)\,dy \overset{f\ C^0}{=} \int_S \lim_{x\to x_0} f(x,y)\,dy$$

③ If $f: T\times S \to \mathbb{R}$ is C^0, T open, S compact and Jordan measurable

and $\frac{\partial f}{\partial x_i}(x,y)$ exists and is C^0 on $T\times S$

(that's stronger than asking $f_y: x \mapsto f(x,y)$ to be C^1)

then F is C^1 and we can swap $\int$ and $\frac{\partial}{\partial x_i}$:

$$\frac{\partial \int_S f(x,y)\,dy}{\partial x_i} = \frac{\partial F}{\partial x_i}(x) = \int_S \frac{\partial f}{\partial x_i}(x,y)\,dy$$

The above theorem can be very useful to compute some integrals that are difficult to compute directly.

<u>Ex</u>: $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta$

We introduce $f(t) = \int_0^{2\pi} e^{t\cos\theta} \cos(t\sin\theta)\, d\theta$

then $f'(t) = \int_0^{2\pi} e^{t\cos\theta} \left(\cos\theta \cos(t\sin\theta) - \sin\theta \sin(t\sin\theta)\right) d\theta$

$= 0$ ← If you are in physics, you recognized a line integral, otherwise wait for next week

so f is constant and

$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta = f(1) = f(0) = \int_0^{2\pi} d\theta = 2\pi$$

<u>Ex</u>: $F(x) = \int_0^x e^{-t^2} dt$, $G(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2} dt$

- F is C^1 by FTC, G is C^1 by the above thm
- $G'(x) = \int_0^1 -2x\, e^{-x^2(1+t^2)} dt \underset{s=xt}{=} -2\int_0^x e^{-x^2-s^2} ds = -2F'(x)F(x) = -(F^2)'$

so $(G+F^2)' = 0$

and $(G+F^2)(x) = (G+F^2)(0) = 0 + \int_0^1 \frac{1}{1+t^2} dt = \frac{\pi}{4}$

Since $\lim_{x\to+\infty} G(x) = 0$

$\int_0^{+\infty} e^{-t^2} dt = \lim_{x\to+\infty} F(x) = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$, ie $\int_0^{+\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$

(We'll see another proof using polar coordinates on Thursday)

Improper integrals

Comment: I strongly suggest you to review the material from MAT137 in the one-variable case
I posted some notes online

Goal: What can we do if a function or its domain is not bounded?

Disclaimer: I use a slightly different approach from the textbook
It is equivalent and it doesn't change anything in practice.

Definition: $U \subset \mathbb{R}^n$ open, $f: U \to \mathbb{R}$ non-negative

We say that $\int_U f$ is convergent (or that f is improperly integrable on U)

when ① $\forall S \subset U$ compact and Jordan measurable (∂S has ZC)

$$\int_S f \text{ exists}$$

② $\sup_S \int_S f < +\infty$ where S are like above

And then we set $\int_U f := \sup_S \int_S f$

Fact: For $U \subset \mathbb{R}^n$ open, there exists a sequence $C_1, \dots, C_i, \dots$ of compact Jordan measurable subsets of U s.t.

① $U = \bigcup C_i$

② $C_i \subset C_{i+1}^\circ$

We say that $(C_i)_i$ is an exhaustion of U

Ex: • $\mathcal{U} = \mathbb{R}^n$, $C_i = B(\vec{0}, i)$

• $\mathcal{U} = \mathbb{R}^n \setminus \{0\}$, $C_i = \left\{ x \in \mathbb{R}^n : \frac{1}{i} \leq \|x\| \leq i \right\}$

Theorem: $\mathcal{U} \subset \mathbb{R}^n$ open, $f: \mathcal{U} \longrightarrow \mathbb{R}$ non-negative, $\int_S f$ exists for S compact Jordan measurable, $(C_i)_{i \in \mathbb{N}}$ as above

$$\int_{\mathcal{U}} f \text{ is convergent} \Leftrightarrow \exists M \geq 0, \forall i, \int_{C_i} f < M$$

and then $\int_{\mathcal{U}} f = \lim_{i \to +\infty} \int_{C_i} f$

Δ $u_i = \int_{C_i} f$ is non-decreasing so it converges iff $\left\{ \int_{C_i} f \right\}$ is bounded

$\Rightarrow$: $\int_{C_i} f \leq \int_{\mathcal{U}} f = \sup_S \int_S f$

$\Leftarrow$: Let $S \subset \mathcal{U}$ compact Jordan measurable.
by compactness $S \subset \bigcup_{i=1}^{K} \mathring{C_i} \subset \bigcup_{i=1}^{K} C_i$ (finitely many)

so $\int_S f \leq \int_{C_K} f \leq \lim_{i \to +\infty} \int_{C_i} f$

$\square$

The above theorem is the reason why I first restricted to the non-negative case:
if (C_i) is an exhausting approximation of $\mathcal{U}$ then the value of $\lim_{i \to +\infty} \int_{C_i} f$ doesn't depend on the choice of C_i

That's false for f not non-negative.

Cf: if $f \geq 0$ then you will always get the same value for $\int_{\mathcal{U}} f$, no matter how you try to compute it

Ex: in the one variable case, we define $\int_a^{+\infty} f$ by $\lim_{c \to +\infty} \int_a^c f$

so that $\int_0^{+\infty} \frac{\sin(x)}{x} dx = \lim_{m \to +\infty} \int_0^m \frac{\sin(x)}{x} dx = \lim_{m \to +\infty} \int_{[0,m]} \frac{\sin(x)}{x} dx = \frac{\pi}{2}$

where we used $C_m = [0,m]$ (that's the convention in the 1-var case)

But for another choice of C_m, we could have obtained something else

$$D_m = [0, 2m\pi - \pi] \cup \bigcup_{k=m}^{2m} [2k\pi, 2k\pi + \pi]$$

then $\int_{D_m} \frac{\sin(x)}{x} dx = \int_0^{2m\pi - \pi} \frac{\sin(x)}{x} dx + \sum_k \underbrace{\int_{2k\pi}^{2k\pi+\pi} \frac{\sin(x)}{x} dx}_{\geqslant \frac{1}{2k\pi + \pi} \int \sin(x) dx}$

so $\lim_{m \to +\infty} \int_{D_m} \frac{\sin(x)}{x} dx > \frac{\pi}{2} + \lim_{m \infty} \sum_{k=m}^{2m} \frac{1}{k+1} = \frac{\pi}{2} + \frac{\ln 2}{\pi} > \frac{\pi}{2}$

Ex: $\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dy\, dx = \int_0^1 \left[\frac{y}{x^2+y^2} \right]_0^1 dx = \int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}$

$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dx\, dy = \int_0^1 \left[-\frac{x}{x^2+y^2} \right]_0^1 dy = \int_0^1 -\frac{1}{1+y^2} dy = -\frac{\pi}{4}$

ie with $C_m = [1/m, 1] \times [0,1]$ we get $\pi/4$

with $C_m = [0,1] \times [1/m, 1]$ we get $-\pi/4$

and we can prove that with $[1/m, 1] \times [1/m, 1]$ we get 0. (Do it)

What should we do when f is not non-negative?

We will see that if $\int_\Omega |f|$ is CV then $\int_\Omega f$ is CV and doesn't depend on choices. So we will only work with absolute CV!!!

Notice that you already met this phenomenon for series in MAT137

① If $\sum a_n$ is abs cv (ie $\sum |a_n| < +\infty$) then you can compute $\sum a_n$ by grouping terms or permuting terms and you will always get the same value

② If $\sum a_n$ is CV but not ACV, by Riemann rearrangement theorem, you can get any value (even $+\infty$ or $-\infty$) by permuting terms

Let's go back to integrals.

Notation: for $f: U \to \mathbb{R}$, $U \subset \mathbb{R}^n$, we define $f_+, f_-: U \to \mathbb{R}$

by $f_+(x) = \max(f(x), 0)$ and $f_-(x) = \max(-f(x), 0)$

so that $f_+, f_- \geq 0$ and $f = f_+ - f_-$ and $|f| = f_+ + f_-$

Definition: $U \subset \mathbb{R}^n$ open, $f: U \to \mathbb{R}$
We say that $\int_U f$ is (absolutely) convergent or that

f is improperly integrable on U if f_+ and f_- are integrable on U and then we set

$$\int_U f := \int_U f_+ - \int_U f_-$$

Remark: this definition is equivalent to the one in the textbook! The absolute CV aspect is hidden in the fact that S can be any Jordan measurable set and not only a ball!

Theorem: $f: \mathcal{U} \to \mathbb{R}$, $\mathcal{U} \subset \mathbb{R}^n$ open, $\int_S f$ exists for any $S \subset \mathcal{U}$ Jordan measurable

$\int_{\mathcal{U}} f$ is improperly integrable if and only if $\int_{\mathcal{U}} |f|$ is integrable

⚠ that's different from the one-variable case where $\int_0^{+\infty} \frac{\sin x}{x}$ exists!

Δ $\Rightarrow$ if f is improperly integrable then f_+ and f_- are too by definition but then $|f| = f_+ + f_-$

If S is compact Jordan measurable:

$$\int_S |f| = \int_S f_+ + \int_S f_- \leq \int_{\mathcal{U}} f_+ + \int_{\mathcal{U}} f_-$$

so $\sup \int_S |f| < +\infty$

This assumption is important!

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{otherwise} \end{cases}$$

⚠ $\int_S \frac{|f| + f}{2}$ exists since $\int_S f$ exists
$\parallel$

$\Leftarrow$: $0 \leq f_+ \leq f_+ + f_- = |f|$

so if S is Jordan measurable compact $\int_S f_+ \leq \int_S |f| \leq \int_{\mathcal{U}} |f|$ and similarly for f_-

$\square$

The following theorem ensures that if f is integrable then we can compute $\int_{\mathcal{U}} f$ without being too careful since we force $\int f$ to be ACV in our definition!

Theorem: Assume that f is integrable on $\mathcal{U}$ and let C_i be a sequence of compact Jordan measurable sets s.t.

① $\mathcal{U} = \bigcup_i C_i$ ② $C_i \subset \mathring{C}_{i+1}$

then $\int_{\mathcal{U}} f = \lim_{i \to +\infty} \int_{C_i} f$

$$\Delta \int_{\mathcal{U}} f = \int_{\mathcal{U}} f_+ - \int_{\mathcal{U}} f_- = \lim_i \int_{C_i} f_+ - \lim_i \int_{C_i} f_-$$

$$= \lim_i \int_{C_i} f_+ - f_- = \lim_i \int_{C_i} f \quad \square$$

(Connection with the definition from the textbook)

<u>Theorem</u>: $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ s.t. $\forall S \subset \mathbb{R}^n$ Jordan measurable $\int_S$ exists

$\int f$ is improperly convergent

$\Leftrightarrow \exists L > 0, \forall \varepsilon > 0, \exists r > 0, \forall S \subset \mathbb{R}^n$ Jordan measurable

$\bar{B}(0,r) \subset S \Rightarrow |\int_S f - L| < \varepsilon$

$\Delta \Rightarrow$ you can work separately with f_+, f_- which are non-negative and then $|\int_S f - \int_{\mathbb{R}^n} f_+ - \int_{\mathbb{R}^n} f_-| \leqslant |\int_S f_+ - \int_{\mathbb{R}^n} f_+| + |\int_S f_- - \int_{\mathbb{R}^n} f_-|$

$\Leftarrow$ <u>Claim 1</u>: if (C_i) is an exhaustion then
$\lim_{i \to +\infty} \int_{C_i} f = L$

Δ By Heine-Borel, $\bar{B}(0,r) \subset \bigcup_{i=1}^{K} C_i = C_K$

so $\forall i > K, |\int_{C_i} f - L| < \varepsilon$ $\square$

<u>Claim 2</u>: if $\lim \int_{C_i} f$ doesn't depend on the exhaustion then $\int |f|$ is integrable

Δ <u>Idea</u>: by contradiction either $\int_{\mathbb{R}^n} f_+ = +\infty$ or $\int_{\mathbb{R}^n} f_- = +\infty$

since $f_+ f_- = 0$ we may find an exhaustion that makes $\int f_+$ or $\int f_-$ fast enough so that the other can't compensate.. see the example for $\int_{D_m} \frac{\sin(x)}{x}$ $\square$

~~the~~ actual proof of Claim 2 relies on the fact that if $\int_S f$ exists for S Jordan measurable, $\exists T \subset S$ Jordan measurable s.t. $\int_S |f| \leqslant 3 |\int_T f|$ and we can prove that the latter is bounded $\square$

Formal proof of Claim 2: (You can safely skip it)

Lemma 1: Claim 1 $\Rightarrow \exists M \geq 0, \forall S$ Jordan measurable, $|\int_S f| \leq M$

$\triangleleft$ Assume $\{\int_S f\}$ is not bounded and let (C_n) an exhaustion

for $n \in \mathbb{N}$, $\exists T_n$ s.t. $|\int_{T_n} f| \geq n + \int_{U_n} |f|$, $U_n = C_n \cup T_1 \cup \dots \cup T_{n-1}$

Define $S_n = T_n \cup U_n$ so that (S_n) is an exhaustion

then $|\int_{S_n} f| = |\int_{T_n} f + \int_{S_n \setminus T_n} f| \geq |\int_{T_n} f| - \int_{U_n} |f| \geq n \xrightarrow[n \to +\infty]{} +\infty$, $|L| \neq +\infty$

contradiction $\square$

Lemma 2: $S \subset \mathbb{R}^n$, Jordan measurable, $\int_S f$ integrable

then $\exists T \subset S$ Jordan measurable s.t. $\int_S |f| \leq 3 |\int_T f|$

$\triangleleft$ $\delta := \int_S |f| = \int_S f_+ + \int_S f_-$ exists by assumption, so either $\int_S f_+$ or $\int_S f_- \geq \frac{\delta}{2}$

Let say $\int_S f_+ \geq \delta/2 > \delta/3$ (I assume $\delta > 0$ otherwise the Lemma is trivial)

So $\exists R \supset S$ a rectangle with a partition so that $L_P(f_+) > \delta/3$

Define T as the union of the subrectangles where $\inf(f_+) > 0$

f_+ is extended by 0

then $\int_T f = \int_T f_+ > 0$

↑ Here I assume f is C^0 so if $\inf f_+ > 0$ then $f > 0$ on the rectangle that's not a big assumption: so this proof works if the discontinuity set has Z.C

then $|\int_T f| = \int_T f = \int_T f_+ = \int_R f_+ > \delta/3 = \frac{1}{3} \int_S |f|$ $\square$

Proof of claim 2: $\exists M$ s.t. $\forall T$, $|\int_T f| \leq M$ by Lemma 1

so $\forall S$, $\int_S |f| \leq 3M$ and $\{\int_S |f|\}$ is bounded $\square$

What do we do in practice?

Step 0: check that $\int_S f$ exists for S Jordan measurable (ex: f C^0)

Step 1: Compute $\int_U |f|$

since $|f| \geq 0$, you don't have to be careful, the result won't depend on how you compute it

For instance you can compute $\lim_{r \to +\infty} \int_{B(\vec{0}, r)} |f|$ if $U = \mathbb{R}^n$

Case 1: $\int_U |f| = +\infty$: you stop... f is not integrable on U

Case 2: $\int_U |f| < +\infty$: f is integrable, go to step 2

Step 2: we know that $\int_U f$ is (absolutely) integrable

then we can compute it (the result won't depend on how you compute it: every operation that seems legit is legit. for instance you can compute $\lim_{r} \int_{B(0,r)} f$ is $U = \mathbb{R}^n$ or compute the $\int dx \int dy$ separately...)

Basic properties: $U \subset \mathbb{R}^n$ open, $f, g : U \to \mathbb{R}$ improperly integrable

then:

① $af + bg$ is improperly integrable and $\int af + bg = a\int f + b\int g$

② $\forall x \in U,\ f(x) \leq g(x) \Rightarrow \int_U f \leq \int_U g$

③ $|f|$ is integrable on U and $\left|\int_U f\right| \leq \int_U |f|$

Δ ① $|af+bg| \leqslant |a||f| + |b||g|$

and if S is compact Jordan measurable

$$\int_S |af+bg| \leqslant |a|\int_S |f| + |b|\int_S |g| \leqslant |a|\int_U |f| + |b|\int_U |g|$$

then we obtain the equality by computing $\lim_i \int_{C_i} af+bg$

② For C_i compact and Jordan measurable

$\int_{C_i} f \leqslant \int_{C_i} g$ and then take the limit

③ $$\left.\begin{array}{l} f \leqslant |f| \Rightarrow \int f \leqslant \int |f| \\ -f \leqslant |f| \Rightarrow -\int f \leqslant \int |f| \end{array}\right\} \Rightarrow \left|\int f\right| \leqslant \int |f|$$

□

Improper integrals : examples

① A first example with unbounded domain

We want to compute $\int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx dy$

Since $f(x,y) = e^{-(x^2+y^2)}$ is continuous and non-negative

$\lim_{n\to+\infty} \int_{C_n} e^{-(x^2+y^2)}$ doesn't depend on the choice of the exhaustion

(but it could be $+\infty$)

$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx\,dy = \lim_{n\to+\infty} \int_{\overline{B}(0,n)} e^{-(x^2+y^2)} dx dy$$

$$= \lim_{n\to+\infty} \int_{[0,n]\times[-\pi,\pi]} r e^{-r^2} dr d\theta$$

$$= \lim_{n\to+\infty} 2\pi \left[-\frac{e^{-r^2}}{2}\right]_0^n$$

$$= \pi$$

$$\boxed{\int_{\mathbb{R}^2} e^{-(x^2+y^2)} = \pi}$$

Now let's take another exhaustion:

$$\pi = \int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx dy = \lim_{n\to+\infty} \int_{[-n,n]\times[-n,n]} e^{-(x^2+y^2)} dx dy = \lim_{n\to+\infty} \int_{-n}^{n} e^{-x^2} dx \int_{-n}^{n} e^{-y^2}$$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2$$

So : $\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}}$ (Gaussian/Euler-Poisson integral)

Exercise : $\int_{\mathbb{R}^n} e^{-\alpha \|x\|^2} = \left(\frac{\pi}{\alpha}\right)^{n/2}$

$\alpha > 0$

② Define $f : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$ by $f(x) = \frac{1}{\|x\|^p}$

(i) $\int_{\{\|x\| < a\}} \|x\|^{-p}$ is convergent $\Leftrightarrow p < n$ (Unbounded function)

(ii) $\int_{\{\|x\| > a\}} \|x\|^{-p}$ is convergent $\Leftrightarrow p > n$ (Unbounded domain)

Δ I prove it for $n = 2$ here (see next page for the general case)

Since f is C^0 and non-negative:

(i) $$\int_{\{\|x\| < a\}} \|x\|^{-p} = \lim_{k \to +\infty} \int_{\{\frac{1}{k} \leq \|x\| \leq a\}} \|x\|^{-p} = \lim_{k \to +\infty} \int_{[1/k, a] \times [-\pi, \pi]} r^{1-p} \, dr \, d\theta$$

$$= \lim_{k \to +\infty} 2\pi \int_{1/k}^{a} \frac{1}{r^{p-1}} dr$$

$$= \begin{cases} +\infty & \text{if } p-1 \geq 1 \\ < +\infty & \text{if } p-1 < 1 \end{cases}$$

(ii) $$\int_{\{\|x\| > a\}} \|x\|^{-p} = \lim_{k \to +\infty} \int_{\{\|x\| \leq k\}} \|x\|^{-p} = \lim_{k \to +\infty} \int_{[0, k] \times [-\pi, \pi]} r^{1-p} \, dr \, d\theta$$

$$= \lim_{k \to +\infty} 2\pi \int_0^k \frac{1}{r^{p-1}} \, dr = \begin{cases} +\infty & \text{if } p-1 \leq 1 \\ < +\infty & \text{if } p-1 > 1 \end{cases}$$

$\square$

A useful corollary:

Theorem: $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ C^0

if $\exists C > 0, p > n$ s.t. $\|x\|^p |f(x)| \leq C$ then $\int_{\mathbb{R}^n} f$ exists

General case: $m \geq 2$

Spherical coordinates in $\mathbb{R}^m$: $(r, \theta, \varphi_1, \dots, \varphi_{m-2})$, $r>0$, $\theta \in (0,2\pi)$, $\varphi_i \in (0,\pi)$

$$x_1 = r\cos\varphi_1$$
$$x_2 = r\sin\varphi_1 \cos\varphi_2$$
$$x_3 = r\sin\varphi_1 \sin\varphi_2 \cos\varphi_3$$
$$\dots$$
$$x_{m-1} = r\sin\varphi_1 \cdots \sin\varphi_{m-2}\cos\theta$$
$$x_m = r\sin\varphi_1 \cdots \sin\varphi_{m-2}\sin\theta$$

Notice that $r = \|x\| = \sqrt{\sum x_i^2}$ and that $\left|\det D\Phi\right| = r^{m-1}\prod_{i=1}^{m-2}\sin^{m-i-1}(\varphi_k)$

$$= r^{m-1}\sin^{m-2}(\varphi_1)\sin^{m-3}(\varphi_2)\cdots\sin(\varphi_{m-2})$$

So that: $$\int_{\{\|x\|>a\}\subset\mathbb{R}^m} \|x\|^{-p} = \lim_{k\to+\infty}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}\int_a^k r^{m-p-1}\prod\sin^{m-i-1}(\varphi_k)\,dr\,d\theta\,d\varphi_1\dots d\varphi_m$$

$$= \lim_{k\to+\infty} C\int_a^k r^{m-p-1}\,dr = \begin{cases} +\infty & \text{if } p+1-m \leq 1 \\ <+\infty & \text{if } p+1-m > 1\end{cases}$$

(3) A first non-improperly-integrable function (Unbounded domain)

We want to know if $\int_{\{x\geq 0, y\geq 0\}} \sin(x^2+y^2)\,dx\,dy$ is convergent

$$\lim_{k\to+\infty} \int_{\{x^2+y^2\leq k^2, x\geq 0, y\geq 0\}} \sin(x^2+y^2)\,dx\,dy = \lim_{k\to+\infty} \int_0^k \int_0^{\pi/2} r\sin(r^2)\,dr\,d\theta$$

$$= \frac{\pi}{4} \lim_{k\to+\infty} \int_0^{k^2} \sin(u)\,du$$

$$= \frac{\pi}{4} \lim_{k\to+\infty} \left(1-\cos(k^2)\right) \text{ DNE}$$

So $\int_{\{x\geq 0, y\geq 0\}} \sin(x^2+y^2)$ is not improperly convergent.

⚠ Here we were able to conclude because we found an exhaustion so that the limit DNE
If we had found a well-defined limit that wouldn't have been enough to conclude: a function whose sign changes which is not improperly CV can be cv for some exhaustion.
See the following example

$$\lim_{k\to+\infty} \int_{[0,k]\times[0,k]} \sin(x^2+y^2)\,dx\,dy = \lim_{k\to+\infty} \int_{[0,k]\times[0,k]} \sin(x^2)\cos(y^2)+\cos(x^2)\sin(y^2)\,dx\,dy$$

Fresnel integrals, we admit it

$$= \lim_{k\to+\infty} 2 \int_0^k \sin(x^2)\,dx \int_0^k \cos(y^2)\,dy = 2\sqrt{\frac{\pi}{8}}\sqrt{\frac{\pi}{8}} = \frac{\pi}{4}$$

But we have already seen that it is not CV
(so the value depends on the choice)

④ An example where the sign changes (unbounded function)

We want to know whether $\displaystyle\int_{\{x^2+y^2<4\}} \log\left((x^2+y^2)^{1/2}\right) dx\,dy$ converges

Here the sign changes, so it is $\boxed{\text{not}}$ enough to check the limit on an exhaustion to prove the improper convergence

$$\lim_{k\to+\infty} \int_{\{1/k \leq x^2+y^2 \leq 4\}} \left|\log\sqrt{x^2+y^2}\right| dx\,dy$$

$$= \lim_{k\to+\infty} \int_0^{2\pi}\int_{1/k}^{2} r\,|\log r|\,dr\,d\theta$$

$$= \lim_{k\to+\infty} 2\pi \int_{1/k}^{2} r\,|\log r|\,dr$$

$$= \lim_{k\to+\infty} 2\pi\left(\int_{1/k}^{1} r(-\log r)\,dr + \int_1^2 r\log r\,dr\right)$$

$$= \lim_{k\to+\infty} -2\pi\left(\left[\frac{1}{2}r^2\log r - \frac{1}{4}r^2\right]_{1/k}^{1} - \left[1/2\, r^2\log r - \frac{1}{4}r^2\right]_1^2\right)$$

$$= \pi(\log 16 - 1) < +\infty$$

So $\displaystyle\int_{\{x^2+y^2<4\}} \log\sqrt{x^2+y^2}\,dx\,dy$ is improperly convergent

(5) Study $\int_{(0,1)\times(0,1)} \frac{1}{x^2y^2}\,dxdy$ (Unbounded function)

Since $f(x,y) = \frac{1}{x^2y^2}$ is non-negative and C^0

$$\int_{(0,1)\times(0,1)} \frac{1}{x^2y^2}\,dxdy = \lim_{k\to+\infty} \int_{[1/k,1]\times[1/k,1]} \frac{1}{x^2y^2}\,dxdy$$

$$= \lim_{k\to+\infty} \int_{1/k}^{1} \frac{1}{x^2}\,dx \int_{1/k}^{1} \frac{1}{y^2}\,dy$$

$$= \lim_{k\to+\infty} (1-k)^2 = +\infty$$

So $\int_{(0,1)\times(0,1)} \frac{1}{x^2y^2}\,dxdy$ is not improperly convergent

(6) Study $\int_{(1,\infty)\times(1,\infty)} \frac{1}{x^2y^2}\,dxdy$ (Unbounded domain)

Since f is continuous and non-negative

$$\int_{(1,\infty)\times(1,\infty)} \frac{1}{x^2y^2}\,dxdy = \lim_{k\to+\infty} \int_{[1,k]\times[1,k]} \frac{1}{x^2y^2}\,dxdy$$

$$= \lim_{k\to+\infty} \left(1-\frac{1}{k}\right)^2 = 1$$

So $\int_{(1,\infty)\times(1,\infty)} \frac{1}{x^2y^2}\,dxdy$ is improperly CV

7) Study $\displaystyle\int_{\mathbb{R}^2} \frac{1}{1+x^2+y^2}\,dxdy$ (Unbounded domain)

Since $f(x,y) = \dfrac{1}{1+x^2+y^2}$ is C^0 and non-negative

$$\int_{\mathbb{R}^2} \frac{1}{1+x^2+y^2}\,dxdy = \lim_{k\to+\infty} \int_{\overline{B}(0,k)} \frac{1}{1+x^2+y^2}\,dxdy$$

$$= \lim_{k\to+\infty} \int_{[0,k]\times[-\pi,\pi]} \frac{1}{1+r^2}\,r\,dr\,d\theta$$

$$= \lim_{k\to+\infty} 2\pi \int_0^k \frac{r}{1+r^2}\,dr$$

$$= \lim_{k\to+\infty} \pi \ln(1+k^2) = +\infty$$

So $\displaystyle\int_{\mathbb{R}^2} \frac{1}{1+x^2+y^2}\,dxdy$ is not improperly CV

(I like this one because it looks like $\displaystyle\int_{\mathbb{R}} \frac{1}{1+x^2}\,dx$ which is improperly CV)

Line integrals

Convention: In this section, by a "curve", I mean a simple regular parametrized C^1 curve, i.e.

$$C = \{\sigma(t) : t \in [a,b]\} \subset \mathbb{R}^m$$

where $\sigma : [a,b] \longrightarrow \mathbb{R}^m$ is C^1 and satisfies
(i) σ is injective on (a,b) "simple points only"
(ii) $\forall t \in (a,b)$, $\sigma'(t) \neq \vec{0}$ "regular"

We say that the curve is closed when $\sigma(a) = \sigma(b)$

Remark: Why these assumptions?

1. We don't want C^0 curves to avoid phenomena like space-filling curves

2. We don't want differentiable but C^1 curves to ensure that our curves are rectifiable (finite length)

3. We want $\sigma'(t) \neq \vec{0}$ to have regular curves and to ensure that the parametrization captures the geometry of C

$$\{\underbrace{(t,t)}_{=\sigma_1(t)} : t \in [-1,1]\} = \{\underbrace{(t^3,t^3)}_{=\sigma_2(t)} : t \in [-1,1]\}$$

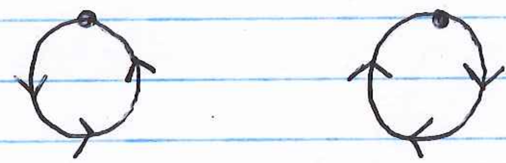
4. We want σ to be injective on (a,b) to have some "correspondance" between the parametrizations and the curves.

eg: $\sigma_1 : [0,1] \longrightarrow \mathbb{R}^2$, $t \mapsto (\cos(2\pi t), \sin(2\pi t))$ and $\sigma_2 : [0,1] \longrightarrow \mathbb{R}^2$, $t \mapsto (\cos(4\pi t), \sin(4\pi t))$ have the same image but not the same length: σ_2 turns twice around 0, we don't want that!

Nevertheless, we won't have an exact correspondance:

(a) We may have 2 parametrizations with different speeds for the same C

(b) We may have 2 parametrizations going in different directions

e.g. 

We say that they have different orientations (there are only two possible orientations)

And this phenomenon will be important for line integrals of vector fields.

⚠ Here I adopted the above conventions from the textbook but notice that usually we define a "curve" in this context as a "parametrization up to diffeomorphisms" and not as the image C.
For this reason we write $\int_\sigma f$ and not $\int_C f$ in this context
This is important in several areas of mathematics (complex analysis, algebraic topology) where we want to distinguish σ_1 and σ_2 from the previous example.
In MAT237 we want our results to only depend on C (up to orientation)
→ ⚠ Be careful if you use other resources/books!

For (2): $\sigma : [-1,1] \longrightarrow \mathbb{R}^2$, $t \mapsto (t, t^2\cos(\pi/t^2))$, $\sigma(0)=(0,0)$ is differentiable, not C^1,

but of infinite length: $\int_{-1}^{1} \|\sigma'(t)\|\, dt = +\infty$

Def: Let $C = \operatorname{Im} \sigma = \{\sigma(t) : t \in [a,b]\}$ be a curve (as above) and $f : C \longrightarrow \mathbb{R}$ continuous.

We define $\displaystyle\int_C f := \int_a^b f(\sigma(t)) \cdot \|\sigma'(t)\| \, dt$

"the line integral of f along C"

Rem: It doesn't depend on the parametrization σ.
Indeed, let $\sigma_1 : [a,b] \longrightarrow \mathbb{R}^m$ and $\sigma_2 : [c,d] \to \mathbb{R}^m$ be two parametrizations of C s.t. $\sigma_2 = \sigma_1 \circ \varphi$ where $\varphi : (c,d) \to (a,b)$ is a C^1-diffeomorphism then

$$\int_c^d f(\sigma_2(t)) \, \|\sigma_2'(t)\| \, dt = \int_c^d f(\sigma_1(\varphi(t))) \, \|(\sigma_1 \circ \varphi)'(t)\| \, dt$$

$$= \int_c^d f(\sigma_1(\varphi(t))) \cdot \|\sigma_1'(\varphi(t))\| \cdot |\varphi'(t)| \, dt$$

$$s = \varphi(t) \rightsquigarrow = \int_a^b f(\sigma_1(s)) \cdot \|\sigma_1'(s)\| \, ds$$

⚠ Notice that φ is either increasing $(\varphi'(t) > 0)$ or decreasing $(\varphi'(t) < 0)$
It will be useful later!

↳ σ_1, σ_2 have same orientation

or opposite orientation

Ex: $C =$ 

$\sigma_1(t) = (\cos t, \sin t)$ for $t \in [0, \pi]$
$\sigma_2(t) = (t, \sqrt{1-t^2})$ for $t \in [-1, 1]$

$\sigma_2(t) = \sigma_1(\varphi(t))$ for $\varphi : \begin{array}{ccc} [-1,1] & \longrightarrow & [0,\pi] \\ t & \longmapsto & \arccos(t) \end{array}$

and $\displaystyle \varphi'(t) = -\frac{1}{\sqrt{1-t^2}} < 0$

Definition: Let $C = \{\sigma(t) : t \in [a,b]\}$ be a curve as above

We define the arclength of C by

$$\mathcal{L}(C) := \sup_{P} \left\{ \sum_{j=0}^{m-1} \|\sigma(t_{j+1}) - \sigma(t_j)\| \right\}$$

where P goes through all the partitions

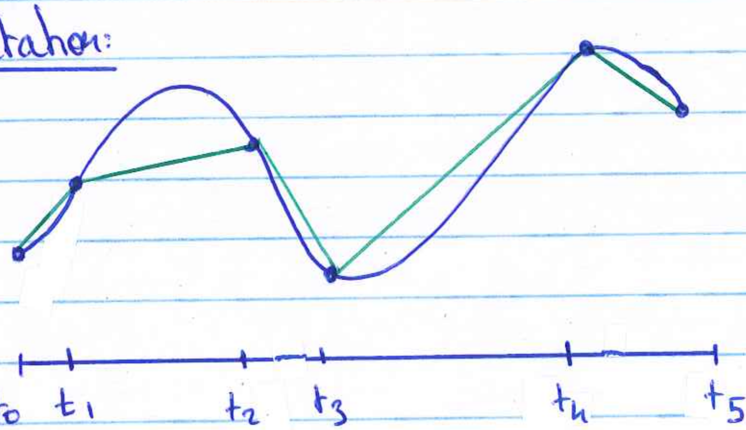
$P = \{a = t_0 < t_1 < \cdots < t_m = b\}$ of $[a,b]$

Comment: this supremum is finite (we say that C is rectifiable) since σ is Lipschitz as a C^1-function on $[a,b]$ compact

ie $\exists K \geq 0$, $\forall t,s \in [a,b]$, $\|\sigma(t) - \sigma(s)\| \leq K|t-s|$

so $\displaystyle\sum_{j=0}^{m-1} \|\sigma(t_{j+1}) - \sigma(t_j)\| \leq \sum_{j=0}^{m-1} K|t_{j+1} - t_j| = K(b-a)$

Geometric interpretation:



The following theorem is very useful since

① it allows to compute $\mathcal{L}(C)$ using integration techniques

② it shows that $\mathcal{L}(C)$ doesn't depend on σ

Theorem: $\boxed{\mathcal{L}(C) = \int_C 1 = \int_a^b \|\sigma'(t)\| \, dt}$

See claim 2 of proof 2 of "A MVT-like inequality"

$\Rightarrow$ Proof : $\boxed{\text{You can skip it}}$

① $\|\sigma(t_{i+1}) - \sigma(t_i)\| = \left\| \int_{t_i}^{t_{i+1}} \sigma'(t)\, dt \right\| \leqslant \int_{t_i}^{t_{i+1}} \|\sigma'(t)\| dt$

so $\sum_{i=0}^{m-1} \|\sigma(t_{i+1}) - \sigma(t_i)\| \leqslant \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \|\sigma'(t)\| dt = \int_a^b \|\sigma'(t)\| dt$

for any partition, ie $\mathcal{L}(C) \leqslant \int_a^b \|\sigma'(t)\| dt$

② $\sigma\ C^1 \Rightarrow \sigma'\ C^0$

since $[a,b]$ is compact, σ' is uniformly continuous

ie $\forall \varepsilon > 0, \exists \delta > 0, \forall t, s \in [a,b], |t-s| < \delta \Rightarrow \|\sigma'(t) - \sigma'(s)\| < \varepsilon$

Let $\varepsilon > 0$, take δ as above and P a partition so that $t_{i+1} - t_i < \delta/2$

then $s \in [t_i, t_{i+1}] \Rightarrow \|\sigma'(s)\| - \|\sigma'(t_{i+1})\| \leqslant \|\sigma'(s) - \sigma'(t_{i+1})\| < \varepsilon$

$\Rightarrow \int_{t_i}^{t_{i+1}} \|\sigma'(s)\| ds \leqslant \int_{t_i}^{t_{i+1}} \|\sigma'(t_{i+1})\| + \varepsilon\, ds$

$b = a + b - a$ $\longrightarrow$ $= \left\| \int_{t_i}^{t_{i+1}} \sigma'(s) + \sigma'(t_{i+1}) - \sigma'(s)\, ds \right\| + \varepsilon(t_{i+1} - t_i)$

triangle ineq $\longrightarrow$ $\leqslant \left\| \int_{t_i}^{t_{i+1}} \sigma'(s)\, ds \right\| + \left\| \int_{t_i}^{t_{i+1}} \sigma'(t_{i+1}) - \sigma'(s)\, ds \right\| + \varepsilon(t_{i+1} - t_i)$

$\|\sigma'(t_{i+1}) - \sigma'(s)\| \leqslant \varepsilon$ $\longrightarrow$ $\leqslant \|\sigma(t_{i+1}) - \sigma(t_i)\| + 2\varepsilon(t_{i+1} - t_i)$

$\Rightarrow \int_a^b \|\sigma'\| = \sum \int_{t_i}^{t_{i+1}} \|\sigma'\| \leqslant \sum \|\sigma(t_{i+1}) - \sigma(t_i)\| + 2\varepsilon(b-a)$

$\leqslant L(C) + 2\varepsilon(b-a)$

and then $\varepsilon \longrightarrow 0$

so $\int_a^b \|\sigma'\| \leqslant L(C)$

⚠ In practice $\int \|\sigma'(t)\| dt$ may be difficult to compute $\square$

$$F : C \subset \mathbb{R}^{(m)} \longrightarrow \mathbb{R}^{(m)}$$ same dimension ⚠

Line integral for a vector field

Def: Let $C = \{\sigma(t) : t \in [a,b]\} \subset \mathbb{R}^m$ as above
Let $\vec{F} : C \longrightarrow \mathbb{R}^m$ be continuous "a vector field"

The line integral of $\vec{F}$ along C is defined by:

$$\underbrace{\int_C \vec{F} \cdot d\vec{x}}_{\text{That's just a notation}} := \int_a^b \underbrace{\vec{F}(\vec{\sigma}(t)) \cdot \vec{\sigma}'(t)}_{\in \mathbb{R} \text{ as the dot product of 2 vectors of } \mathbb{R}^m} \, dt$$

Physics notation: $\vec{F} = (F_1, \dots, F_m)$, $\vec{\sigma} = (\sigma_1, \dots, \sigma_m)$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{x} &= \int_a^b \vec{F}(\vec{\sigma}(t)) \cdot \vec{\sigma}'(t) \, dt \\ &= \int_a^b \Big(F_1(\vec{\sigma}(t)) \, \vec{\sigma}_1'(t) + \dots + F_m(\vec{\sigma}(t)) \, \sigma_m'(t) \Big) \, dt \\ &= \sum_{i=1}^m \int_a^b \underbrace{F_i(\vec{\sigma}(t))}_{=: F_i} \underbrace{\sigma_i'(t) dt}_{=: dx_i} \end{aligned}$$

So a convenient notation / mnemonic device is

$$\int_C \vec{F} \cdot d\vec{x} = \int_C F_1 \, dx_1 + \dots + F_m \, dx_m$$

where $\int_C F_i \, dx_i := \int_a^b F_i(\vec{\sigma}(t)) \, \sigma_i'(t) \, dt$

⚠ That's just a notation, for instance

$$\int_C -y \, dx + x \, dy = \int_C \vec{F} \cdot d\vec{x} = \int_a^b \vec{F}(\vec{\sigma}(t)) \cdot \vec{\sigma}'(t) \, dt$$

where $\vec{F}(x,y) = (-y, x)$

Do NOT try to compute $\int -y \, dx$ directly, it's a notation !!! that's all !

⚠ Orientation dependence of $\int_C \vec{F} \cdot d\vec{x}$

Let $\sigma_1 : [a,b] \to \mathbb{R}^n$, $\sigma_2 : [c,d] \to \mathbb{R}^n$ as above such that
$C = \operatorname{Im} \sigma_1 = \operatorname{Im} \sigma_2$

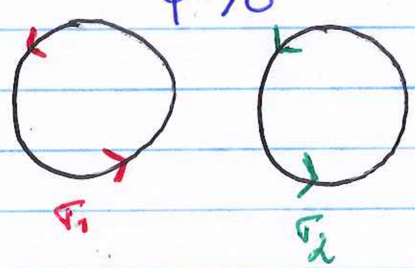
then $\sigma_2 = \sigma_1 \circ \varphi$ where $\varphi : (c,d) \to (a,b)$ is a C^1-diffeomorphism

Notice that φ is either always increasing $(\varphi' > 0)$
or always decreasing $(\varphi' < 0)$ ⚠

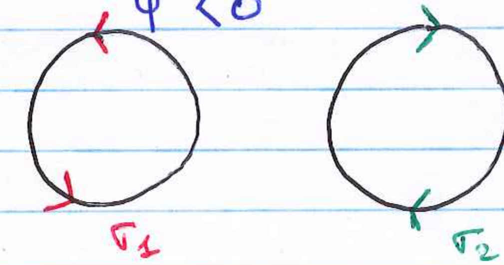
If $\varphi' > 0$ we say that σ_1 and σ_2 have <u>same</u> orientation

If $\varphi' < 0$ we say that σ_1 and σ_2 have <u>opposite</u> orientation

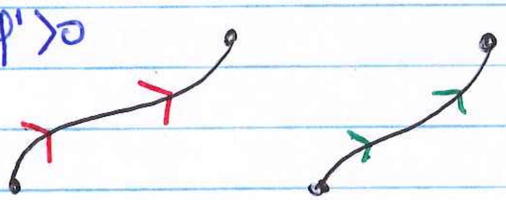
(1) $\varphi' > 0$



(2) $\varphi' < 0$



(3) $\varphi' > 0$



(4) $\varphi' < 0$

Let's see what happen to $\int_C \vec{F} \cdot d\vec{x}$:

$$\int_c^d \vec{F}(\vec{\sigma}_2(t)) \cdot \vec{\sigma}_2{}'(t)\, dt = \int_c^d \vec{F}(\vec{\sigma}_1(\varphi(t))) \cdot \left(\varphi'(t)\, \vec{\sigma}_1{}'(\varphi(t))\right) dt$$

since the dot product is linear and $\varphi' \in \mathbb{R}$

$$= \int_c^d \left(\vec{F}(\vec{\sigma}_1(\varphi(t))) \cdot \vec{\sigma}_1{}'(\varphi(t))\right) \underbrace{\varphi'(t)}\, dt$$

there is absolute value for the CoV

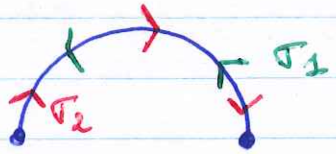
$$= \begin{cases} + \int_a^b \vec{F}(\vec{\sigma}_1(s)) \cdot \vec{\sigma}_1{}'(s)\, ds & \text{if } \varphi' > 0 \\ \text{or} \\ - \int_a^b \vec{F}(\vec{\sigma}_1(s)) \cdot \vec{\sigma}_1{}'(s)\, ds & \text{if } \varphi' < 0 \end{cases}$$

Conclusion :

$$\int_c^d \vec{F}(\vec{\sigma_2}(t)) \cdot \vec{\sigma_2}'(t)\, dt = \begin{cases} \int_a^b \vec{F}(\vec{\sigma_1}(t)) \cdot \vec{\sigma_1}'(t)\, dt & \text{if } \sigma_1 \text{ and } \sigma_2 \text{ have same orientation} \\ -\int_a^b \vec{F}(\vec{\sigma_1}(t)) \cdot \vec{\sigma_1}'(t)\, dt & \text{if } \sigma_1 \text{ and } \sigma_2 \text{ have opposite orientation} \end{cases}$$

⚠ $\int_C \vec{F} \cdot d\vec{x}$ depends on the orientation of σ
(and not only on its image $C = \operatorname{Im} \sigma$)

Ex : $\sigma_1 : \begin{array}{ccc} [0,\pi] & \longrightarrow & \mathbb{R}^2 \\ t & \longmapsto & (\cos t, \sin(t)) \end{array}$ $\quad \sigma_2 : \begin{array}{ccc} [-1,1] & \longrightarrow & \mathbb{R}^2 \\ t & \longmapsto & (t, \sqrt{1-t^2}) \end{array}$

$C =$ 

$\sigma_2 = \sigma_1 \circ \varphi$
$\varphi(t) = \arccos t$, $\varphi'(t) = -\frac{1}{\sqrt{1-t^2}} < 0$

so σ_1, σ_2 have opposite orientation

① $\int_C -y\,dx + x\,dy$ with σ_1 :

$$\int_C -y\,dx + x\,dy = \int_0^\pi \underbrace{(-\sin t)}_{-y}\underbrace{(-\sin t)}_{dx} + \underbrace{(\cos t)}_{x}\underbrace{(\cos t)}_{dy}\, dt$$

$$= \int_0^\pi \sin^2 t + \cos^2 t \, dt = \int_0^\pi dt = \pi$$

② $\int_C -y\,dx + x\,dy$ with σ_2 :

$$\int_C -y\,dx + x\,dy = \int_{-1}^1 -\sqrt{1-t^2} - t\frac{t}{\sqrt{1-t^2}}\, dt = -\int_{-1}^1 \frac{1-t^2+t^2}{\sqrt{1-t^2}}\, dt$$

$$= -\int_{-1}^1 \frac{1}{\sqrt{1-t^2}}\, dt = -\left[\arcsin(t)\right]_{-1}^1 = -\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= -\pi$$

When $\vec{F}$ is a gradient field, ie $\vec{F} = \nabla f$ for some $f : \mathbb{R}^n \to \mathbb{R}$ C^1 then $\int_C \vec{F} \cdot d\vec{x}$ is easy to compute as showed in the next theorem

In physics we say that the vector field $\vec{F}$ is conservative when $\vec{F} = \nabla f$ for some f C^1

<u>Theorem</u>: (Gradient theorem / FTC for line integrals)

Let $C = \{\sigma(t) : t \in [a,b]\} \subset \mathbb{R}^m$ be a curve as above

Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ be C^1, set $F : \mathbb{R}^m \longrightarrow \mathbb{R}^m$, $x \mapsto \nabla f(x)$

then $$\int_C \vec{F} \cdot d\vec{x} = f(\sigma(b)) - f(\sigma(a))$$

or in a more concise way: $\int_C \vec{\nabla f} \cdot d\vec{x} = f(\sigma(b)) - f(\sigma(a))$

⚠ If $\vec{F} = \nabla f$ then $\int_C \vec{F} \cdot d\vec{x}$ only depends on the values of f at the endpoints

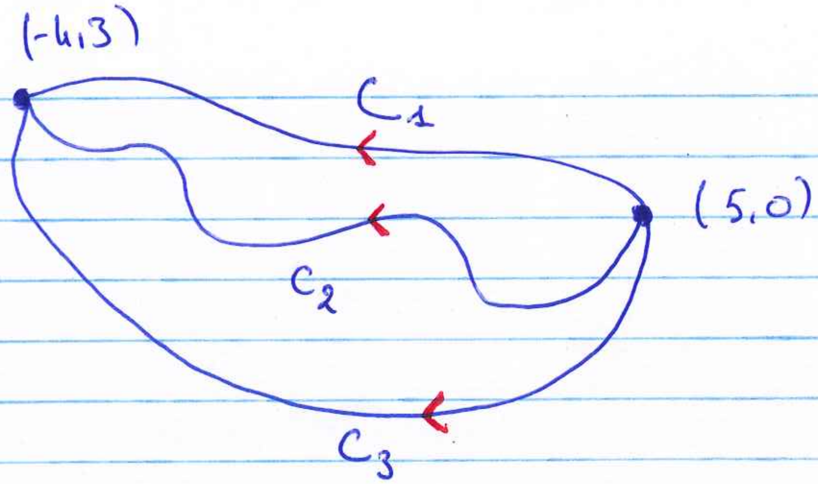
Δ $$\int_C \vec{F} \cdot d\vec{x} = \int_a^b \nabla f(\sigma(t)) \cdot \sigma'(t)\, dt = \int_a^b (f \circ \sigma)'(t)\, dt$$
$$= [f \circ \sigma(t)]_a^b \overset{FTC}{=} f(\sigma(b)) - f(\sigma(a)) \quad \square$$

<u>Corollary</u>: If $C = \{\sigma(t) : t \in [a,b]\} \subset \mathbb{R}^m$ is closed (ie $\sigma(a) = \sigma(b)$)
and $\vec{F} = \nabla f$ for $f : \mathbb{R}^m \to \mathbb{R}$ C^1

then $$\int_C \vec{F} \cdot d\vec{x} = 0$$

Δ $\int_C \vec{F} \cdot d\vec{x} = f(\sigma(b)) - f(\sigma(a)) = f(p) - f(p) = 0$ where $p = \sigma(a) = \sigma(b)$ $\square$

Ex:



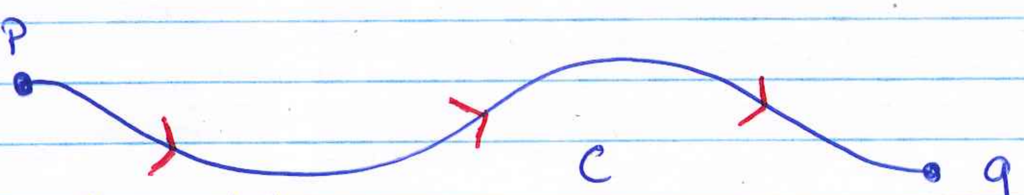
Compute $\int_{C_i} y\,dx + x\,dy = \int_{C_i} \vec{F} \cdot d\vec{x}$

where $\vec{F}(x,y) = (y, x)$

Notice that $\vec{F} = \nabla f$ where $f(x,y) = xy$

So $\int_{C_i} y\,dx + x\,dy = f(-4,3) - f(5,0) = -4 \cdot 3 - 5 \cdot 0 = -12$

Ex:



$\alpha \geq 1$

Compute $\int_C \|\vec{x}\|^{\alpha-1} \vec{x} \cdot d\vec{x}$ ie $\vec{F}(\vec{x}) = \|\vec{x}\|^{\alpha-1} \vec{x}$

Notice that $\vec{F} = \nabla f$ where $f(\vec{x}) = \dfrac{\|\vec{x}\|^{\alpha+1}}{\alpha+1}$

So $\int_C \|\vec{x}\|^{\alpha-1} \vec{x} \cdot d\vec{x} = f(q) - f(p) = \dfrac{\|q\|^{\alpha+1} - \|p\|^{\alpha+1}}{\alpha+1}$

Ex: $\int_C \|\vec{x}\|^{-2} \vec{x} \cdot d\vec{x} = \log\|q\| - \log\|p\|$

Since $\|\vec{x}\|^{-2} \vec{x} = \nabla(\log\|\vec{x}\|)$

Green's theorem

Def: $S \subset \mathbb{R}^n$ is a regular region if it is compact and $S = \overline{\mathring{S}}$

Remark: $S = \overline{\mathring{S}}$ means that $\forall x \in \partial S, \forall r > 0, B(x,r) \cap \mathring{S} \neq \emptyset$

Eg:  $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ is a regular region

$\{(x,0) : x \in [-1,1]\}$ is not a regular region in $\mathbb{R}^2$

Def: We say that $C \subset \mathbb{R}^n$ is a simple regular piecewise-C^1 closed curve

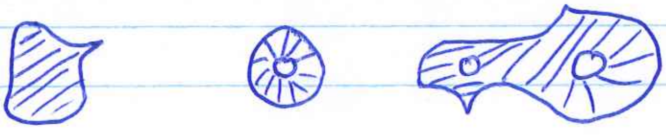
if $C = \{\sigma(t) : t \in [a,b]\}$ where $\sigma : [a,b] \longrightarrow \mathbb{R}^n$ satisfies:

1. σ is C^0
2. σ is injective on (a,b) "simple"
3. $\sigma(a) = \sigma(b)$ "closed"
4. There are finitely many $t_k \in [a,b]$, $a = t_0 < t_1 < \dots < t_K = b$ s.t.
   - σ is C^1 on (t_k, t_{k+1}) piecewise C^1
   - $\forall s \in (t_k, t_{k+1})$, $\sigma'(s) \neq \vec{0}$ regular
   - $\lim_{s \to t_k^+} \sigma'(s)$ and $\lim_{s \to t_{k+1}^-} \sigma'(s)$ exist

Remark: notice that line integrals are well-defined for piecewise C^1 curve.

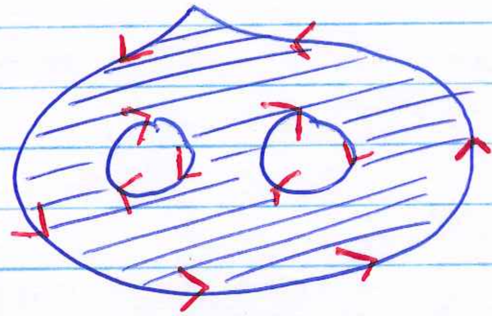
Def: We say that a regular region $S \subset \mathbb{R}^2$ has a piecewise smooth boundary if

$\partial S = C_1 \cup \dots \cup C_s$ where the C_i are disjoint simple regular piecewise-C^1 closed curves.

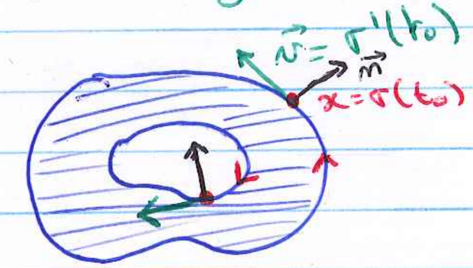
Ex: 

Def. Let $S \subset \mathbb{R}^2$ be a regular region with piecewise smooth boundary $\partial S = C_1 \cup \cdots \cup C_s$. We say that ∂S is *positively oriented* if the parametrization σ_i of C_i keeps $\mathring{S}$ on the left for $i = 1 \ldots s$.

Ex:



More formally: if $x = \sigma(t_0) = (\sigma_1(t_0), \sigma_2(t_0)) \in \partial S$
we set $\vec{v} = (v_1, v_2) = \sigma'(t_0) = (\sigma_1'(t_0), \sigma_2'(t_0))$
and we want that $\vec{n} = (v_2, -v_1)$ points outward of S
(we get $\vec{n}$ by rotating $\vec{v}$ by $\frac{\pi}{2}$ clockwise)



Definition: if $\partial S = C_1 \cup \cdots \cup C_s$ we write

$$\int_{\partial S} \vec{F} \cdot d\vec{x} := \sum_{i=1}^{s} \int_{C_i} \vec{F} \cdot d\vec{x}$$

↳ positively oriented

Theorem (Green's theorem)

Let $S \subset \mathbb{R}^2$ be a regular region with piecewise smooth boundary and $F : \mathcal{U} \to \mathbb{R}^2$ be a C^1-vector field where $\mathcal{U} \subset \mathbb{R}^2$ open satisfies $S \subset \mathcal{U}$.

Then $$\int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \quad \text{where } \vec{F} = (F_1, F_2)$$
↳ positively oriented

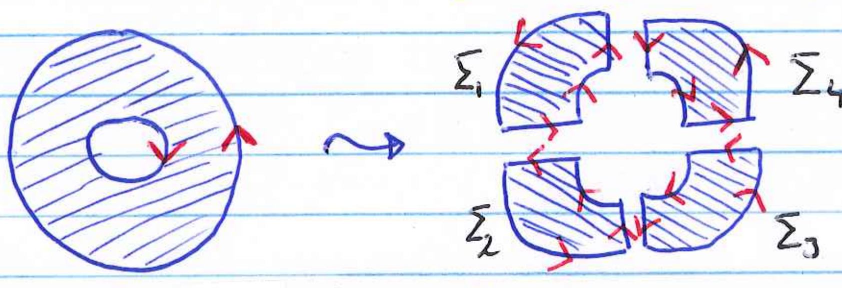
or using the convenient (but dangerous) notation:
$$\int_{\partial S} P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \quad \text{where } \vec{F} = (P, Q)$$
↳ positively oriented

Remark: It's another special case of a general result:
Stokes theorem: $\int_{\Omega} d\omega = \int_{\partial \Omega} \omega$

Δ We prove Green's theorem in a special case: for S that can be broken into finitely many elementary regions.

We say that $\Sigma \subset \mathbb{R}^2$ is elementary if
$$\Sigma = \{(x,y) : a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\} \quad \varphi_i\ C^0 \text{ and piecewise } C^1$$
and $$= \{(x,y) : c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\} \quad \psi_i\ C^0 \text{ and piecewise } C^1$$

eg: S



Notice that $\iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \sum_{i=1}^{N} \iint_{\Sigma_i} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ (additivity of the domain)

and that $\int_{\partial S} \vec{F} \cdot d\vec{x} = \sum_{i=1}^{N} \int_{\partial \Sigma_i} \vec{F} \cdot d\vec{x}$ since the additional edges are added twice with opposite orientation and cancel each other

Hence, it is enough to prove the result for $\Sigma \subset \mathbb{R}^2$ elementary region:

$$\iint_\Sigma \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \iint_\Sigma \frac{\partial F_2}{\partial x} - \iint_\Sigma \frac{\partial F_1}{\partial y}$$

$$= \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x,y)\,dx\,dy - \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial F_1}{\partial y}(x,y)\,dy\,dx$$

$$= \int_c^d F_2(\psi_2(y),y) - F_2(\psi_1(y),y)\,dy$$

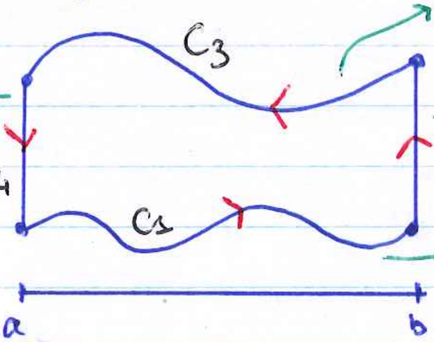
$$- \int_a^b F_1(x,\varphi_2(x)) - F_1(x,\varphi_1(x))\,dx$$

$$= \int_{\partial\Sigma} F_2\,dy + \int_{\partial\Sigma_1} F_1\,dx = \int_{\partial\Sigma_1} F_1 dx + F_2 dy$$

For the last equality:

$\sigma_3(t) = (t,\varphi_2(t))$, $t \in [a,b]$, opposite orientation

$\sigma_4(t) = (a,t)$
$t \in [\varphi_1(a),\varphi_2(a)]$
opposite orientation



$\sigma_2(t) = (b,t)$, $t \in [\varphi_1(b),\varphi_2(b)]$

$\sigma_1(t) = (t,\varphi_1(t))$, $t \in [a,b]$

so $$\int_{\partial\Sigma} F_1 dx = \int_{C_1} F_1 dx + \int_{C_2} F_1 dx + \int_{C_3} F_1 dx + \int_{C_4} F_1 dx$$

$$= \int_a^b F_1(t,\varphi_1(t))\cdot 1\,dt + \int_{\varphi_1(b)}^{\varphi_2(b)} F_1(b,t)\cdot 0\,dt$$

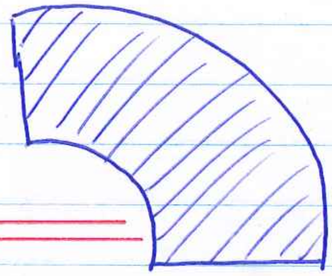
$$- \int_a^b F_1(t,\varphi_2(t))\cdot 1\,dt - \int_{\varphi_1(a)}^{\varphi_2(a)} F_1(a,t)\cdot 0\,dt$$

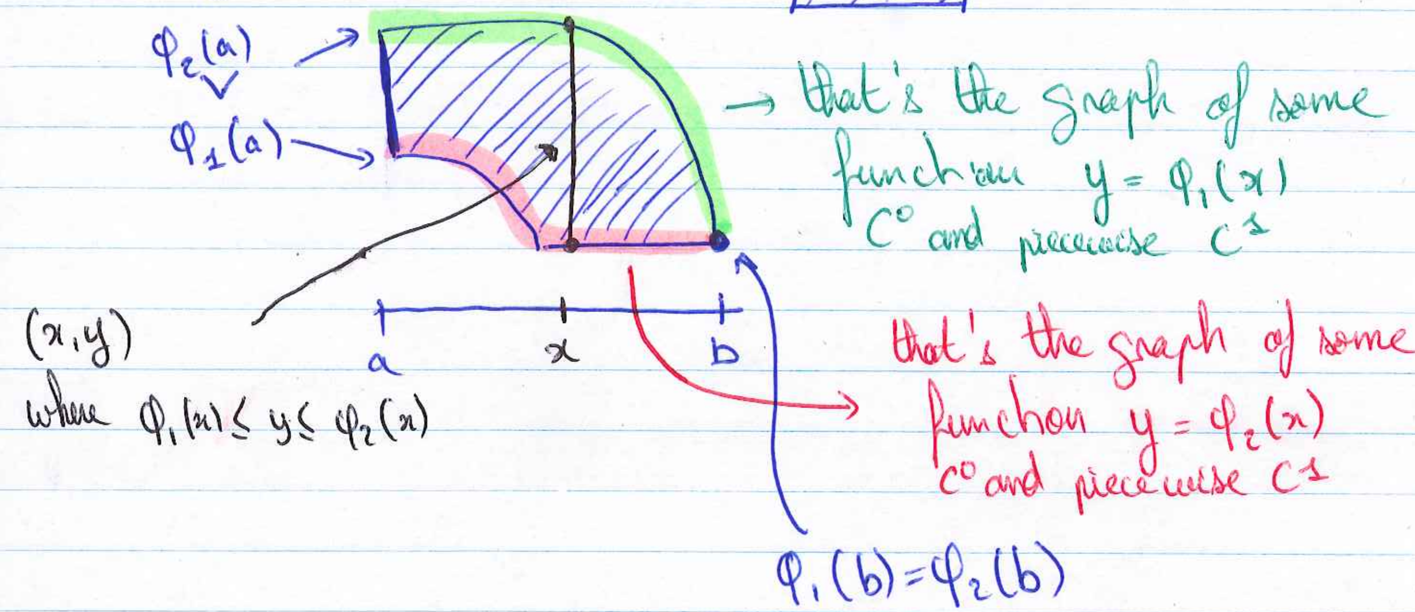
$$= \int_a^b F_1(t,\varphi_1(t)) - F_1(t,\varphi_2(t))\,dt$$

And similarly for $\int_{\partial\Sigma} F_2\,dy = \int_c^d F_2(\psi_2(t),t) - F_2(\psi_1(t),t)\,dt$

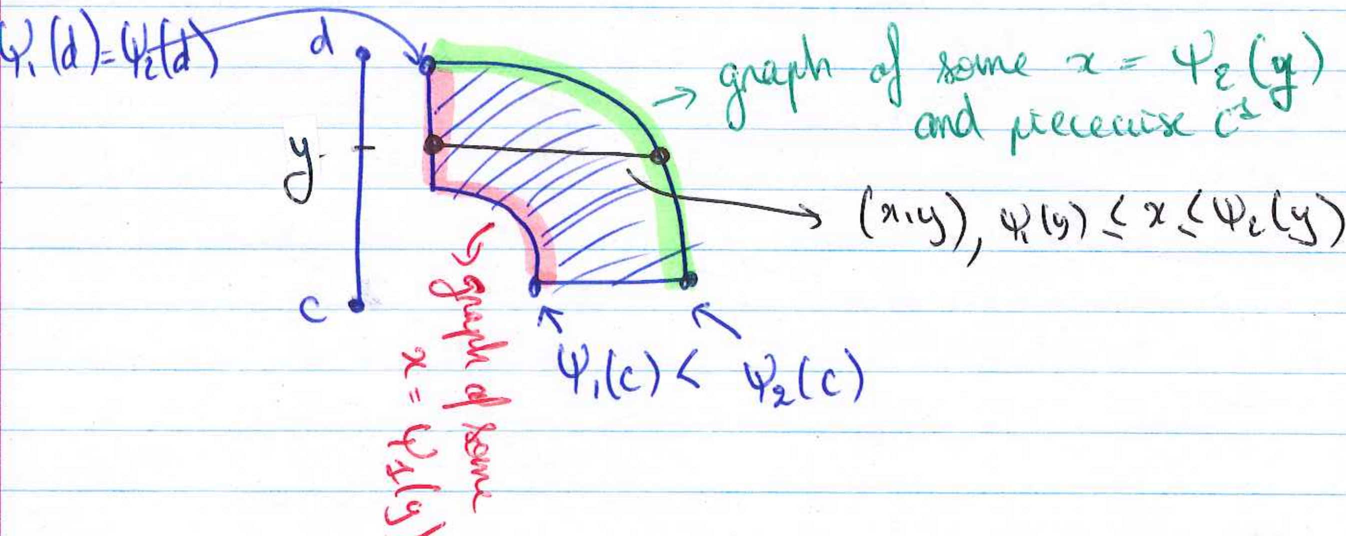
using $\Sigma_1 = \{(x,y) : c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$ $\square$

ADDENDUM:

Why is Σ =  **an elementary region?**



So $\Sigma = \{(x,y) : a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\}$



So we also have that $\Sigma = \{(x,y) : c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$

Ccl: $\exists\, \varphi_1, \varphi_2, \psi_1, \psi_2$ C^0 and piecewise C^1 such that

$$\Sigma = \{(x,y) : a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\} = \{(x,y) : c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$$

Comment: Unfortunately, some regular regions with piecewise smooth boundaries can't be broken into finitely many elementary regions: We only proved a special case of Green's theorem:

Ex: $\{(x,y) : 0 \leq x \leq 1, \ 0 \leq y \leq 1 + x^3 \sin(1/x)\}$

because of the oscillations around $x = 0^+$

Here is an alternative statement:

Theorem: "Green's theorem for flux $\int_{\partial S} \vec{F} \cdot \vec{n} = \iint \operatorname{div}(\vec{F})$"

$S \subset \mathbb{R}^2$ regular region with piecewise smooth boundaries

$F : U \to \mathbb{R}^2$ C^1, U open, $S \subset U \subset \mathbb{R}^2$

$$\iint_S \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \int_{\partial S} \vec{F} \cdot \vec{n}$$

line integral for the real-valued function $x \mapsto F(x) \cdot n(x)$

positively oriented

where $\vec{n}(x)$ is the normal outward pointing unit vector at $\vec{x} \in \partial S$

Δ $\partial S = \bigcup C_i$, σ a parametrization of $C := C_i$ positively oriented

then $\vec{n}(\sigma(t)) = \dfrac{1}{\|\sigma'(t)\|} \left(\sigma_2'(t), -\sigma_1'(t)\right)$

Hence
$$\int_{\partial S} \vec{F} \cdot \vec{n} = \int_a^b \frac{\left(F_1(\sigma(t))\, \sigma_2'(t) - F_2(\sigma(t))\, \sigma_1'(t)\right) \cancel{\|\sigma'(t)\|}}{\cancel{\|\sigma'(t)\|}} \, dt$$

$\partial S \to \mathbb{R}$

$x \mapsto \vec{F}(x) \cdot \vec{n}(x)$

line integral for real valued

$$= \int_{\partial S} -F_2 \, dx + F_1 \, dy = \iint_S \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \qquad \square$$

line integral for vector field $\vec{F} = (-F_2, F_1)$

Green's thm

Ex: Use Green's theorem to compute a difficult $\int_C$ thanks to an easy $\iint_S$: $S = \overline{B}(0,1)$, $C = \partial S$ positively oriented

$$\int_C y e^{-x} dx + \left(\tfrac{1}{2}x^2 - e^{-x}\right) dy = \iint_S (x + e^{-x}) - e^{-x} = \iint_S x = \int_{-\pi}^{\pi} \int_0^1 r^2 \cos\theta \, dr \, d\theta$$
$$= 0$$

Ex 2. Green's theorem allows to compute an area thanks to a line integral!

→ Type "planimeter" on google or your favorite search engine.

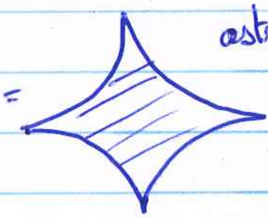
Take P, Q s.t. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then

$$\text{Area}(S) = \iint_S 1 = \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \int_{\partial S} P\,dx + Q\,dy$$

Ex: $\int_{\partial S} x\,dy$, $\int_{\partial S} -y\,dx$, $\int_{\partial S} \frac{-y}{2}\,dx + \frac{x}{2}\,dy$

i.e. $\vec{F} = (0, x)$, $\vec{F} = (-y, 0)$, $\vec{F} = (-y/2, x/2)$

For instance: $S =$



astroid , $\partial S : (\cos^3 t, \sin^3 t)$

$$\text{Area}(S) = \frac{1}{2}\int_{\partial S} -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} \frac{3}{2}\sin^4 t\cos^2 t + \frac{3}{2}\cos^4 t\sin^2 t\,dt$$

$$= \frac{3}{2}\int_0^{2\pi} \cos^2 t \sin^2 t\,dt$$

$$= \frac{3}{8}\int_0^{2\pi} \sin^2(2t)\,dt$$

$$= \frac{3}{16}\int_0^{2\pi} 1 - \cos(4t)\,dt$$

$$= \frac{3\pi}{8}$$

Ex3: Gradient fields / Conservative fields

Green's theorem allows to prove the following result to check if a vector field is conservative or not

Proposition: $\cdot\ F : \mathbb{R}^2 \to \mathbb{R}^2$ a C^1 vector field

$\cdot\ F = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}$ C^2

if and only if $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$

Δ $\underline{\Rightarrow}$: $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial^2 f}{\partial x \partial y} \underset{\text{Clairaut}}{=} \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial F_1}{\partial y}$

$\underline{\Leftarrow}$: $\oint_C \vec{F} \cdot d\vec{x} \underset{\text{Green}}{=} \iint_S \left(\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} \right) = 0$ so $\oint_C \vec{F} \cdot d\vec{x}$ for any closed curve

$\Rightarrow \oint_C \vec{F} \cdot d\vec{x}$ independent on C for any curve C $\square$

Ex4 :

$$\int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

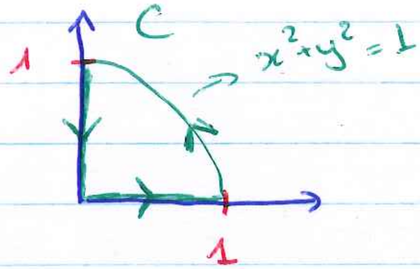
$$\| \quad \int_C \vec{F} \cdot d\vec{x} + \int_D \vec{F} \cdot d\vec{x}$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{x} = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) - \int_D \vec{F} \cdot d\vec{x}$$

that could be easier to compute

rather than

Ex 5 :



Compute $\int_C xy^2\,dx + 2xy\,dy$

Method 1 : directly, we have to parametrize the 3 edges $(t,0)$, $(\cos t, \sin t)$, $(0,t)$ and compute 3 integrals.

Method 2 : with Green :

$$\int_C xy^2\,dx + 2xy\,dy = \iint_S 2y - 2xy$$

since C is positively oriented for $C = \partial S$ where $S = \{x \geq 0, y \geq 0, x^2+y^2 \leq 1\}$

$$= \int_0^1 \int_0^{\pi/2} 2r^2 \sin\theta - r^3 \sin(2\theta)\, d\theta\, dr$$

$$= \int_0^1 2r^2 - r^3\, dr = 5/12$$

Ex 6 : $S = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$, $\partial S = \{(a\cos\theta, b\sin\theta), \theta \in [-\pi, \pi]\}$

$$A(S) = \int_{\partial S} x\,dy$$

$$= \int_{-\pi}^{\pi} ab\cos^2\theta\, d\theta = \int_{-\pi}^{\pi} ab\,\frac{\cos(2\theta)+1}{2}\, d\theta$$

$$= \frac{ab}{2}\int_{-\pi}^{\pi} (\cos(2\theta)+1)\, d\theta = \pi ab$$

Surface integrals

Convention: in this section, by a surface I mean:

$$S = \{\sigma(t) : t \in T\} \subset \mathbb{R}^3$$

where $\sigma : \mathcal{U} \to \mathbb{R}^3$ is C^1, $\mathcal{U} \subset \mathbb{R}^2$ is open, $T \subset \mathcal{U}$ is Jordan measurable, σ is injective on T, and $(\partial_1\sigma, \partial_2\sigma)$ are linearly independant except on a set having zero content

Notation: $\partial_1\sigma = \left(\frac{\partial\sigma_1}{\partial x}, \frac{\partial\sigma_2}{\partial x}, \frac{\partial\sigma_3}{\partial x}\right)$, $\partial_2\sigma = \left(\frac{\partial\sigma_1}{\partial y}, \frac{\partial\sigma_2}{\partial y}, \frac{\partial\sigma_3}{\partial y}\right)$

Def: Let $S \subset \mathbb{R}^3$ be as above, $f : S \to \mathbb{R}$ C^0
We define the surface integral of f over S by

$$\iint_S f = \iint_T f(\sigma(u,v)) \left\| \partial_1\sigma(u,v) \times \partial_2\sigma(u,v) \right\| du\, dv$$

Comment 1: It doesn't depend on the parametrization of S

Comment 2: In general S may not admit a "global" parametrization
It is possible to give a more general definition
but it outreaches MAT237

The most general case we will consider is

$S = \bigcup_{i=1}^{N} S_i$ where

1. S_i is as above
2. $S_i \cap S_j = \emptyset$ or $S_i \cap S_j$ is a curve

then $\int_S f = \sum_{i=1}^{N} \int_{S_i} f$ (By 2 the curves counted twice have a zero integral)

Def: $S \subset \mathbb{R}^3$ be a surface as before

The area of S is $A(S) := \iint_S 1 = \iint_T \|\partial_1 \sigma \times \partial_2 \sigma\|$

Ex: $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$

$\sigma(\varphi, \theta) = (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi) \qquad \theta \in [0, 2\pi], \varphi \in [0, \pi]$

$$\|\partial_1 \sigma \times \partial_2 \sigma\| = \left\| \begin{pmatrix} -\sin\theta \sin\varphi \\ \cos\theta \sin\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\theta \cos\varphi \\ \sin\theta \cos\varphi \\ -\sin\varphi \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} -\cos\theta \sin^2\varphi \\ -\sin\theta \sin^2\varphi \\ -\sin\varphi \cos\varphi \end{pmatrix} \right\|$$

$$= \left(\cos^2\theta \sin^4\varphi + \sin^2\theta \sin^4\varphi + \sin^2\varphi \cos^2\varphi \right)^{1/2}$$

$$= \left(\sin^4\varphi + \sin^2\varphi \cos^2\varphi \right)^{1/2} = \sin\varphi$$

So $A(S) = \int_0^{2\pi} \int_0^{\pi} \sin\varphi \, d\varphi \, d\theta = 4\pi$

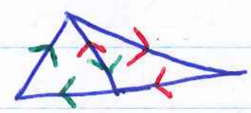
Orientation of a surface in $\mathbb{R}^3$

There are several equivalent ways to define the orientability of a surface:
$S \subset \mathbb{R}^3$ is orientable if

1. There exists a continuous normal vector field:
$$\vec{n} : S \to \mathbb{R}^3 \quad C^0$$
s.t. $\forall p \in S$, $\vec{n}(p) \neq \vec{0}$ and $\vec{n}(p)$ is orthogonal to S at p.

2. We may cover S by local parametrizations whose orientations agree on their intersections

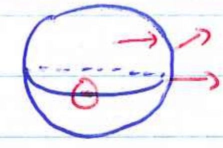
3. We may decompose S into triangles with compatible orientation along a common edge:

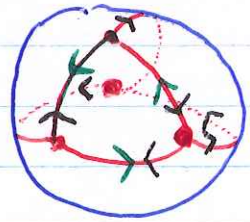


we want both triangle to give an opposite orientation on the common edge

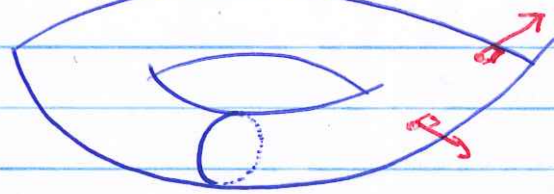
<u>Remark</u>: It's a subtle notion, so I prefer to keep it informal to avoid technical details

<u>Eg. Sphere</u>: $S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$ is orientable
take $\vec{n}(p)$ = outward pointing unit normal vector at p



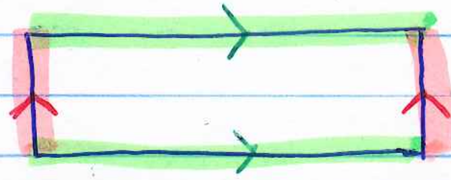


a triangulation with 4 triangles

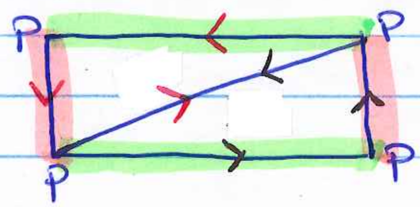
Eg: Torus : T = 

$\vec{n}(p)$ = unit normal vector at p which is pointing outward

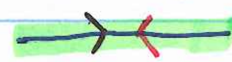
T may be obtained by gluing the edges of a band of paper like that:

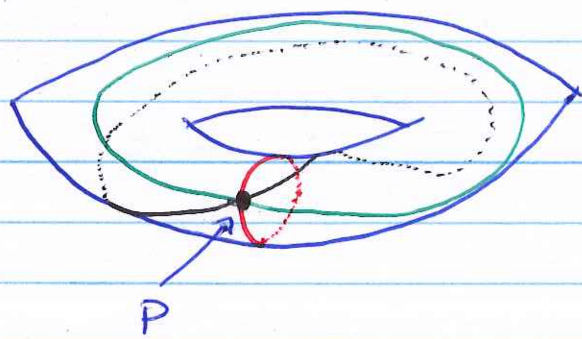


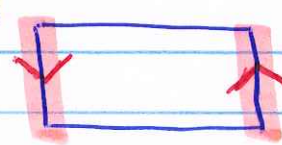
It is easier to see a triangulation this way:



(notice that the opposite orientation condition is satisfied by the edges glued together)

 and 



Eg: some surfaces are not orientable, for instance the Mobius band that you can construct yourself: 

Comment: If a surface is orientable, it admits only two orientations in each connected component.

So $\#$ of orientation $= 2^{\#\text{connected components}}$

Def. Let $S \subset \mathbb{R}^3$ be an oriented surface whose orientation is given by $\vec{n} : S \to \mathbb{R}^3$ a continuous unit ($\|\vec{n}\| = 1$) normal vector field
Let $F : S \to \mathbb{R}^3$ be a C^0 vector field

The surface integral of F along S oriented by $\vec{n}$ is

$$\iint_S \vec{F} \cdot d\vec{S} := \iint_S \vec{F} \cdot \vec{n}$$

(Here $x \mapsto \vec{F}(x) \cdot \vec{n}(x)$ is a real valued function, so the surface integral of $\vec{F} \cdot \vec{n}$ is well defined)

Comment: The surface integral of a vector field is not defined along non-orientable surfaces.

Def. Let $S \subset \mathbb{R}^3$ be a surface as above together with an orientation given by a continuous normal vector field $\vec{n} : S \to \mathbb{R}^3$

Notice that $\partial_1 \sigma \times \partial_2 \sigma$ is normal to S, so either

① $\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v) = \lambda \vec{n}(\sigma(u,v))$, $\lambda > 0$
ie the parametrization is compatible with the orientation

or ② $\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v) = \lambda \vec{n}(\sigma(u,v))$, $\lambda < 0$
ie the parametrization gives the opposite orientation

Proposition: If $S \subset \mathbb{R}^3$ is an oriented surface together with a parametrization $\sigma : T \to S$ compatible with its orientation, then

$$\iint_S \vec{F} \cdot d\vec{S} := \iint_S \vec{F} \cdot \vec{n} = \iint_T F(\sigma(u,v)) \cdot (\partial_1 \sigma(u,v) \times \partial_2 \sigma(u,v))\, du\, dv$$

Δ Indeed, then $\vec{n}(\sigma(u,v)) = \dfrac{\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)}{\|\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)\|}$, thus

$$\iint_S \vec{F}\cdot\vec{n} = \iint_T \big(F(\sigma(u,v))\cdot n(\sigma(u,v))\big)\,\|\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)\|\,du\,dv$$

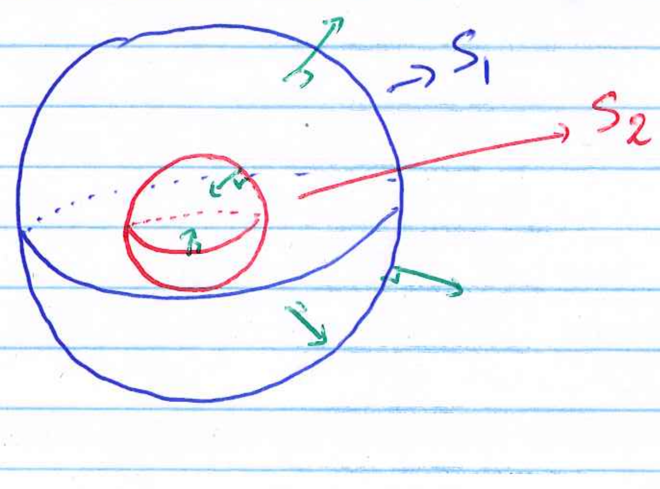
$$= \iint_T F(\sigma(u,v))\cdot\big(\partial_1\sigma(u,v)\times\partial_2\sigma(u,v)\big)\,du\,dv$$

$\square$

<u>Comment</u>: If a surface is part of the boundary of a regular region in $\mathbb{R}^3$ then it is always orientable.
The usual orientation consists in taking pointwise the normal vector pointing outward.

<u>Ex</u> : $R = \{(x,y,z) : 1 \leq x^2+y^2+z^2 \leq 4\}$

$\partial R = S_1 \cup S_2$

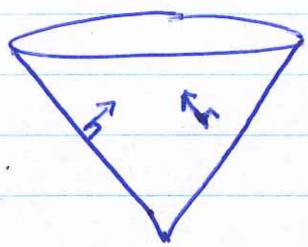


Example : $S = \{ x^2 + y^2 = z^2 ,\ 0 \leq z \leq 1 \}$

- orientation given by $\vec{n}$ pointing to the z-axis
- $F(x,y,z) = (xz, yz, y)$

Compute $\iint_S \vec{F} \cdot \vec{n}$

Δ



(we could also have used $\sigma(x,y) = (x, y, \sqrt{x^2+y^2})$ for $x^2+y^2 \leq 1$)

$\sigma(r,\theta) = (r\cos\theta, r\sin\theta, r)$, $r \in [0,1]$, $\theta \in [-\pi, \pi]$

$$\partial_1 \sigma = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix} \qquad \partial_2 \sigma = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix}$$

$$\partial_1 \sigma \times \partial_2 \sigma = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{pmatrix}$$

Gives the good orientation. (check it)

$$\iint_S \vec{F} \cdot \vec{n} = \int_{-\pi}^{\pi} \int_0^1 \begin{pmatrix} r^2\cos\theta \\ r^2\sin\theta \\ r\sin\theta \end{pmatrix} \cdot \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{pmatrix} dr\, d\theta$$

$$= \int_{-\pi}^{\pi} \int_0^1 \underbrace{-r^3\cos^2\theta - r^3\sin^2}_{-r^3} + r^2\sin\theta \, dr\, d\theta$$

$$= \int_{-\pi}^{\pi} -\frac{1}{4} + \frac{1}{3}\sin\theta \, d\theta$$

$$= -\frac{\pi}{2}$$

Addendum 1: Why is there a cross product in the surface integral of a real-valued function?

Answer 1: from a mathematics point of view:

For the same reason we have a Jacobian determinant in the change of variables formula:

$$\iint_S f = \iint_T f(\sigma(u,v)) \underbrace{\|\partial_1\sigma(u,v) \times \partial_2\sigma(u,v)\|}\, du\, dv$$

this factor ensures that the value doesn't depend on the "speed" of the parametrization (and hence on the choice of the parametrization)

ⓐ You can repeat the heuristic idea I gave at the beginning of the CoV (p58 of the notes) and see that it is the "good" factor to add.

ⓑ You can compute directly:

$\sigma_1 : T_1 \longrightarrow S$, $\sigma_2 : T_2 \longrightarrow S$, parametrizations

$\varphi : T_2 \longrightarrow T_1$ C^1-diffeomorphism (actually $\varphi : \mathcal{U}_2 \to \mathcal{U}_1$ open s.t. $\varphi(T_2) = T_1$ ...)

s.t. $\sigma_2 = \sigma_1 \circ \varphi$

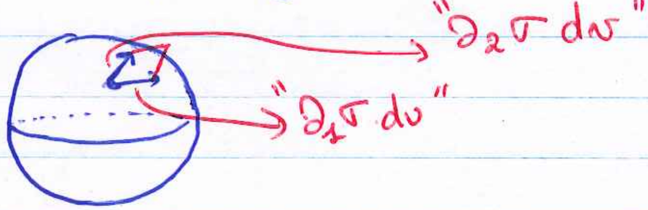
$$\iint_{T_2} f\circ\sigma_2 \, \|\partial_1\sigma_2 \times \partial_2\sigma_2\| = \iint_{T_2} f(\sigma_1(\varphi)) \, \|\partial_1\sigma_1(\varphi) \times \partial_2\sigma_1(\varphi)\| \, |\det D\varphi|$$

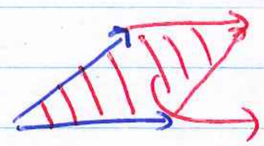
$$\boxed{\text{CoV} \longrightarrow} = \iint_{T_1} f\circ\sigma_1 \, \|\partial_1\sigma_1 \times \partial_2\sigma_1\|$$

Compute $\partial_1(\sigma_1\circ\varphi)$
$\partial_2(\sigma_1\circ\varphi)$
and simplify...

Answer 2: from a "physics" point of view.

We use the parametrization to "locally flatten" S by approximating it with parallelograms:





the area here is
$\|(\partial_1\sigma_1 du) \times (\partial_2\sigma_2 dv)\|$

So we get some kind of Riemann sum

$$\sum_R f(p)\, \partial(R)$$

$p \in R$

↳ R is the parallelogram

and the limit when $\partial(R) \longrightarrow 0$

gives the integral

x——x

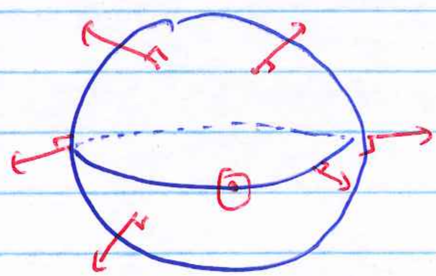
Not part of MAT 237: come back here when you'll learn about "the first differential form" and "Gauss Theorema Egregium" in Riemannian geometry:

$$\|\partial_1\sigma \times \partial_2\sigma\| = \sqrt{(\partial_1\sigma\cdot\partial_1\sigma)(\partial_2\sigma\cdot\partial_2\sigma) - (\partial_1\sigma\cdot\partial_2\sigma)^2}$$
$$= \sqrt{EG - F^2}$$

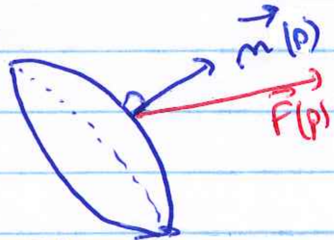
Addendum 2: What's the physics interpretation of the surface integral of vector field?

Assume that we have a fluid (in motion) in the space and denote by $\vec{F}(p)$ the velocity of the fluid at p

We have S an oriented surface, let's say a sphere with orientation given by outward pointing normal (unit) vector

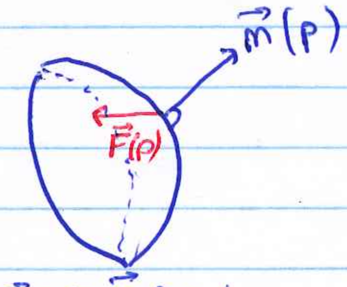


Take $p \in S$ then locally:



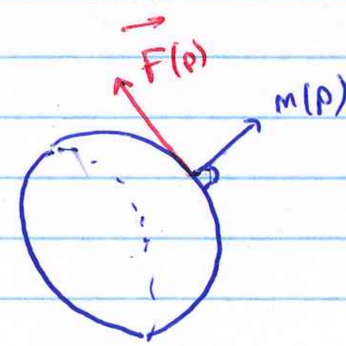
$\vec{n}(p) \cdot \vec{F}(p) > 0$

"the fluid goes out" at p



$\vec{n}(p) \cdot \vec{F}(p) < 0$

"the fluid goes in" at p



$\vec{n} \cdot \vec{F}(p) = 0$

"the fluid doesn't cross S at p"

Gradient, curl, divergence

Def: $U \subset \mathbb{R}^m$ open, $f : U \longrightarrow \mathbb{R}$ C^1 → real valued

The gradient of f at $x \in U$ is

$$\nabla f(x) := \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_m}(x) \right) \in \mathbb{R}^m$$

so that $\nabla f : U \longrightarrow \mathbb{R}^m$ C^0

→ domain and codomain have same dimension

Def: $U \subset \mathbb{R}^m$ open, $F : U \longrightarrow \mathbb{R}^m$ C^1

The divergence of F at $x \in U$ is

$$\operatorname{div} F(x) := \frac{\partial F_1}{\partial x_1}(x) + \frac{\partial F_2}{\partial x_2}(x) + \dots + \frac{\partial F_m}{\partial x_m}(x) \in \mathbb{R}$$

so that $\operatorname{div} F : U \longrightarrow \mathbb{R}$ C^0

Notation/Mnemonic device:

if you write $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_m)$ and $F = (F_1, \dots, F_m)$

then $\operatorname{div} F = \underbrace{\nabla \cdot F}_{\text{dot product}} = \sum_{i=1}^{m} \frac{\partial}{\partial x_i}(F_i)$

So it is common to denote $\operatorname{div}(F) = \nabla \cdot F$

Be careful, that's just an abuse of notation and a good mnemonic device: $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_m)$ is not really a vector and $\partial/\partial x_i$ is not a scalar only for $n=3$ codomain is also $\mathbb{R}^3$

Def: $\mathcal{U} \subset \mathbb{R}^3$ open, $F: \mathcal{U} \longrightarrow \mathbb{R}^3$ C^1

The curl of F at $x \in \mathcal{U}$ is

$$\operatorname{curl} F(x) := \begin{pmatrix} \frac{\partial F_3}{\partial x_2}(x) - \frac{\partial F_2}{\partial x_3}(x) \\ \frac{\partial F_1}{\partial x_3}(x) - \frac{\partial F_3}{\partial x_1}(x) \\ \frac{\partial F_2}{\partial x_1}(x) - \frac{\partial F_1}{\partial x_2}(x) \end{pmatrix} \in \mathbb{R}^3$$

So that $\operatorname{curl} F: \mathcal{U} \longrightarrow \mathbb{R}^3$ C^0

Notation / Mnemonic device:

if you write $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, $F = (F_1, F_2, F_3)$

then $\operatorname{curl} F = \nabla \times F$
↳ cross product

So it is common to write $\operatorname{curl} F = \nabla \times F$

(Again, it is just an abuse of notation and a very good mnemonic device)

Def: $\mathcal{U} \subset \mathbb{R}^n$ open, $f: \mathcal{U} \to \mathbb{R}$ C^2 C^2!

We define the Laplacian or Laplace operator of f at x by

$$\Delta f(x) := \frac{\partial^2 f}{\partial x_1^2}(x) + \cdots + \frac{\partial^2 f}{\partial x_n^2}(x) \in \mathbb{R}$$

so that $\Delta f: \mathcal{U} \to \mathbb{R}$ C^0

Comment: $\Delta f = \operatorname{div}(\operatorname{grad} f) = \nabla \cdot (\nabla f)$
so it is common to use the notation $\nabla^2 f = \Delta f$

Def. $U \subset \mathbb{R}^3$ open, $F : U \to \mathbb{R}^3$ C^2

The vector Laplacian of F at $x \in U$ is

$$\Delta F(x) = (\Delta F_1(x), \Delta F_2(x), \Delta F_3(x)) \in \mathbb{R}^3$$

where $F = (F_1, F_2, F_3)$

Comment: $\Delta F = \operatorname{grad}(\operatorname{div} F) - \operatorname{curl}(\operatorname{curl} F) = \nabla(\nabla \cdot F) - \nabla \times (\nabla \times F)$

Product rules: $f, g : \mathbb{R}^n \to \mathbb{R}$, $F, G : \mathbb{R}^n \to \mathbb{R}^n$ C^1
($n = 3$ when a curl is involved)

$$\left.\begin{aligned}
\nabla(fg) &= f \nabla g + g \nabla f \\
\operatorname{div}(fG) &= f \operatorname{div}(G) + (\nabla f) \cdot G \\
\operatorname{curl}(fG) &= f \operatorname{curl} G + (\nabla f) \times G \\
\operatorname{div}(F \times G) &= G \cdot (\operatorname{curl} F) - F \cdot (\operatorname{curl} G)
\end{aligned}\right\} \text{essentials}$$

a little bit less

$$\left\{\begin{aligned}
\operatorname{curl}(F \times G) &= (G \cdot \nabla) F + (\operatorname{div} G) F - (F \cdot \nabla) G - (\operatorname{div} F) G \\
\nabla(F \cdot G) &= (G \cdot \nabla) F + G \times (\operatorname{curl} F) + (F \cdot \nabla) G + F \times (\operatorname{curl} G)
\end{aligned}\right.$$

$$(F \cdot \nabla) G = \left(\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \cdot \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix} \right) G = \left(F_1 \frac{\partial}{\partial x_1}, F_2 \frac{\partial}{\partial x_2}, F_3 \, \partial/\partial x_3 \right) G$$

(Mnemonic device)

$$= F_1 \frac{\partial G}{\partial x_1} + F_2 \frac{\partial G}{\partial x_2} + F_3 \frac{\partial G}{\partial x_3} \in \mathbb{R}^3$$

$$\partial G / \partial x_i = \left(\frac{\partial G_1}{\partial x_i}, \frac{\partial G_2}{\partial x_i}, \frac{\partial G_3}{\partial x_i} \right) \in \mathbb{R}^3$$

Assume that $n = 3$ then:

For $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ C^2, $\operatorname{curl}(\nabla f) = \vec{0}$

For $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ C^2, $\operatorname{div}(\operatorname{curl} F) = 0$

That you may summarize by noticing that two successive arrows in the following diagram give 0:

$$\left\{\begin{matrix}\text{functions}\\ \mathbb{R}^3 \to \mathbb{R}\end{matrix}\right\} \xrightarrow{\nabla} \left\{\begin{matrix}\text{vector fields}\\ \mathbb{R}^3 \to \mathbb{R}^3\end{matrix}\right\} \xrightarrow{\text{curl}} \left\{\begin{matrix}\text{vector fields}\\ \mathbb{R}^3 \to \mathbb{R}^3\end{matrix}\right\} \xrightarrow{\text{div}} \left\{\begin{matrix}\text{functions}\\ \mathbb{R}^3 \to \mathbb{R}\end{matrix}\right\}$$

Not part of MAT 237: When you will learn differential forms and de Rham cohomology, come back here and compare the above diagram with:

$$\Omega^0(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3)$$

where $\Omega^k(\mathbb{R}^3) = \{ k\text{-differential forms on } \mathbb{R}^3 \}$

and d is the exterior derivative and $d \circ d = 0$
(exact forms are closed)

Spoiler: ∇, curl, div correspond to the exterior derivative using the following bases:

$\Omega^1(\mathbb{R}^3) = \langle dx, dy, dz \rangle$

$\Omega^2(\mathbb{R}^3) = \langle dydz, dzdx, dxdy \rangle$

HOMEWORK: Questions from 5.4

The divergence theorem

(also called Gauss theorem or Green - Ostrogradski theorem)

[Not part of MAT237: it is again a consequence of the general Stokes theorem $\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$

Theorem: $R \subset \mathbb{R}^3$ a regular region with piecewise smooth boundary (ie ∂R is as in the surface integral section)
We assume that ∂R is oriented by the normal outward pointing unit vector $\vec{n}(x)$

$$F : \mathcal{U} \longrightarrow \mathbb{R}^3 \quad C^1, \ \mathcal{U} \subset \mathbb{R}^3 \text{ open}, \ R \subset \mathcal{U}$$

Then
$$\underbrace{\iint_{\partial R} \vec{F} \cdot \vec{n}}_{\text{Surface integral for vector fields}} = \underbrace{\iiint_{R} \operatorname{div} F}_{\text{usual integral for 3 variables and } \operatorname{div} F : R \to \mathbb{R}}$$

Δ We are not proving this theorem

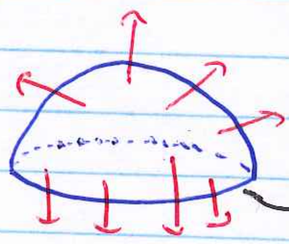
① If each part of the boundary is elementary
ie $\{\varphi_1(x,y) \leq z \leq \varphi_2(x,y)\} = \{\psi_1(y,z) \leq x \leq \psi_2(y,z)\}$
$= \{\tau_1(x,z) \leq y \leq \tau_1(x,z)\}$
then we may follow the proof of Green theorem

② The general case is too difficult for MAT237

□

Ex: Let $R = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 2,\ z \geq 0\}$

and $S = \partial R$ oriented by taking the unit outward pointing normal vector $\vec{n}$

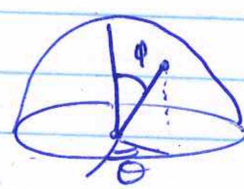


on this circle the normal vector is not well defined but it's a curve so it has an integral of 0 (zero content)

$F(x,y,z) = ((2x+y)z, 0, z)$

Compute $\iint_S \vec{F} \cdot \vec{n}$.

$$\iint_S \vec{F}\cdot\vec{n} = \iiint_R \operatorname{div} F \overset{\text{divergence theorem}}{=} \iiint_R 2z+1$$



$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}} (2r\cos\varphi + 1)\, r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$

$$= 2\pi \int_0^{\pi/2}\int_0^{\sqrt{2}} r^3 \sin(2\varphi) + r^2 \sin\varphi \, dr\, d\varphi$$

$$= 2\pi \int_0^{\sqrt{2}} r^3 + r^2 \, dr$$

$$= 2\pi\left(1 + \frac{2\sqrt{2}}{3}\right)$$

The divergence theorem allows to give a physical interpretation of the divergence operator.

$\mathcal{U} \subset \mathbb{R}^3$ open, $F : \mathcal{U} \longrightarrow \mathbb{R}^3$ C^1, $p \in \mathcal{U}$

then if $r > 0$ is small $\operatorname{div} F(x) \simeq \operatorname{div} F(p)$ for $x \in \overline{B}(p,r)$ (continuity)

so that:

$$\iiint_{\overline{B}(p,r)} \operatorname{div} F \simeq \operatorname{div} F(p) \cdot \iiint_{\overline{B}(p,r)} 1 = \frac{4}{3}\pi r^3 \operatorname{div} F(p)$$

ie
$$\operatorname{div} F(p) \simeq \frac{3}{4\pi r^3} \iiint_{\overline{B}(p,r)} \operatorname{div} F$$

$$= \frac{3}{4\pi r^3} \iint_{\partial \overline{B}(p,r)} \vec{F} \cdot \vec{n}$$

By the way this is independant of the coordinates so $\operatorname{div} F$ is the same in any coordinate systems

Ccl: $\operatorname{div} F(p) = \lim_{r \to 0} \frac{3}{4\pi r^3} \iint_{\partial \overline{B}(p,r)} \vec{F} \cdot \vec{n}$

ie $\operatorname{div} F(p)$ measures the outward flux near p per unit of volume per unit of time

so if $\operatorname{div} F(p) > 0$, p is a source, the flux is outgoing 

if $\operatorname{div} F(p) < 0$, p is a sink, the flux is ingoing 

<u>Gauss' Law</u>: $R \subset \mathbb{R}^3$ a regular region with piecewise smooth boundary s.t. $\vec{0} \notin \partial R$
We assume that ∂R is oriented by the normal outward pointing unit vector field.

Then $$\iint_{\partial R} \frac{\vec{r}}{\|\vec{r}\|^3} \cdot \vec{n} = \begin{cases} 4\pi & \text{if } \vec{0} \in R \\ 0 & \text{otherwise} \end{cases}$$

where $\vec{r}(x,y,z) = (x,y,z)$

Δ <u>First case</u>: $\vec{0} \notin R$

$$\operatorname{div}\left(\frac{\vec{r}}{\|\vec{r}\|^3}\right) = \operatorname{div}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \|\vec{r}\|^2 - 3x^2 + \|\vec{r}\|^2 - 3y^2 + \|\vec{r}\|^2 - 3z^2$$
$$= 3\|\vec{r}\|^2 - 3\|\vec{r}\|^2$$
$$= 0$$

and it is true on R because $0 \notin R$

By the divergence theorem: $\iint_{\partial R} \frac{\vec{r}}{\|\vec{r}\|^3} = \iiint_R 0 = 0$

<u>Second case</u>: $\vec{0} \in R$

We can no longer directly apply the divergence theorem since $\frac{\vec{r}}{\|\vec{r}\|}$ is not defined at $\vec{0} \in R$

Since $\vec{0} \in R \setminus \partial R$, we know that $\vec{0} \in \mathring{R}$
hence $\exists \varepsilon > 0$ s.t.

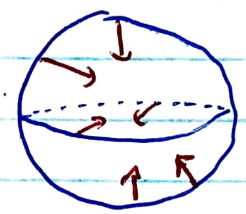
$$B(\vec{0}, \varepsilon) \subset \mathring{R} \subset \overline{\mathring{R}} = R$$

↳ $\mathring{R}$ open ↳ always true ↳ R is a regular region

We define $R' = R \setminus B(0,\varepsilon)$

then $\partial R' = \partial R \cup \partial B(0,\varepsilon)$ with the orientation pointing outward from R' so

$\partial B(0,\varepsilon)$:



(inward for the ball) ⚠

By the divergence theorem:

$$0 = \iiint_{R'} 0 = \iiint_{R'} \mathrm{div}\left(\frac{\vec{r}}{\|\vec{r}\|^3}\right) = \iint_{\partial R} \frac{\vec{r}}{\|\vec{r}\|^3} - \iint_{\partial B(0,\varepsilon)} \frac{\vec{r}}{\|\vec{r}\|^3}$$

↳ outward pointing for the ball here

$$\Rightarrow \iint_{\partial R} \frac{\vec{r}}{\|\vec{r}\|^3} = \iint_{\partial B(0,\varepsilon)} \frac{\vec{r}}{\|\vec{r}\|^3}$$

$\partial B(0,\varepsilon) = \{(\varepsilon \cos\theta \sin\varphi, \varepsilon \sin\theta \sin\varphi, \varepsilon \cos\varphi)\}$

$\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$

$\partial_\theta \sigma \times \partial_\varphi \sigma = -\sin\varphi\, \varepsilon\, \vec{r}(\sigma(\theta,\varphi))$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{\vec{r}(\sigma(\theta,\varphi))}{\|\vec{r}(\sigma(\theta,\varphi))\|^3} \cdot \left(\sin\varphi\, \varepsilon\, \vec{r}(\sigma(\theta,\varphi))\right)$$

< 0 inward

but we want outward so $\times -1$

$\|\vec{r}(\sigma(\theta,\varphi))\| = \varepsilon$ ⟶

$$= \int_0^{\pi}\int_0^{2\pi} \sin\varphi \frac{\cancel{\|\vec{r}\|^2 \varepsilon}}{\cancel{\|\vec{r}\|^3}}\, d\theta\, d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi} \sin\varphi\, d\theta\, d\varphi = 2\pi \left[-\cos\varphi\right]_0^{\pi} = 4\pi$$

□

$$\iint_{\partial B(0,1)} \frac{\vec{r}}{\|\vec{r}\|^3} = 4\pi \neq 0$$

whereas $\operatorname{div}\left(\frac{\vec{r}}{\|\vec{r}\|^3}\right) = 0$

because we can't apply the divergence theorem to $B(\vec{0},1)$ since $\vec{0} \in B(\vec{0},1)$ but $\frac{\vec{r}}{\|\vec{r}\|^3}$ is not defined at $\vec{0}$

$\rightarrow$ Be careful before applying the divergence theorem

<u>Example</u>: We can use the divergence theorem to compute a volume using a surface integral (in the same way that we use Green's thm to compute an area using a line integral)

Let $R \subset \mathbb{R}^3$ be a regular region with a piecewise smooth boundary.
We assume that ∂R is oriented using the <u>outward</u> normal unit vector $\vec{n}$.
Let $\vec{r}(x,y,z) = (x,y,z)$

$$V(R) = \iiint_R 1 = \iiint_R \frac{\operatorname{div}(\vec{r})}{3}$$

$$= \frac{1}{3}\iiint_R \operatorname{div}(\vec{r})$$

$$= \frac{1}{3}\iint_{\partial R} \vec{r}\cdot\vec{n} \quad \text{by the divergence theorem}$$

$\hookrightarrow$ outward orientation

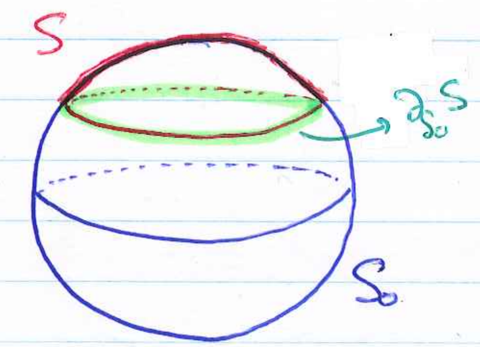
$$\boxed{V(R) = \frac{1}{3}\iint_{\partial R} \vec{r}\cdot\vec{n}}$$

Stokes theorem (Also known as Kelvin-Stokes or curl theorem)

Setup: • $S_0 \subset \mathbb{R}^3$ a smooth surface

• S a surface lying in S_0

• We denote the relative boundary of S in S_0 by $\partial_{S_0} S$ (we assume it is a curve or $\emptyset$)



(We often drop the S_0 and simply write ∂S, but be careful $\partial_{S_0} S \neq \partial S$)

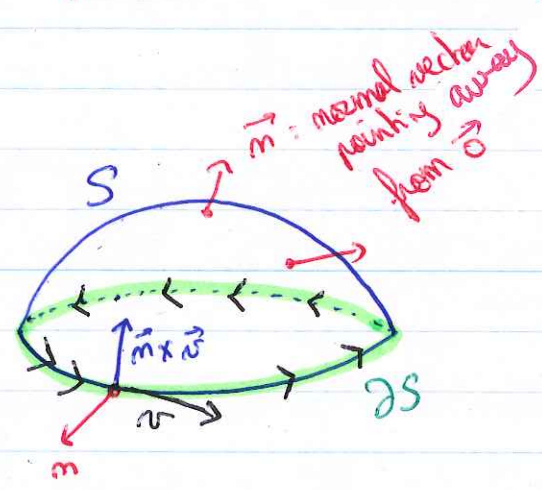
Comment: by "relative" we mean that we forget $\mathbb{R}^3$ and see the ambiant space as S_0 for some topology.
I won't give details because it outreaches MAT237 but that's a very intuitive notion.

If you want a formal definition for $\partial_{S_0} S$:

$$\partial_{S_0} S = \{x \in S_0 : \forall \varepsilon > 0,\ B(x,\varepsilon) \cap S \neq \emptyset \text{ and } B(x,\varepsilon) \cap (S_0 \setminus S) \neq \emptyset\}$$

• We assume that S is oriented by $\vec{n} : S \to \mathbb{R}^3$ a C^0 normal unit vector field

• We assume that $\partial_{S_0} S$ is positively oriented: if $\vec{v}$ gives the orientation of $\partial_{S_0} S$ then $\vec{n} \times \vec{v}$ points to S



Theorem (Stokes, Kelvin-Stokes or curl)

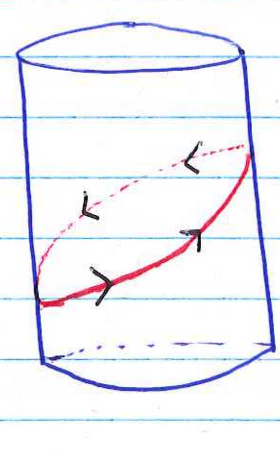
$F: \mathcal{U} \longrightarrow \mathbb{R}^3$ C^1, $\mathcal{U} \subset \mathbb{R}^3$ open, $S \subset \mathcal{U}$, then

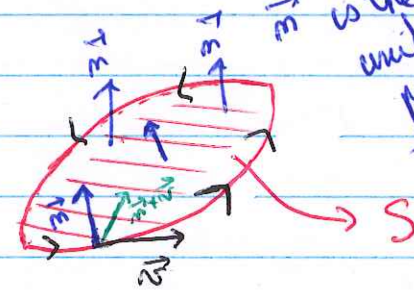
$$\int_{\partial_{S_0} S} \vec{F} \cdot d\vec{x} = \iint_S (\operatorname{curl} \vec{F}) \cdot \vec{m}$$

Comment: That's a "3D" version of Green's theorem: $\to S_0 =$ plane
the latter is a special case when the surface is "flat"
(ie inside a plane)

Comment: That's (again!) a special case of the general Stokes theorem for differential forms.

Ex: We define C as the ellipse obtained by intersecting the cylinder $x^2+y^2=1$ with the plane $y-z=0$ oriented counterclockwise as viewed from above.
Compute $\int_C (x-z)dx + (x+y)dy + (y+z)dz$





$\vec{m}$ is the normal unit vector pointing to the top

9.
$$\int_C (x-z)\,dx + (x+y)\,dy + (y+z)\,dz$$

$$= \iint_S \operatorname{curl} \vec{F} \cdot \vec{m} \text{ where } \vec{F}(x,y,z) = (x-z, x+y, y+z)$$

$$\operatorname{curl} \vec{F} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\vec{m}$ is the unit vector normal to $y-z=0$ pointing to the top

so.
$$\vec{m} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

and $\operatorname{curl} \vec{F} \cdot \vec{m} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

so $\displaystyle\int_C \vec{F} \cdot d\vec{x} = \iint_S \sqrt{2}$

then notice that $S = \{ \underbrace{(r\cos\theta, r\sin\theta, r\sin\theta)}_{\sigma(r,\theta)} : r \in [0,1], \theta \in [-\pi,\pi] \}$

$$\partial_1 \sigma \times \partial_2 \sigma = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \sin\theta \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ r\cos\theta \end{pmatrix} = \begin{pmatrix} 0 \\ -r \\ r \end{pmatrix}$$

Hence $\displaystyle\int_C \vec{F} \cdot d\vec{x} = \int_{-\pi}^{\pi} \int_0^1 \sqrt{2}\, \|(0,-r,r)\|\, dr\, d\theta$

$$= 4\pi \int_0^1 r\,dr$$

$$= 2\pi.$$

In this corollary, I am not very formal.
It is possible to define properly what do I mean by "closed", but I think that's not necessary for our purpose.
You can believe your intuition: "closed" means that there is no "boundary" in the above sense, so the line integral is 0.

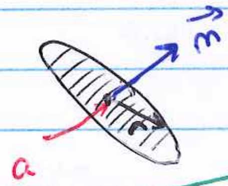
Corollary:

Assume that S is "closed"

then $\displaystyle\iint_S (\operatorname{curl}\vec{F})\cdot\vec{m} = 0$

Stokes theorem allows to give a physics interpretation of curl.

Let $a \in \mathbb{R}^3$, $\vec{m}$ a unit vector and D_r the disk centered at a normal to $\vec{m}$ and of radius r



intuitively the average goes to the exact value! we may prove it with a "MVT for $\int$" theorem for surface integrals

then $$(\operatorname{curl}\vec{F}(a))\cdot\vec{m} = \lim_{r\to 0}\frac{1}{\pi r^2}\iint_{D_r}(\operatorname{curl}\vec{F})\cdot\vec{m}$$

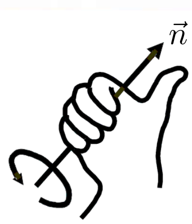
$$= \lim_{r\to 0}\frac{1}{\pi r^2}\int_{C_r}\vec{F}\cdot d\vec{x} \qquad (*)$$

If $\vec{F}$ is a force field, then $\int_{C_r}\vec{F}\cdot d\vec{x}$ is the work of F on a particle moving along C_r:

So $\operatorname{curl}\vec{F}(a)\cdot\vec{m} > 0$: the force pushes the particle counterclockwise

< 0 ——————————— clockwise

By "counterclockwise", I mean that the force pushes the particle in the direction of the orientation of the boundary, as in the "right hand rule" on the right:

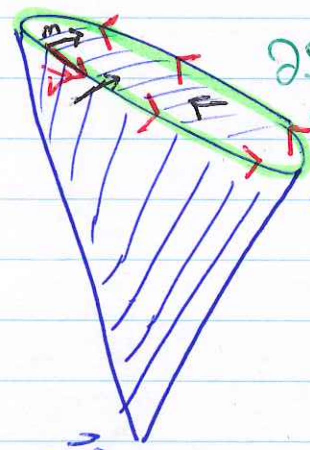


(Image from Wikipedia)

By the way, (*) doesn't depend on the coordinate system!

Another example of usage:

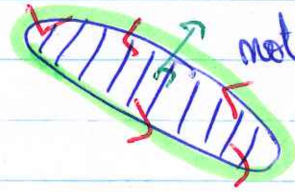
Let S be the part of the cone $z = \sqrt{x^2+y^2}$ below $x+z=1$ oriented by $\vec{n}$ going upward (in the inside)



∂S positively oriented

$\leftarrow S$

we want to compute $\iint_S (\operatorname{curl}\vec{F})\cdot\vec{n}$



notice the the ellipse T has the same boundary compatible orientation if $\vec{n}$ goes upward

so $$\iint_S \operatorname{curl}\vec{F}\cdot\vec{n} = \int_C \vec{F}\cdot d\vec{x} = \iint_T \operatorname{curl}\vec{F}\cdot\vec{n}$$

that could be easier to compute

It is false in general that
$$\iint_{S_1} \vec{G}\cdot\vec{n} = \iint_{S_2} \vec{G}\cdot\vec{n}$$ for G a vector field

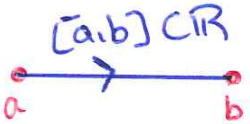
if $\partial S_1 = \partial S_2$

(it is true when $\vec{G} = \operatorname{curl}\vec{F}$) !!!!!

In this chapter, you met the following special cases of the general Stokes theorem: $\int_\Omega d\omega = \int_{\partial\Omega} \omega$

FTC:

"curve in $\mathbb{R}$"



$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

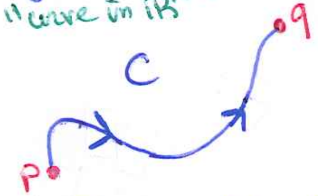
① $\int_a^b$ is the usual one-variable Riemann-Darboux integral

② $F : [a,b] \to \mathbb{R}$ C^1

③ $[a,b]$ is a segment line in $\mathbb{R}$

Gradient theorem

"curve in $\mathbb{R}^n$"



$$\int_C \vec{\nabla} f \cdot d\vec{x} = f(q) - f(p)$$

① $\int_C$ is the line integral for vector fields

② C is an oriented curve in $\mathbb{R}^n$

③ $f : \mathcal{U} \to \mathbb{R}$ C^1

$\mathcal{U} \subset \mathbb{R}^n$ is an open subset containing C

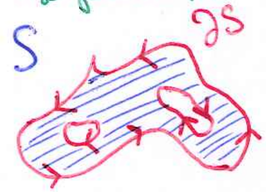
- We want the surface to be on the left

$\Leftrightarrow$ • If $\vec{v}$ is tangent compatible with the orientation, we want $\vec{n} = (v_2, -v_1)$ to point outward

$\vec{n}$ = rotation of $\vec{v}$ by $\frac{\pi}{2}$ clockwise

Green's theorem

"surface in $\mathbb{R}^2$"



$$\int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

or

$$\int_{\partial S} P\,dx + Q\,dy = \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

① $S \subset \mathbb{R}^2$ is a planar regular region (surface in $\mathbb{R}^2$)

② ∂S is piecewise smooth and positively oriented

③ $\int_{\partial S}$ is the line integral for vector fields

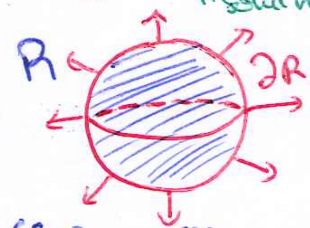
④ $\iint_S$ is the usual integral for 2-variable function $f : S \to \mathbb{R}$, $S \subset \mathbb{R}^2$

⑤ $\vec{F} : \mathcal{U} \to \mathbb{R}^2$ C^1

$\mathcal{U} \subset \mathbb{R}^2$ open with $S \subset \mathcal{U}$

Divergence theorem

"solid in $\mathbb{R}^3$"



$$\iint_{\partial R} \vec{F} \cdot \vec{n} = \iiint_R \operatorname{div}(F)$$

① $R \subset \mathbb{R}^3$ is a regular region ("solid")

② ∂R is a piecewise smooth surface oriented by $\vec{n}$ the outward pointing normal unit vector.

③ $\iint_{\partial R}$ surface integral for vector fields

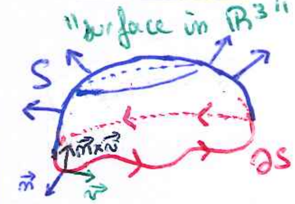
④ $\iiint_R$ usual integral for 3-variable functions $f : R \to \mathbb{R}$, $R \subset \mathbb{R}^3$

⑤ $\vec{F} : \mathcal{U} \to \mathbb{R}^3$ C^1

$\mathcal{U} \subset \mathbb{R}^3$ open, $R \subset \mathcal{U}$

Stokes theorem

"surface in $\mathbb{R}^3$"



$$\int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S (\operatorname{curl} F) \cdot \vec{n}$$

① $S \subset \mathbb{R}^3$ is an oriented surface

② ∂S is the relative boundary of S with the positive orientation

It is an oriented curve in $\mathbb{R}^3$

③ $\int_{\partial S}$ is the line integral for vector fields

④ $\iint_S$ is the surface integral for vector fields

⑤ $F : \mathcal{U} \to \mathbb{R}^3$ C^1

$\mathcal{U} \subset \mathbb{R}^3$ open, $S \subset \mathcal{U}$

We want $\vec{n} \times \vec{v}$ to point to the surface S

Conservative vector fields

Theorem: $U \subset \mathbb{R}^n$ open, $n \geq 2$, $F: U \to \mathbb{R}^n$ C^0

TFAE: ① $\exists f: U \to \mathbb{R}$ C^1 s.t. $F = \nabla f$

② $\int_C \vec{F} \cdot d\vec{x} = 0$ for any closed piecewise smooth oriented curve C in U

③ $\int_{C_1} \vec{F} \cdot d\vec{x} = \int_{C_2} \vec{F} \cdot d\vec{x}$ "path-independence"

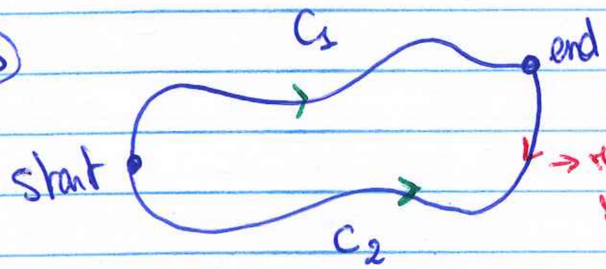
for any two oriented piecewise smooth curves in U with same start point and same end point

Def: In the above case, we say that $\vec{F}$ is conservative

Δ ① $\Rightarrow$ ② by the Gradient theorem:

$$\int_C \vec{F} \cdot dx = f(\text{endpoint}) - f(\text{startpoint}) = f(p) - f(p) = 0$$

② $\Rightarrow$ ③



$\to$ reverse orientation for C_2
I will use the notation $-C_2$

$C = C_1 \cup (-C_2)$

then $0 = \int_{C_1} \vec{F} \cdot d\vec{x} + \int_{-C_2} \vec{F} \cdot d\vec{x} = \int_{C_1} \vec{F} d\vec{x} - \int_{C_2} \vec{F} \cdot d\vec{x}$

③ ⇒ ① Pick $q \in \mathcal{U}$ and for $p \in \mathcal{U}$ set

$$f(p) = \int_C \vec{F} \cdot d\vec{x}$$ where C is any curve from q to p (the integral doesn't depend on the choice by assumption)

$$f(p+te_i) = \int_{q \to p+tv} \vec{F} \cdot d\vec{x} = \int_{q \to p} \vec{F} \cdot d\vec{x} + \int_{p \to p+te_i} \vec{F} \cdot d\vec{x}$$

$$= f(p) + \int_{p,\, p+te_i} \vec{F} \cdot d\vec{x}$$

So $$\frac{f(p+te_i) - f(p)}{t} = \frac{1}{t} \int_0^t F(p+se_i) \cdot e_i \, ds$$

$$= \frac{1}{t} \int_0^t F_i(p+se_i)\, ds \xrightarrow[t \to 0]{} F_i(p)$$

So $\nabla f = F$ C^0, and f C^1 $\square$

We are now going to give several special case $(n=2,3)$ of a very general result called "Poincaré lemma".

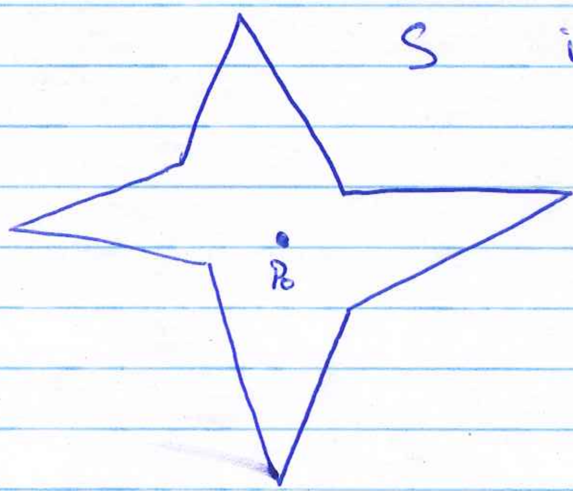
<u>Def</u>: We say that $S \subset \mathbb{R}^n$ is star-shaped if

$\exists p_0 \in S$ st. $\forall p \in S$ the line segment from p_0 to p lies in S

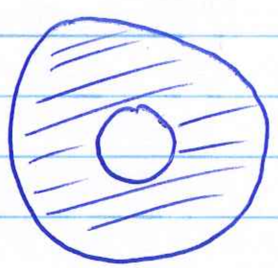
⚠ Obviously "$\neq \emptyset$ + convex $\Rightarrow$ star-shaped" but the converse is false
(here we have p_0 a fixed start point)

| $\emptyset$ is convex not star shaped

<u>Ex</u>: S is star shaped but not convex



<u>Ex</u>:



is not star shaped

($n=2$)

Theorem: $\mathcal{U} \subset \mathbb{R}^2$ open, $F : \mathcal{U} \to \mathbb{R}^2$ C^1

① If $\vec{F} = \nabla f$ for $f : \mathcal{U} \to \mathbb{R}$ C^2 then $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$

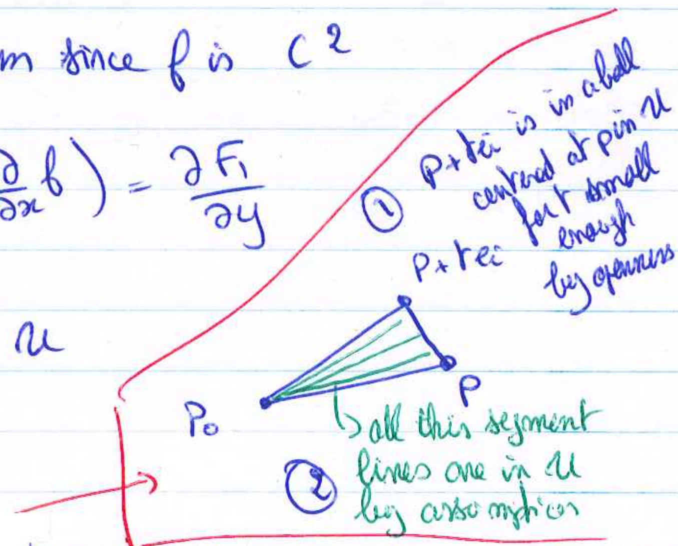
↳ "$\vec{F}$ is conservative"

② The converse is true when $\mathcal{U}$ is star-shaped:
If $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$ on $\mathcal{U}$ star-shaped then $\exists f : \mathcal{U} \to \mathbb{R}$ C^2

such that $\vec{F} = \nabla f$

Δ ① it's simply by Clairaut's theorem since f is C^2

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial F_1}{\partial y}$$



② $\exists p_0 \in \mathcal{U}$ st. $\forall p \in \mathcal{U}$, $[p_0, p] \subset \mathcal{U}$
Let $f(p) = \displaystyle\int_{[p_0,p]} \vec{F} \cdot d\vec{x}$

For $t \geq 0$ small enough the triangle $P_0 \to p + te_i \to P \to P_0$ and its interior lies in $\mathcal{U}$

So $$0 = \iint_T \underbrace{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}_{=0} \overset{\text{Green}}{=} \int_{\partial T} \vec{F} \cdot d\vec{x} = \int_{P_0 \to P} \vec{F} \cdot d\vec{x} + \int_{P \to p+te_i} \vec{F} \cdot d\vec{x} + \int_{p+te_i \to P_0} \vec{F} \cdot d\vec{x}$$

$$= f(P) + \int_0^t F(p+se_i) \cdot e_i \, ds - f(P+te_i)$$

So $$\frac{f(p+te_i) - f(p)}{t} = \frac{1}{t} \int_0^t F_i(p+se_i)\, ds \xrightarrow[t \to 0]{} F_i(p)$$

□

The star-shaped assumption is important

Ex: Define $F(x,y) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2} \right)$ on $\mathbb{R}^2 \setminus \{0\}$

then $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ but F is $\boxed{\text{not}}$ conservative.

Indeed, take C the unit circle counterclockwised oriented then

$$\int_C \vec{F} \cdot dx = \int_0^{2\pi} \underbrace{\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}}_{= F(\sigma(t))} \cdot \underbrace{\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}}_{= \sigma'(t)} dt \quad , \sigma = (\cos t, \sin t)$$

$$= \int_0^{2\pi} -1 \, dt = -2\pi \neq 0$$

whereas C is closed!

Theorem. $(n=3)$ $\mathcal{U} \subset \mathbb{R}^3$ open, $F: \mathcal{U} \to \mathbb{R}^3$ C^1

(1) If $\vec{F} = \nabla f$ for $f: \mathcal{U} \to \mathbb{R}$ C^2 then $\operatorname{curl} \vec{F} = \vec{0}$

(2) The converse hold on a starshaped domain:
If $\mathcal{U}$ is starshaped and $\operatorname{curl} \vec{F} = \vec{0}$
then $\vec{F} = \nabla f$ for $f: \mathcal{U} \to \mathbb{R}$ C^2

Δ Same proof as before by replacing Green by Stokes $\square$

Assume that $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1-vector field s.t. $\text{curl } \vec{F} = \vec{0}$

How can we find a potential $f: \mathbb{R}^3 \to \mathbb{R}$ C^2 s.t. $\vec{F} = \nabla f$?

We just follow the first proof of this section!

Take $q = (a,b,c)$ and $p = (x,y,z)$

We know that $f(p) = \int_{C_{q \to p}} \vec{F} \cdot d\vec{x}$ works

doesn't depend on the curve from q to p!

We take the following segment line:

$$(a,b,c) \xrightarrow{L_1} (x,b,c) \xrightarrow{L_2} (x,y,c) \xrightarrow{L_3} (x,y,z)$$

$\sigma_1(t) = (t,b,c)$, $t \in [a,x]$ $\quad$ $\sigma_2(t) = (x,t,c)$, $t \in [b,y]$ $\quad$ $\sigma_3(t) = (x,y,t)$, $t \in [c,z]$

Then $f(p) = \int_{q \to p} \vec{F} \cdot d\vec{x} = \int_{L_1} \vec{F} \cdot d\vec{x} + \int_{L_2} \vec{F} \cdot d\vec{x} + \int_{L_3} \vec{F} \cdot d\vec{x}$

$$\boxed{f(p) = \int_a^x F_1(t,b,c)\,dt + \int_b^y F_2(x,t,c)\,dt + \int_c^z F_3(x,y,t)\,dt}$$

Vector potentials (or vector fields that are curls)

Here is another special case of Poincaré Lemma:

Theorem: $\mathcal{U} \subset \mathbb{R}^3$ open, $F : \mathcal{U} \to \mathbb{R}^3$ C^1

① If $F = \operatorname{curl} G$ for $G : \mathcal{U} \to \mathbb{R}^3$ C^2 then $\operatorname{div} F = 0$

② The converse is true when $\mathcal{U}$ is star-shaped.
If $\mathcal{U}$ is star-shaped and $\operatorname{div} F = 0$ then there exists $G : \mathcal{U} \to \mathbb{R}^3$ C^2 s.t. $F = \operatorname{curl} G$

We say that G is a vector potential of F

Δ ① $\operatorname{div} F = \operatorname{div}(\operatorname{curl} G) = 0$

② Since $\mathcal{U}$ is star-shaped, $\exists p_0 \in \mathcal{U}$ s.t. $\forall q \in \mathcal{U}$ the line segment from p_0 to q is in $\mathcal{U}$.

For $q = (x, y, z) \in \mathcal{U}$ we set: we apply the $\int$ componentwise

$$G(q) = \int_0^1 F(\underbrace{(1-t)p_0 + tq}_{\in\, \mathcal{U}}) \times (t(q - \vec{p_0}))\, dt$$

$\in \mathcal{U}$ so well defined

Using the theorem to differentiate under the integral, we get that G is C^2 and that $\operatorname{curl} G = \int_0^1 \operatorname{curl}(\ \times\ )\, dt$

where the curl are w.r.t. x, y, z of course

To simplify the notations, I set:

$$\tilde{F}(x,y,z) = F((1-t)p_0 + tq) \qquad (\text{remember } q = (x,y,z))$$

$$= F((1-t)p_{0,x} + tx,\ (1-t)p_{0,y} + ty,\ (1-t)p_{0,z} + tz)$$

$$= F(p_0 + tr(x,y,z))$$

and

$$r(x,y,z) = q - p_0 = (x - p_{0,x},\ y - p_{0,y},\ z - p_{0,z})$$

then we use the formula $\operatorname{curl}(F \times G) = (G\cdot\nabla)F + (\operatorname{div} G)F - (F\cdot\nabla)G - (\operatorname{div} F)G$

(Be careful, I've just realized that the formula in the online text book is false)

$$\operatorname{curl}(\tilde{F} \times (tr)) = \underset{①}{(tr\cdot\nabla)\tilde{F}} + \underset{②}{(\operatorname{div}(tr))\tilde{F}} - \underset{③}{(\tilde{F}\cdot\nabla)(tr)} - \underset{④}{\operatorname{div}(\tilde{F})G}$$

① $(tr\cdot\nabla)\tilde{F} = t\left((x-p_{0,x})\frac{\partial}{\partial x} + (y-p_{0,y})\frac{\partial}{\partial y} + (z-p_{0,z})\frac{\partial}{\partial z}\right)\tilde{F}$

$$= t^2 \frac{d}{dt}(F((1-t)p_0 + tq))$$

② $\operatorname{div}(tr)\tilde{F} = 3t\, F((1-t)p_0 + tq)$

③ $(\tilde{F}\cdot\nabla)(tr) = t\, F((1-t)p_0 + tq)$

④ $\operatorname{div}(\tilde{F}) = t\frac{\partial F_1}{\partial x}(-) + t\frac{\partial F_2}{\partial y}(-) + t\frac{\partial F_3}{\partial z}(-) = t\,\overbrace{\operatorname{div} F((1-t)p_0 + tq)}^{=0} = 0$

Hence $\operatorname{curl} G(q) = \int_0^1 2t\, F((1-t)p_0 + tq) + t^2 \frac{d}{dt}(F((1-t)p_0 + tq))\, dt$

$$= \int_0^1 \frac{d}{dt}\left(t^2 F((1-t)p_0 + tq)\right) dt$$

$$= \left[t^2 F((1-t)p_0 + tq)\right]_{t=0}^{t=1}$$

$$= F(q)$$

$\square$

Notice that the above proof gives a formula to find a suitable G:

$$G(x,y,z) = \int_0^1 F\left((1-t)P_0 + t(x,y,z)\right) \times \left(t\left((x,y,z) - P_0\right)\right) dt$$

where $P_0 \in \mathcal{U}$ s.t. $\forall q \in \mathcal{U}$ the line segment from P_0 to q is included in $\mathcal{U}$

Quite often, in practice, $\mathcal{U}$ is centered at O so that $P_0 = \vec{0}$ works

The above theorem fails with no assumption on the domain

Define $F(x,y,z) = \dfrac{1}{(x^2+y^2+z^2)^{3/2}}(x,y,z)$ on $\mathcal{U} = \mathbb{R}^3 \setminus \{\vec{0}\}$

then $\operatorname{div} F = 0$ but there is no $G : \mathcal{U} \to \mathbb{R}^3$ C^2 s.t. $F = \operatorname{curl} G$

Indeed, otherwise we would have $\displaystyle\iint_{\substack{\{x^2+y^2+z^2=1\} \\ \text{outward}}} \vec{F}\cdot\vec{n} = 0$

but $\displaystyle\iint \vec{F}\cdot\vec{n} = 4\pi \neq 0$

University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus

Poincaré lemma: summary of the last two sections

Jean-Baptiste Campesato

April 2nd, 2020

Contents

1 Star-shaped sets

Definition 1. We say that $S \subset \mathbb{R}^n$ is *star-shaped* if there exists $p_0 \in S$ such that for any $q \in S$ the line segment from p_0 to q is included in S.

Proposition 2. *A non-empty convex set is star-shaped.*

Beware: the empty set is convex but not star-shaped (because "$\forall p \in \varnothing,\ P(p)$" is vacuously true whereas "$\exists p \in \varnothing,\ P(p)$" is always false).

2 Conservative vector fields

Theorem 3. *Let $U \subset \mathbb{R}^n$ be open $(n \geq 2)$, and $\mathbf{F} : U \to \mathbb{R}^n$ be a continuous vector field.*
The following are equivalent:

1. *There exists $f : U \to \mathbb{R}$ C^1 such that $\mathbf{F} = \nabla f$.*
2. *For any piecewise smooth closed oriented curve C in U, we have $\int_C \mathbf{F} \cdot d\mathbf{x} = 0$.*
3. *$\mathbf{F}$ satisfies the* path-independence property*: for any two piecewise smooth oriented curves C_1 and C_2 with same startpoint and same endpoint, we have $\int_{C_1} \mathbf{F} \cdot d\mathbf{x} = \int_{C_2} \mathbf{F} \cdot d\mathbf{x}$.*

Then we say that $\mathbf{F}$ is *conservative* and that f is a *potential* of $\mathbf{F}$ (beware: in physics we usually say that $-f$ is a potential of $\mathbf{F}$).

2.1 $n = 2$

Proposition 4. *Let $U \subset \mathbb{R}^2$ be open and $\mathbf{F} : U \to \mathbb{R}^2$ be a C^1 vector field.*
If $\mathbf{F}$ is conservative then $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$ on U.

When U is star-shaped, the converse is true:

Theorem 5 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^2$ be a* ***star-shaped*** *open set and $\mathbf{F} : U \to \mathbb{R}^2$ be a C^1 vector field.*
If $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$ on U then there exists $f : U \to \mathbb{R}$ C^2 such that $\mathbf{F} = \nabla f$.

Beware: the above theorem is false with no assumption on the domain.
Indeed, define $\mathbf{F}(x, y) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$ on $U = \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Then $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$ but $\displaystyle\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = -2\pi \neq 0$ where C is the unit circle with the counterclockwise orientation, so $\mathbf{F}$ is not conservative.

Alexis Clairaut proved a first version of this Poincaré lemma in 1739 but there was a flaw since he didn't realize that he needed some assumptions on the domain. Then Jean Le Rond d'Alembert gave this counter-example in 1768. Notice that the notation $\displaystyle\int_C P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$ was introduced by A. Clairaut in the above cited paper.

Proposition 6. *Let $U \subset \mathbb{R}^2$ be an open rectangle and $\mathbf{F} : U \to \mathbb{R}^2$ be a C^1 conservative vector field.*
Let $p = (a, b) \in U$ and define $f : U \to \mathbb{R}$ by $f(x, y) = \displaystyle\int_a^x F_1(t, b)\mathrm{d}t + \int_b^y F_2(x, t)\mathrm{d}t$.
Then f is C^2 and $\mathbf{F} = \nabla f$.

2.2 $n = 3$

Proposition 7. *Let $U \subset \mathbb{R}^3$ be open and $\mathbf{F} : U \to \mathbb{R}^3$ be a C^1 vector field.*
If $\mathbf{F}$ is conservative then $\operatorname{curl} \mathbf{F} = \mathbf{0}$ *on U.*

When U is star-shaped, the converse is true:

Theorem 8 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^3$ be a* ***star-shaped*** *open set and $\mathbf{F} : U \to \mathbb{R}^3$ be a C^1 vector field.*
If $\operatorname{curl} \mathbf{F} = \mathbf{0}$ *on U then there exists $f : U \to \mathbb{R}$ C^2 such that $\mathbf{F} = \nabla f$.*

Proposition 9. *Let $U \subset \mathbb{R}^3$ be an open rectangle and $\mathbf{F} : U \to \mathbb{R}^3$ be a C^1 conservative vector field.*
Let $p = (a, b, c) \in U$ and define $f : U \to \mathbb{R}$ by $f(x, y, z) = \displaystyle\int_a^x F_1(t, b, c)\mathrm{d}t + \int_b^y F_2(x, t, c)\mathrm{d}t + \int_c^z F_2(x, y, t)\mathrm{d}t$.
Then f is C^2 and $\mathbf{F} = \nabla f$.

2.3 General case

This subsection is not part of MAT237.
The proof of the above Poincaré Lemma for $n = 2$ relied on the Gradient Theorem, and the proof of the above Poincaré Lemma for $n = 3$ relied on Kelvin–Stokes theorem. They admit a generalization to any n from which we may derive the following general results.

Proposition 10. *Let $U \subset \mathbb{R}^n$ be open and $\mathbf{F} : U \to \mathbb{R}^n$ be a C^1 vector field.*
If $\mathbf{F}$ is conservative then $\forall i, j = 1, \dots, n$, $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$ on U.

When U is star-shaped, the converse is true:

Theorem 11 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^n$ be a* ***star-shaped*** *open set and $\mathbf{F} : U \to \mathbb{R}^n$ be a C^1 vector field.*
If $\forall i, j = 1, \dots, n$, $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$ on U then there exists $f : U \to \mathbb{R}$ C^2 such that $\mathbf{F} = \boldsymbol{\nabla} f$.

Proposition 12. *Let $U \subset \mathbb{R}^n$ be an open rectangle and $\mathbf{F} : U \to \mathbb{R}^n$ be a C^1 conservative vector field.*
Let $p = (a_1, \dots, a_n) \in U$ and define $f : U \to \mathbb{R}$ by $f(x_1, \dots, x_n) = \sum_{i=1}^{n} \int_{a_i}^{x_i} F_i(x_1, \dots, x_{i-1}, t, a_{i+1}, \dots, a_n)\mathrm{d}t$.
Then f is C^2 and $\mathbf{F} = \nabla f$.

3 Vector potentials

Proposition 13. *Let $U \subset \mathbb{R}^3$ be open and $\mathbf{F} : U \to \mathbb{R}^3$ be a C^1 vector field.*
If $\mathbf{F} = \operatorname{curl} \mathbf{G}$ for $\mathbf{G} : U \to \mathbb{R}^3$ C^2 then $\operatorname{div} \mathbf{F} = 0$ on U.

Then we say that $\mathbf{G}$ is a *vector potential* of $\mathbf{F}$.

When U is star-shaped, the converse is true:

Theorem 14 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^3$ be a **star-shaped** open set and $\mathbf{F} : U \to \mathbb{R}^3$ be a C^1 vector field.*
If $\operatorname{div} \mathbf{F} = 0$ on U then there exists $\mathbf{G} : U \to \mathbb{R}^3$ C^2 such that $\mathbf{F} = \operatorname{curl} \mathbf{G}$.

Beware: the above theorem is false with no assumption on the domain.
Indeed, define $\mathbf{F} = \frac{\mathbf{r}}{\|\mathbf{r}\|^3}$ on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ where $\mathbf{r}(x, y, z) = (x, y, z)$.
Then $\operatorname{div} \mathbf{F} = 0$ but there is no $\mathbf{G} : \mathbb{R}^3 \setminus \{\mathbf{0}\} \to \mathbb{R}^3$ such that $\mathbf{F} = \operatorname{curl} \mathbf{G}$.

Proposition 15. *Let $U \subset \mathbb{R}^3$ be a **star-shaped** open set and $\mathbf{F} : U \to \mathbb{R}^3$ be a C^1 vector field such that $\operatorname{div} \mathbf{F} = 0$.*
Let $p_0 = (x_0, y_0, z_0) \in U$ such that for any $q \in U$ the line segment from p_0 to q is included in U.
Define $\mathbf{G} : U \to \mathbb{R}^3$ by

$$\mathbf{G}(x, y, z) = \int_0^1 \mathbf{F}\begin{pmatrix} x_0 + t(x - x_0) \\ y_0 + t(y - y_0) \\ z_0 + t(z - z_0) \end{pmatrix} \times \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} t\mathrm{d}t$$

Then $\mathbf{G}$ is C^2 and $\mathbf{F} = \operatorname{curl} \mathbf{G}$.

Quite often (but not always), U is *centered* at $\mathbf{0}$ so that we can take $p_0 = \mathbf{0}$, then the above formula may rewritten

$$\mathbf{G} = \int_0^1 \mathbf{F}\big(t\mathbf{r}\big) \times \big(t\mathbf{r}\big)\mathrm{d}t$$

where $\mathbf{r}(x, y, z) = (x, y, z)$.

4 The general Poincaré lemma

In all the above results named *Poincaré Lemma*, we may relax the assumption on the domain by assuming that U is open and contractible (which is weaker than open and star-shaped since a star-shaped set is contractible) but this notion is not part of MAT237.

Then, they are all special cases of the following very general result:

Theorem 16 (Poincaré Lemma)**.**
If M is a contractible manifold then $H^n_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$,
i.e. the closed differential forms on M are exact.

You should come back here after you learn differential forms and de Rham cohomology.