University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus

Poincaré lemma: potentials and vector potentials

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Contents

1 Star-shaped sets

Definition 1. We say that $S \subset \mathbb{R}^n$ is *star-shaped* if there exists $p_0 \in S$ such that for any $q \in S$ the line segment from p_0 to q is included in S.

Proposition 2. *A non-empty convex set is star-shaped.*

Beware: the empty set is convex but not star-shaped (because "$\forall p \in \varnothing,\ P(p)$" is vacuously true whereas "$\exists p \in \varnothing,\ P(p)$" is always false).

2 Conservative vector fields

Theorem 3. *Let $U \subset \mathbb{R}^n$ be open ($n \geq 2$), and $\mathbf{F} : U \to \mathbb{R}^n$ be a continuous vector field.*
The following are equivalent:

1. *There exists $f : U \to \mathbb{R}$ C^1 such that $\mathbf{F} = \nabla f$.*
2. *For any piecewise smooth closed oriented curve C in U, we have $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$.*
3. *$\mathbf{F}$ satisfies the* path-independence property*: for any two piecewise smooth oriented curves C_1 and C_2 with same startpoint and same endpoint, we have $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$.*

Then we say that $\mathbf{F}$ is *conservative* and that f is a *potential* of $\mathbf{F}$ (beware: in physics we usually say that $-f$ is a potential of $\mathbf{F}$).

2.1 $n = 2$

Proposition 4. *Let $U \subset \mathbb{R}^2$ be open and $\mathbf{F} : U \to \mathbb{R}^2$ be a C^1 vector field.*
If $\mathbf{F}$ is conservative then $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ on U.

When U is star-shaped, the converse is true:

Theorem 5 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^2$ be a **star-shaped** open set and $\mathbf{F} : U \to \mathbb{R}^2$ be a C^1 vector field.*
If $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ on U then there exists $f : U \to \mathbb{R}$ C^2 such that $\mathbf{F} = \nabla f$.

Beware: the above theorem is false with no assumption on the domain.
Indeed, define $\mathbf{F}(x, y) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$ on $U = \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Then $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ but $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = -2\pi \neq 0$ where C is the unit circle with the counterclockwise orientation, so $\mathbf{F}$ is not conservative.

Alexis Clairaut proved a first version of this Poincaré lemma in 1739 but there was a flaw since he didn't realize that he needed some assumptions on the domain. Then Jean Le Rond d'Alembert gave this counter-example in 1768. Notice that the notation $\int_C P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$ was introduced by A. Clairaut in the above cited paper.

Proposition 6. *Let $U \subset \mathbb{R}^2$ be an open rectangle and $\mathbf{F} : U \to \mathbb{R}^2$ be a C^1 conservative vector field.*
Let $p = (a, b) \in U$ and define $f : U \to \mathbb{R}$ by $f(x, y) = \int_a^x F_1(t, b)\mathrm{d}t + \int_b^y F_2(x, t)\mathrm{d}t$.
Then f is C^2 and $\mathbf{F} = \nabla f$.

2.2 $n = 3$

Proposition 7. *Let $U \subset \mathbb{R}^3$ be open and $\mathbf{F} : U \to \mathbb{R}^3$ be a C^1 vector field.*
If $\mathbf{F}$ is conservative then $\operatorname{curl} \mathbf{F} = \mathbf{0}$ on U.

When U is star-shaped, the converse is true:

Theorem 8 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^3$ be a **star-shaped** open set and $\mathbf{F} : U \to \mathbb{R}^3$ be a C^1 vector field.*
If $\operatorname{curl} \mathbf{F} = \mathbf{0}$ on U then there exists $f : U \to \mathbb{R}$ C^2 such that $\mathbf{F} = \nabla f$.

Proposition 9. *Let $U \subset \mathbb{R}^3$ be an open rectangle and $\mathbf{F} : U \to \mathbb{R}^3$ be a C^1 conservative vector field.*
Let $p = (a, b, c) \in U$ and define $f : U \to \mathbb{R}$ by $f(x, y, z) = \int_a^x F_1(t, b, c)\mathrm{d}t + \int_b^y F_2(x, t, c)\mathrm{d}t + \int_c^z F_2(x, y, t)\mathrm{d}t$.
Then f is C^2 and $\mathbf{F} = \nabla f$.

2.3 General case

This subsection is not part of MAT237.
The proof of the above Poincaré Lemma for $n = 2$ relied on the Gradient Theorem, and the proof of the above Poincaré Lemma for $n = 3$ relied on Kelvin–Stokes theorem. They admit a generalization to any n from which we may derive the following general results.

Proposition 10. *Let $U \subset \mathbb{R}^n$ be open and $\mathbf{F} : U \to \mathbb{R}^n$ be a C^1 vector field.*
If $\mathbf{F}$ is conservative then $\forall i, j = 1, \dots, n$, $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ on U.

When U is star-shaped, the converse is true:

Theorem 11 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^n$ be a **star-shaped** open set and $\mathbf{F} : U \to \mathbb{R}^n$ be a C^1 vector field.*
If $\forall i, j = 1, \dots, n$, $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ on U then there exists $f : U \to \mathbb{R}$ C^2 such that $\mathbf{F} = \boldsymbol{\nabla} f$.

Proposition 12. *Let $U \subset \mathbb{R}^n$ be an open rectangle and $\mathbf{F} : U \to \mathbb{R}^n$ be a C^1 conservative vector field.*
Let $p = (a_1, \dots, a_n) \in U$ and define $f : U \to \mathbb{R}$ by $f(x_1, \dots, x_n) = \displaystyle\sum_{i=1}^{n} \int_{a_i}^{x_i} F_i(x_1, \dots, x_{i-1}, t, a_{i+1}, \dots, a_n)\mathrm{d}t$.
Then f is C^2 and $\mathbf{F} = \nabla f$.

3 Vector potentials

Proposition 13. *Let $U \subset \mathbb{R}^3$ be open and $\mathbf{F} : U \to \mathbb{R}^3$ be a C^1 vector field.*
If $\mathbf{F} = \operatorname{curl} \mathbf{G}$ for $\mathbf{G} : U \to \mathbb{R}^3$ C^2 then $\operatorname{div} \mathbf{F} = 0$ on U.

Then we say that $\mathbf{G}$ is a *vector potential* of $\mathbf{F}$.

When U is star-shaped, the converse is true:

Theorem 14 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^3$ be a* ***star-shaped*** *open set and $\mathbf{F} : U \to \mathbb{R}^3$ be a C^1 vector field.*
If $\operatorname{div} \mathbf{F} = 0$ on U then there exists $\mathbf{G} : U \to \mathbb{R}^3$ C^2 such that $\mathbf{F} = \operatorname{curl} \mathbf{G}$.

Beware: the above theorem is false with no assumption on the domain.
Indeed, define $\mathbf{F} = \frac{\mathbf{r}}{\|\mathbf{r}\|^3}$ on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ where $\mathbf{r}(x, y, z) = (x, y, z)$.
Then $\operatorname{div} \mathbf{F} = 0$ but there is no $\mathbf{G} : \mathbb{R}^3 \setminus \{\mathbf{0}\} \to \mathbb{R}^3$ such that $\mathbf{F} = \operatorname{curl} \mathbf{G}$.

Proposition 15. *Let $U \subset \mathbb{R}^3$ be a* ***star-shaped*** *open set and $\mathbf{F} : U \to \mathbb{R}^3$ be a C^1 vector field such that $\operatorname{div} \mathbf{F} = 0$.*
Let $p_0 = (x_0, y_0, z_0) \in U$ such that for any $q \in U$ the line segment from p_0 to q is included in U.
Define $\mathbf{G} : U \to \mathbb{R}^3$ by

$$\mathbf{G}(x, y, z) = \int_0^1 \mathbf{F}\begin{pmatrix} x_0 + t(x - x_0) \\ y_0 + t(y - y_0) \\ z_0 + t(z - z_0) \end{pmatrix} \times \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} t\mathrm{d}t$$

Then $\mathbf{G}$ is C^2 and $\mathbf{F} = \operatorname{curl} \mathbf{G}$.

Quite often (but not always), U is *centered* at $\mathbf{0}$ so that we can take $p_0 = \mathbf{0}$, then the above formula may rewritten

$$\mathbf{G} = \int_0^1 \mathbf{F}(t\mathbf{r}) \times (t\mathbf{r})\mathrm{d}t$$

where $\mathbf{r}(x, y, z) = (x, y, z)$.

4 The general Poincaré lemma

In all the above results named *Poincaré Lemma*, we may relax the assumption on the domain by assuming that U is open and contractible (which is weaker than open and star-shaped since a star-shaped set is contractible) but this notion is not part of MAT237.

Then, they are all special cases of the following very general result:

Theorem 16 (Poincaré Lemma)**.**
If M is a contractible manifold then $H^n_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$,
i.e. the closed differential forms on M are exact.

You should come back here after you learn differential forms and de Rham cohomology.