University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus!

Review questions from sections 5.5, 5.6 and 5.7

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April 4th, 2020

Exercise 1. We want to compute $\iint_S \mathbf{F}\cdot\mathbf{n}$ for $\mathbf{F}(x,y,z)=(2x,-3y,z)$ and S the surface enclosed within $x^2+y^2=1$, $z=0$ and $z=x+2$ for the orientation obtained by outward pointing normal vectors.
(1) Compute it directly. (2) Compute it using the Divergence Theorem.

Exercise 2. Compute $\iint_S \mathbf{F}\cdot\mathbf{n}$ where $\mathbf{F}(x,y,z)=(-a^2y,b^2x,z^2)$ and S is the ellipsoid $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$ oriented by outward pointing normal vectors.

Exercise 3. Let $R\subset\mathbb{R}^3$ a regular region with piecewise smooth boundary ∂R oriented by outward pointing normal vectors.
Prove that $v(R)=\frac{1}{3}\iint_{\partial R}\mathbf{r}\cdot\mathbf{n}$ where $\mathbf{r}(x,y,z)=(x,y,z)$.

Exercise 4. We want to compute $\iint_S x^2+y+z$ where S is the unit sphere in $\mathbb{R}^3$.

1. Why didn't I explicit an orientation on S?
   (*Hint: The answer is not that I forgot to, at least for this question!*)
2. Why can't you use directly the divergence theorem?
3. Use the divergence theorem to compute this integral!

Exercise 5. We denote by S the boundary of the pyramid with vertices $(0,0,1)$, $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,0)$ oriented by outward pointing normal vectors.
Compute $\iint_S \mathbf{F}\cdot\mathbf{n}$ where $\mathbf{F}(x,y,z)=(x^2y,3y^2z,9z^2x)$.

Exercise 6. Let $R=\left\{(x,y,z)\in\mathbb{R}^3\ :\ 1\le x^2+y^2+z^2\le 4\right\}$ (draw it!). We assume that $S=\partial R$ is oriented by outward pointing normal vectors.
Compute $\iint_S \mathbf{F}\cdot\mathbf{n}$ where $\mathbf{F}(x,y,z)=\left(\frac{x}{x^2+y^2+z^2},\frac{y}{x^2+y^2+z^2},0\right)$.

Exercise 7 (Green's identities)**.** Let $R\subset\mathbb{R}^3$ be a regular region with piecewise smooth boundary oriented by outward pointing normal vectors. Let $f,g:U\to\mathbb{R}$ be two C^2 functions where $U\subset\mathbb{R}^3$ is open such that $R\subset U$.

1. Prove that $\iint_{\partial R} f\nabla g\cdot\mathbf{n}=\iiint_R\left(\nabla f\cdot\nabla g+f\nabla^2 g\right)$
2. Prove that $\iint_{\partial R}(f\nabla g-g\nabla f)\cdot\mathbf{n}=\iiint_R\left(f\nabla^2 g-g\nabla^2 f\right)$
3. (*Difficult*) Let $\mathbf{a}\in\mathring{R}$ and $r(\mathbf{x})=\|\mathbf{x}-\mathbf{a}\|$. Prove that $f(\mathbf{a})=-\frac{1}{4\pi}\iiint_R\frac{\nabla^2 f}{r}+\frac{1}{4\pi}\iint_{\partial R}\left(\frac{\nabla f}{r}-f\nabla\left(\frac{1}{r}\right)\right)\cdot\mathbf{n}$.
   (*Hint: Use second's Green identity with $g=\frac{1}{r}$ for the region $R'=R\setminus B(\mathbf{a},\varepsilon)$ and take the limit when $\varepsilon\to 0$*)

Green's identities are useful to study harmonic functions:

Exercise 8. We say that a function $f:U\to\mathbb{R}$ defined on U open is ***harmonic*** if it is C^2 and $\nabla^2 f=0$ on U (Laplace's equation).

1. *The mean value property.*
   Let $f:U\to\mathbb{R}$ be a harmonic function where $U\subset\mathbb{R}^3$ is open. Let $\mathbf{a}\in U$ and $\varepsilon>0$ small enough so that $\overline{B}(\mathbf{a},\varepsilon)\subset U$. Prove that $f(\mathbf{a})=\frac{1}{4\pi\varepsilon^2}\iint_{\partial B(\mathbf{a},\varepsilon)}f=\frac{3}{4\pi\varepsilon^3}\iiint_{\overline{B}(\mathbf{a},\varepsilon)}f$.
   (*Hint: use Green's third identity for the first equality, the second equality derives from radial integration*)
   Comment: we can prove that the converse holds, i.e. a continuous satisfying the mean value property is harmonic (Bôcher 1906, Koebe 1906).

2. *The maximum principle, a special case.*
   Let $f : U \to \mathbb{R}$ be harmonic on $U \subset \mathbb{R}^3$ open. Let $R \subset U$ be a regular region with piecewise smooth boundary oriented. Prove that if $\forall \mathbf{x} \in \partial R,\ f(\mathbf{x}) = 0$ then $\forall \mathbf{x} \in R,\ f(\mathbf{x}) = 0$.
   (*Hint: use Green's first identity*)

Comments. The above results hold in any dimension, not only on $\mathbb{R}^3$. *Harmonic functions are very rigid since, for instance, they are real analytic and satisfy very strong properties such as the* mean value property, *the* maximum principle, Harnack's inequality *and real versions of* Riemann's removable singularity theorem *and* Liouville's theorem. *They are closely related to* holomorphic functions (*i.e. differentiable functions for complex variables*). *They appear in several areas of mathematics and physics.*

Exercise 9. Let $a > 0$ and $C \subset \mathbb{R}^3$ be the curve defined by the intersection of $x^2 + y^2 + z^2 = a^2$ with $y + z = a$ oriented counterclockwise as seen from above.
Compute $\int_C y\mathrm{d}x + y^2\mathrm{d}y + (x + 2z)\mathrm{d}z$.

Exercise 10. Let $S \subset \mathbb{R}^3$ be the surface defined by the portion of $2z = x^2 + y^2$ below the plane $z = 2$. We assume that S is oriented by downward normal vectors. Let $\mathbf{F}(x, y, z) = (3y, -4xz, -4xy)$.
Compute $\iint_S \mathbf{F} \cdot \mathbf{n}$ (1) directly (2) using Stokes' theorem.

Exercise 11. Let $R = [0, 1] \times [0, 1] \times [0, 1]$ and $S = \partial R$ oriented with outward normal vectors.
Let $\mathbf{F}(x, y, z) = (xyz, y^2 + 1, z^3)$.
Compute $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}$ (1) using Stokes' theorem (2) using the Divergence Theorem (3) directly.

Exercise 12. Let $S \subset \mathbb{R}^3$ be a smooth oriented surface with piecewise smooth with piecewise smooth positively oriented (relative) boundary ∂S. Let $f, g : U \to \mathbb{R}$ be two C^2 functions where $U \subset \mathbb{R}^3$ is open and satisfies $S \subset U$.
Prove that $\int_{\partial S} f\nabla g \cdot \mathrm{d}\mathbf{x} = \iint_S (\nabla f \times \nabla g) \cdot \mathbf{n}$ and that $\int_{\partial S} (f\nabla g + g\nabla f) \cdot \mathrm{d}\mathbf{x} = 0$.

Exercise 13. Are the following vector fields $\mathbf{F}$ defined on U conservative? If so, find a potential f such that $\mathbf{F} = \nabla f$.

1. $\mathbf{F}(x, y) = (x^2 + y^2, 2xy)$ on $U = \mathbb{R}^2$.
2. $\mathbf{F}(x, y) = (xy, xy)$ on $U = \mathbb{R}^2$.
3. $\mathbf{F}(x, y) = (2x \cos y, -x^2 \sin y - \sin y)$ on $U = \mathbb{R}^2$.
4. $\mathbf{F}(x, y) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$ on $U = \mathbb{R}^2 \setminus \{\mathbf{0}\}$.
5. $\mathbf{F}(x, y, z) = (xy, y, z)$ on $U = \mathbb{R}^3$.
6. $\mathbf{F}(x, y, z) = (2xyz, x^2z, x^2y)$ on $U = \mathbb{R}^3$.
7. $\mathbf{F}(x, y, z) = (2xyz + \sin x, x^2z, x^2y)$ on $U = \mathbb{R}^3$.
8. $\mathbf{F}(x, y, z) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}, 0\right)$ on $U = \mathbb{R}^3 \setminus \{z = 0\}$.

Exercise 14. For each of the following vector fields $\mathbf{F} : U \to \mathbb{R}^3$, does it exist $\mathbf{G} : U \to \mathbb{R}^3$ C^2 such that $\mathbf{F} = \operatorname{curl} \mathbf{G}$? If so, find such a $\mathbf{G}$.

1. $\mathbf{F}(x, y, z) = (xz, -yz, xy)$ on $U = \mathbb{R}^3$.
2. $\mathbf{F}(x, y, z) = \nabla(x^3 + xyz - z^2)$ on $U = \mathbb{R}^3$.
3. $\mathbf{F}(x, y, z) = (x, y, -z)$ on $U = \mathbb{R}^3$.
4. $\mathbf{F}(x, y, z) = (x^2 + 1, z - 2xy, x^2)$ on $U = \mathbb{R}^3$.
5. $\mathbf{F}(x, y, z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$ on $U = \mathbb{R}^3 \setminus \{\mathbf{0}\}$.