Exercise 1 :

1. Since f is C^1, $\begin{array}{l} K \longrightarrow \mathbb{R} \\ x \longmapsto \|Df(x)\| \end{array}$ is C^0 on K compact

Hence $\exists M > 0$ s.t. $\forall x \in K$, $\|Df(x)\| < M$

By the "MVT-like inequality", since K is convex, for $x, y \in K$

we have $\|f(x) - f(y)\| \leq \left(\sup_{t \in (0,1)} \|Df((1-t)x + ty)\| \right) \|x - y\|$

$\leq M \|x - y\|$

Hence $f_{|K} : K \longrightarrow \mathbb{R}^p$ is Lipschitz.

2. $f_{|K}$ is not Lipschitz means:

$\forall M \geq 0$, $\exists x, y \in K$, $\|f(x) - f(y)\| > M \|x - y\|$

Let $n \in \mathbb{N}$, then $\exists \tilde{x}_n, \tilde{y}_n \in K$, $\|f(\tilde{x}_n) - f(\tilde{y}_n)\| > n \|\tilde{x}_n - \tilde{y}_n\|$

Then $(\tilde{x}_n, \tilde{y}_n)$ is a sequence with terms in $K \times K$ compact

so $\exists \sigma : \mathbb{N} \to \mathbb{N}$ increasing s.t. $(x_n = \tilde{x}_{\sigma(n)}, y_n = \tilde{y}_{\sigma(n)}) \longrightarrow (x, y) \in K \times K$

and $\|f(x_n) - f(y_n)\| > \sigma(n) \|x_n - y_n\|$

$\geq n \|x_n - y_n\|$ since $\forall n$, $\sigma(n) \geq n$

3. Assume that $x \neq y$ then $\|x - y\| - \|x_n - y_n\| \leq \left| \|x-y\| - \|x_n - y_n\| \right| < \frac{\|x-y\|}{2}$

for n big enough, ie $\|x_n - y_n\| > \frac{\|x - y\|}{2}$

Hence $\|f(x_n) - f(y_n)\| \geq n \frac{\|x - y\|}{2} \xrightarrow[n \to \infty]{} \infty$

so $f_{|K}$ is not bounded which is impossible since

$f_{|K}$ is C^0 on a compact

4. Since U is open and $x \in K \subset U$, $\exists \tilde{r} > 0$, $B(x,\tilde{r}) \subset U$

and then $\overline{B}(x,r) \subset U$ for $r = \frac{\tilde{r}}{2}$

Since $x_n \to x$, $\exists N_1$, $n \geq N_1 \Rightarrow x_n \in \overline{B}(x,r)$

$y_n \to \underset{\overset{\shortparallel}{y}}{x}$, $\exists N_2$, $n \geq N_2 \Rightarrow y_n \in \overline{B}(x,r)$

take $N = \max(N_1, N_2)$

5. $f_{|\overline{B}(x,r)}$ is Lipschitz since $\overline{B}(x,r)$ is compact and convex

ie $\exists M$, $\forall a,b \in \overline{B}(x,r)$, $\|f(a) - f(b)\| \leq M\|b-a\|$

but for $n \geq N$, $x_n, y_n \in \overline{B}(x,r)$ and

$\|f(x_n) - f(y_n)\| > n\|x_n - y_n\|$ contradiction

Ex2 (1) • T bounded $\Rightarrow \exists r>0,\ T \subset B(0,r)$

$$\Rightarrow \bar{T} \subset \bar{B}(0,r)$$

so $\bar{T}$ is bounded

and $\partial T = \bar{T} \setminus \mathring{T} \subset \bar{T}$ is bounded too

• $\bar{T}$ is closed

$\partial T = \bar{T} \setminus \mathring{T} = \bar{T} \cap (\underbrace{\mathbb{R}^n \setminus \underbrace{\mathring{T}}_{\text{open}}}_{\text{closed}})$ is closed as the $\cap$ of 2 closed sets

• $\bar{T}$ and ∂T are compact as closed and bounded sets

(2) $\bar{T} \subset U$ so $\Phi(\bar{T})$ is well defined

and $\Phi(\bar{T})$ is compact (hence bounded) as the C^0 image of a compact set

$T \subset \bar{T} \Rightarrow \Phi(T) \subset \Phi(\bar{T})$ so $\Phi(T)$ is bounded

(3) (a) $R = [a_1, b_1] \times \dots \times [a_n, b_n]$

Set $r = \min(b_i - a_i)$

then $\forall i, \exists!\, N_i$ s.t. $a_i + (N_i - 1) r < b_i \leq a_i + N_i r$

this way we have squares with edges of length r covering R

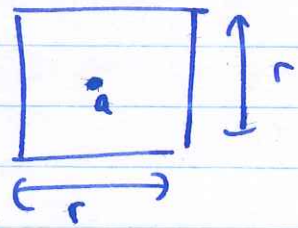
and $\sum \nu(S_i) \leq \nu(R) + r^n \leq \nu(R) + \prod(b_i - a_i) = 2\nu(R)$

(b) Proof of the hint:

$$\|(x_1,\dots,x_m)\| = \sqrt{x_1^2+\dots+x_m^2} \leq \sqrt{m \max(|x_1|,\dots,|x_m|)^2}$$
$$= \sqrt{m}\max(|x_1|,\dots,|x_m|)$$

Question:

$$S = \{x \in \mathbb{R}^m : |x_i - a_i| \leq r/2\}$$



for some $a \in A$ and $r > 0$
↳ center ↳ side length

Take $x \in S \cap A$ then

$$|f_i(x) - f_i(a)| \leq \sqrt{\sum_{j=1}^{m} (f_j(a) - f_j(x))^2}$$

well-defined since $x, a \in A$

$$= \|f(x) - f(a)\|$$
$$\leq C\|x - a\| \quad \leftarrow f \text{ is Lipschitz}$$
$$\leq C\sqrt{m}\max(|x_i - a_i|) \quad \leftarrow \text{Hint}$$
$$\leq \frac{C\sqrt{m}\, r}{2} \quad \leftarrow x \in S$$

So $f(x) \in \left\{ y \in \mathbb{R}^m : |y_i - f_i(a)| \leq \frac{C\sqrt{m}\, r}{2} \right\}$

ie $f(A \cap S) \subset R$!! square centered at $f(a)$ of side length $C\sqrt{m}\, r$

then $v(R) = (C\sqrt{m}\, r)^m = C^m \sqrt{m}^m r^m$

ⓒ $\partial T \subset T \subset \mathcal{U}$, ∂T compact, $\Phi : \mathcal{U} \to \mathbb{R}^m$ C^1
so $\Phi_{|\partial T} : \partial T \to \mathbb{R}^m$ is Lipschitz by exo 1

ie: $\exists C > 0, \forall x, y \in \partial T, \|\Phi(y) - \Phi(x)\| \leq C \|y - x\|$

Let $\varepsilon > 0$. Since ∂T has zero content, $\exists R_1, \dots, R_q$ rectangles
s.t. $\partial T \subset \bigcup_i R_i$ and $\Sigma\, \mathcal{V}(R_i) < \frac{\varepsilon}{C^m \sqrt{m}^m}$

By question ⓐ, up to replacing ε by $\varepsilon/2$, we may assume that each R_i is a square

Then, by ⓑ, $\exists S_i$ square s.t. $\Phi_{|\partial T}(R_i \cap \partial T) \subset S_i$ and
$\mathcal{V}(S_i) = C^m \sqrt{m}^m \mathcal{V}(R_i)$

It is not an issue if R_i overflow ∂T, we just need to know that $R_i \cap \partial T \subset \{x \in \mathbb{R}^m : |x_i - a_i| \leq \delta/2\}$ and that $a \in \partial T$
If $a \notin \partial T$ we may enlarge the square to ensure that $a \in \partial T$ by no more than doubling the edges but then $\mathcal{V}(\tilde{R_i}) \leq 2^m \mathcal{V}(R_i)$ so we divide ε by 2^m and we are good.

Finally $\Phi(\partial T) = \Phi(\bigcup R_i \cap \partial T) = \bigcup \Phi(R_i \cap \partial T) \subset \bigcup S_i$

and $\sum_1^q \mathcal{V}(S_i) = C^m \sqrt{m}^m \sum_1^q \mathcal{V}(R_i) < \varepsilon$

so $\Phi(\partial T)$ has zero content.

(4) • $T \subset \overline{T} \Rightarrow \Phi(T) \subset \Phi(\overline{T})$

$$\Rightarrow \overline{\Phi(T)} \subset \overline{\Phi(\overline{T})} = \Phi(\overline{T}) = (\Phi^{-1})^{-1}(\overline{T})$$

since Φ^{-1} is continuous

ie $\overline{\Phi(T)} \subset \Phi(\overline{T})$

• $\Phi^{-1}(\overline{\Phi(T)})$ is closed since Φ is continuous

and $T \subset \Phi^{-1}(\Phi(T)) \subset \Phi^{-1}(\overline{\Phi(T)})$

$\Rightarrow \overline{T} \subset \Phi^{-1}(\overline{\Phi(T)})$

$\Rightarrow \Phi(\overline{T}) \subset \Phi(\Phi^{-1}(\overline{\Phi(T)})) \subset \overline{\Phi(T)}$

So $\overline{\Phi(T)} = \Phi(\overline{T})$

Similarly $\Phi(T^\circ) = (\Phi(T))^\circ$

• then $\Phi(\partial T) = \Phi(\overline{T} \setminus \mathring{T})$

$= \Phi(\overline{T}) \setminus \Phi(\mathring{T})$ since Φ is bijective

$= \overline{\Phi(T)} \setminus \Phi(T)^\circ$

$= \partial(\Phi(T))$

(5) $\Phi(T)$ is bounded by 2

and $\partial(\Phi(T)) = \Phi(\partial T)$ has zero content by 4 and 3

hence $\Phi(T)$ is Jordan measurable

Ex3 ① σ is $C^1 \Rightarrow \nabla\sigma_i : \mathbb{R}^m \to \mathbb{R}^m$ is C^0

since R is compact, $\exists C_i > 0$, $\forall x \in R$, $\|\nabla\sigma_i(x)\| \leq C_i$

Since S is convex, we may apply the MVT:

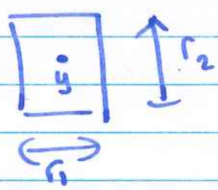
for $x, y \in S$, $\exists c \in S$, $\sigma_i(y) - \sigma_i(x) = \nabla\sigma_i(c) \cdot (y-x)$

$$\Rightarrow |\sigma_i(y) - \sigma_i(x)| = |\nabla\sigma_i(c) \cdot (y-x)|$$

$$\leq \|\nabla\sigma_i(c)\| \cdot \|y-x\| \text{ by CS}$$

$$\leq C_i \|y-x\| \text{ since } S \subset R$$

② Assume that $S = \{x \in \mathbb{R}^m : |x_i - y_i| \leq r_i/2\}$



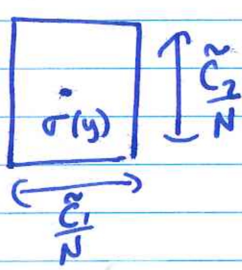
notice that $r_i = \dfrac{(b_i - a_i)}{N}$ by construction of P

then, for $x \in S$, $|\sigma_i(x) - \sigma_i(y)| \leq C_i \|x-y\|$

$$\leq C_i \sqrt{m} \max(|x_i - y_i|)$$

where $\tilde{C}_i = C_i \sqrt{m} \max\limits_{j=1..m}(b_j - a_j)$ $\longrightarrow$ $\leq \dfrac{C_i \sqrt{m} \max(b_j - a_j)}{2N} = \dfrac{\tilde{C}_i}{2N}$

So $\sigma(S) \subset \left\{x \in \mathbb{R}^p : |x_i - \sigma_i(y)| \leq \dfrac{\tilde{C}_i}{2N}\right\}$



which is a rectangle of volume $\prod\limits_{i=1}^{p} \left(\dfrac{\tilde{C}_i}{N}\right) = \dfrac{\prod \tilde{C}_i}{N^p}$

$$= \frac{C}{N^p}$$

where $C = \prod \tilde{C}_i$

③ The partition contains N^m rectangles S_i and each $\sigma(S_i)$ is included in a rectangle R_i of volume $\dfrac{C_i}{N^p}$

$$\sigma(R) = \sigma(\cup S_i) \subset \cup R_i$$

$$\text{and } \sum \mathcal{V}(R_i) = \frac{N^m}{N^p} \sum C_i = \frac{\tilde{C}}{N^{p-m}} \xrightarrow[N\to\infty]{} 0 \quad \text{since } p > m$$

so $\forall \varepsilon > 0$, we may find N s.t. $\dfrac{\tilde{C}}{N^{p-m}} < \varepsilon$

and then we have rectangles $R_1, \dots, R_{N^m}$

s.t. $\sigma(R) \subset \cup R_i$ and $\sum \mathcal{V}(R_i) < \varepsilon$