University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus

Reading week questions

Jean-Baptiste Campesato

February 21st, 2020

Here are a few questions if you want to relax by doing mathematics after a few exhausting days of reading week!

Exercise 1. Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^p$ be a C^1 function. Let $K \subset \mathbb{R}^n$ be a compact subset such that $K \subset U$. We want to prove that $f_{|K} : K \to \mathbb{R}^p$ is Lipschitz.

1. Just for this question, we also assume that K is convex. Prove that the conclusion holds.
(*Hint: you may use the result from* `http://www.math.toronto.edu/campesat/ens/1920/IFT-MVT.pdf`)

Now, we come back to the general statement and the compact set K is no longer supposed to be convex.
We suppose by contradiction that $f_{|K}$ is not Lipschitz.

2. Prove that there exist two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ with terms in K which are respectively convergent to some $x \in K$ and $y \in K$ such that $\forall n \in \mathbb{N},\ \|f(y_n) - f(x_n)\| > n\|y_n - x_n\|$
3. Prove that necessarily $x = y$.
(*Hint: use that f_K is continuous on a compact...*)
4. Prove that there exist $r > 0$ and $N \in \mathbb{N}$ such that $\overline{B}(x,r) \subset U$ and if $n \geq N$ then $x_n, y_n \in \overline{B}(x,r)$.
5. Find a contradiction with Question 1 and conclude.

Exercise 2. Let $U, V \subset \mathbb{R}^n$ be two open sets.
Let $\Phi : U \to V$ be a homeomorphism which is also C^1 (i.e. we assume that Φ is bijective, C^1 and that Φ^{-1} is C^0, but we don't assume that Φ^{-1} is C^1).
Let $T \subset \mathbb{R}^n$ be a Jordan measurable set (i.e. T is bounded and ∂T has zero content) such that $\overline{T} \subset U$.
We want to prove that $\Phi(T)$ is Jordan measurable (i.e. $\Phi(T)$ is bounded and $\partial(\Phi(T))$ has zero content).

1. Prove that $\overline{T}$ and ∂T are compact.
2. Prove that $\Phi(T)$ is bounded.
3. We want to prove that $\Phi(\partial T)$ has zero content
   (a) Prove that a rectangle $R \subset \mathbb{R}^n$ may be covered by finitely many squares $S_1, \dots, S_q$ such that $\sum v(S_i) \leq 2v(R)$.
   A square is a rectangle whose edges have same length, i.e. a set of the form $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$ where $b_i - a_i = r$ for some $r > 0$ or equivalently of the form $S = \left\{x \in \mathbb{R}^n \ :\ |x_i - a_i| \leq r/2\right\}$ for some center $a \in \mathbb{R}^n$ and side length $r > 0$.
   (b) Let $f : A \to \mathbb{R}^n$ be Lipschitz with constant C where $A \subset \mathbb{R}^n$. Let $S \subset \mathbb{R}^n$ be a square of side length r and center $a \in A$. Prove that $f(A \cap S)$ is included in a square of volume $C^n \sqrt{n}^n r^n$.
   Hint: you can start by proving that $\|(x_1, \dots, x_n)\| \leq \sqrt{n} \max(|x_1|, \dots, |x_n|)$.
   Notice that we only assume that the center of S is in A, we don't assume that $S \subset A$.
   (c) Conclude.
   Hint: use the result from Exercise 1
4. Prove that $\partial(\Phi(T)) = \Phi(\partial T)$.

5. Conclude.

Exercise 3. Let $U \subset \mathbb{R}^n$ be open, $\sigma : U \to \mathbb{R}^p$ be C^1 where $n < p$ and a rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset U$.
We want to prove that $\{\sigma(t) : t \in R\}$ has zero content.
Let P be the partition of R defined by subdividing each $[a_i, b_i]$ into N subintervals of the same length.

1. Prove that for $i = 1, \ldots, p$, there exists C_i such that $\forall S$ subrectangle, $\forall x, y \in S$, $|\sigma_i(y) - \sigma_i(x)| \leq C_i \|y - x\|$.
2. Prove that if S is a subrectangle then $\sigma(S)$ is included in a rectangle of volume $\dfrac{C}{N^p}$ where C is a constant.
3. Conclude.

In the above exercises you proved the following theorems:

Theorem 1. *Let $U \subset \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^p$ be a C^1 function and $K \subset U$ a compact set.*
Then $f_{|K} : K \to \mathbb{R}^p$ is Lipschitz.

Remark. Beware, it is false that a C^1 function is Lipschitz (e.g. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$). However, as proved in the above theorem, a C^1 function is locally Lipschitz.

The following result is important to justify that the integral in the change of variables formula is well-defined.

Theorem 2. *Let $U, V \subset \mathbb{R}^n$ be two open sets, $\Phi : U \to V$ be a homeomorphism which is also C^1 and $T \subset \mathbb{R}^n$ a Jordan measurable set such that $\overline{T} \subset U$.*
Then $\Phi(T)$ is Jordan measurable.

Theorem 3. *Let $U \subset \mathbb{R}^n$ be open, $\sigma : U \to \mathbb{R}^p$ be C^1 where $n < p$ and $R \subset \mathbb{R}^n$ be a rectangle such that $R \subset U$.*
Then $\{\sigma(t) : t \in R\}$ has zero content.