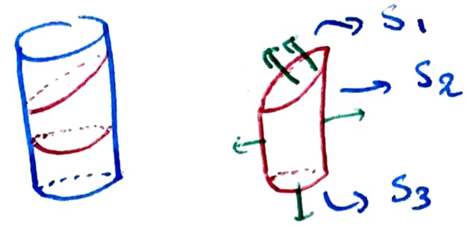
Ex 1:



(1) • $\iint_{S_1} \vec{F}\cdot\vec{n}$? $S_1 = \{\underbrace{(r\cos\theta, r\sin\theta, r\cos\theta + 2)}_{= \sigma(r,\theta)} : \theta \in [0,2\pi], r \in [0,1]\}$

$$\partial_r\sigma \times \partial_\theta\sigma = \begin{pmatrix}\cos\theta\\ \sin\theta\\ \cos\theta\end{pmatrix} \times \begin{pmatrix}-r\sin\theta\\ r\cos\theta\\ -r\sin\theta\end{pmatrix} = \begin{pmatrix}-r\\ 0\\ r\end{pmatrix}$$ good orientation

$$\iint_{S_1} \vec{F}\cdot\vec{n} = \int_0^{2\pi}\int_0^1 F(\sigma(r,\theta))\cdot(-r,0,r)\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 2r - r^2\cos\theta\,dr\,d\theta = 2\pi$$

• $\iint_{S_2} \vec{F}\cdot\vec{n}$? $S_2 = \{\underbrace{(\cos\theta, \sin\theta, \lambda(\cos\theta + 2))}_{= \sigma(\lambda,\theta)} : \theta \in [0,2\pi], \lambda \in [0,1]\}$

$$\partial_\lambda\sigma \times \partial_\theta\sigma = \begin{pmatrix}0\\ 0\\ \cos\theta + 2\end{pmatrix} \times \begin{pmatrix}-\sin\theta\\ \cos\theta\\ -\lambda\sin\theta\end{pmatrix} = \underbrace{-(\cos\theta+2)}_{<0}\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}$$ bad orientation

$$\iint_{S_2} \vec{F}\cdot\vec{n} = \int_0^{2\pi}\int_0^1 F(\sigma(\lambda,\theta))\cdot\big((\cos\theta+2)(\cos\theta, \sin\theta, 0)\big)\,d\lambda\,d\theta$$
$$= \int_0^{2\pi} (\cos\theta + 2)(2\cos^2\theta - 3\sin^2\theta)\,d\theta$$
$$= -2\pi$$

• $\iint_{S_3} \vec{F}\cdot\vec{n}$? $S_3 = \{\underbrace{(r\cos\theta, r\sin\theta, 0)}_{= \sigma(r,\theta)} : r \in [0,1], \theta \in [0,2\pi]\}$

$$\partial_r\sigma \times \partial_\theta\sigma = \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} \times \begin{pmatrix}-r\sin\theta\\ r\cos\theta\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 0\\ r\end{pmatrix}$$ bad orientation

$$\iint_{S_3} \vec{F}\cdot\vec{n} = \int_0^{2\pi}\int_0^1 F(\sigma(r,\theta))\cdot(0,0,r)\,dr\,d\theta = 0$$

Hence: $$\iint_S \vec{F}\cdot\vec{n} = \iint_{S_1} \vec{F}\cdot\vec{n} + \iint_{S_2} \vec{F}\cdot\vec{n} + \iint_{S_3} \vec{F}\cdot\vec{n}$$
$$= 2\pi - 2\pi + 0$$
$$= 0$$

(1) Denote by R the region enclosed by S, since S is oriented by outward pointing normal vectors, we may apply the Divergence Theorem:

$$\iint_S \vec{F} \cdot \vec{n} = \iiint_R \operatorname{div} \vec{F} = \iiint_R 2-3+1 = 0$$

<u>Exo 2</u> : Let $R = \left\{ (x,y,z) \in \mathbb{R}^3 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1 \right\}$

then $\partial R = S$ with the outward pointing normal vector orientation

hence, by the divergence theorem:

$$\iint_S \vec{F} \cdot \vec{n} = \iiint_R \operatorname{div}(\vec{F})$$

$$= \iiint_R 2z$$

$$(u,v,w) = \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) \rightarrow = \iiint_{u^2+v^2+w^2 \leq 1} 2abc^2 w$$

$$\text{Spherical coordinates} \rightarrow = 2abc^2 \int_0^{\pi}\int_0^{2\pi}\int_0^1 r\cos\varphi \, r^2 \sin\varphi \, dr \, d\theta \, d\varphi$$

$$= \pi abc^2 \int_0^{\pi} \cos\varphi \sin\varphi \, d\varphi = 0$$

<u>Exo 3</u>: ∂R is oriented by outward pointing normal vectors, hence, by the divergence theorem:

$$\mathcal{V}(R) = \iiint_R 1 = \frac{1}{3}\iiint_R \operatorname{div}(r) = \frac{1}{3}\iint_{\partial R} \vec{r} \cdot \vec{n}$$

$\hookrightarrow$ Divergence theorem

Exo 4 : (1) It's the surface integral of a scalar field (not a vector field) hence it doesn't depend on the orientation of S

(2) The divergence theorem concerns surface integrals of vector fields.

(3) We know that the unit normal outward pointing vector at $(x,y,z) \in S$ is

$\vec{n}(x,y,z) = (x,y,z)$

Hence, if we set $F(x,y,z) = (x,1,1)$ then

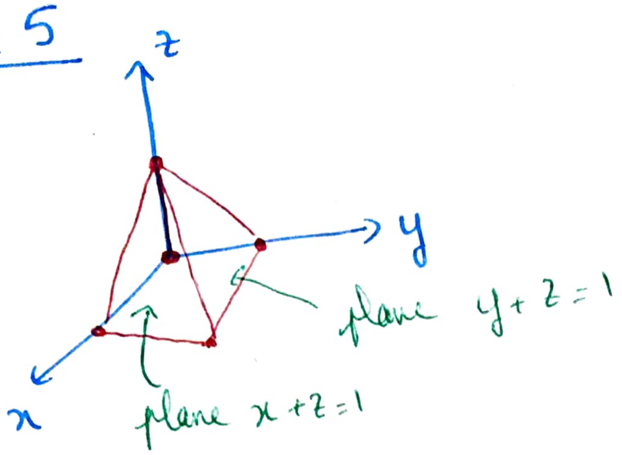
$$\iint_S x^2+y+z = \iint_S \vec{F} \cdot \vec{n}$$

$$= \iiint_{B(0,1)} \operatorname{div}(F)$$

$$= \iiint_{B(0,1)} 1$$

$$= \frac{4\pi}{3}$$

Ex 5



So the filled pyramid is

$$R = \left\{ (x,y,z) : x \in [0,1],\ y \in [0,1],\ z \leq \min(1-x, 1-y) \right\}$$

The orientation on $S = \partial R$ is given by normal outward pointing vectors, so we can apply the Divergence Theorem

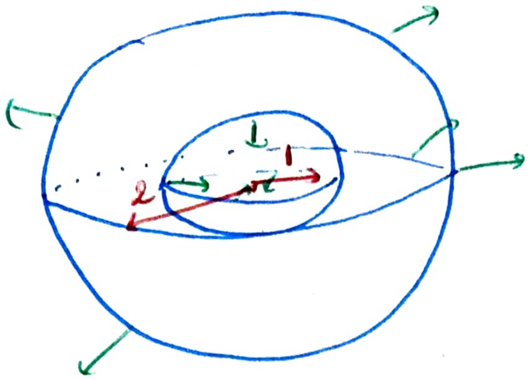
$$\iint_S \vec{F} \cdot \vec{n} = \iiint_R \operatorname{div} F = \iiint_R 2xy + 6yz + 18xz$$

$$= \iiint_{R \cap \{x \leq y\}} 2xy + 6yz + 18xz + \iiint_{R \cap \{x \geq y\}} 2xy + 6yz + 18xz$$

$$= \int_0^1 \int_0^y \int_0^{1-y} 2xy + 6yz + 18xz \, dz\, dx\, dy + \int_0^1 \int_0^x \int_0^{1-x} 2xy + 6yz + 18xz \, dz\, dy\, dx$$

$$= \frac{3}{10} + \frac{4}{10} = \frac{7}{10}$$

Ex6 :



notice that the inside sphere is oriented by vectors pointing to the origin and the outside sphere by vectors pointing away from the origin

$S = \partial R$ is oriented by outward pointing normal vectors

F is defined on $\mathcal{U} = \mathbb{R}^3 \setminus \{(0,0,0)\}$ open and $R \subset \mathcal{U}$

Moreover $S = \partial R$ has the good orientation

Hence, we may apply the Divergence Theorem

$$\iint_S \vec{F} \cdot \vec{n} = \iiint_R \operatorname{div}(\vec{F}) = \iiint_R \frac{2z^2}{(x^2+y^2+z^2)^2}$$

Spherical coordinates $\rightarrow$

$$= \int_0^{\pi}\int_0^{2\pi}\int_1^2 \frac{2r^2\cos^2\varphi}{r^4}\, r^2 \sin\varphi \, dr\, d\theta\, d\varphi$$

$$= 4\pi \int_1^2 1 \, dr \int_0^{\pi} \cos^2\varphi \sin\varphi \, d\varphi$$

$$= 4\pi \Big[r \Big]_1^2 \left[-\frac{\cos^3\varphi}{3} \right]_0^{\pi}$$

$$= 4\pi \left(2 - 1 \right)\left(\frac{1}{3} + \frac{1}{3} \right)$$

$$= \frac{8\pi}{3}$$

Ex 7 :

1. By the divergence theorem :

$$\iint_{\partial R} (f \nabla g)\cdot\vec{n} = \iiint_R \mathrm{div}(f\nabla g)$$

$$\left(\begin{array}{l}\mathrm{div}(fG) \\ = f\,\mathrm{div}\,G + \nabla f\cdot G\end{array}\right) \rightarrow = \iiint_R f\,\mathrm{div}(\nabla g) + \nabla f\cdot\nabla g$$

$$\left(\nabla^2 g = \mathrm{div}(\nabla g)\right) \rightarrow = \iiint_R f\nabla^2 g + \nabla f\cdot\nabla g$$

2. By 1: $\iint_{\partial R} f\nabla g = \iiint_R \nabla f\cdot\nabla g + f\nabla^2 g$ (1)

and $\iint_{\partial R} g\nabla f = \iiint_R \nabla f\cdot\nabla g + g\nabla^2 f$ (2)

(1)-(2) : $\iint_{\partial R} f\nabla g - g\nabla f = \iiint_R f\nabla^2 g - g\nabla^2 f$

3. We apply (2) to f, $g = \frac{1}{r}$ and $R' = R\setminus B(a,\varepsilon)$

where $\varepsilon > 0$ is s.t. $B(a,\varepsilon) \subset \mathring{R}$ ($\exists$ such since $a \in \mathring{R}$)

$$\iint_{\partial R\cup\partial B(a,\varepsilon)} f\nabla(1/r) - \frac{1}{r}\nabla f = \iiint_{R\setminus B(a,\varepsilon)} \underbrace{f\cancel{\nabla^2(1/r)}}_{=0} - \frac{1}{r}\nabla^2 f$$

$$\Rightarrow \underbrace{\iint_{\partial B(a,\varepsilon)} f\nabla(1/r) - \frac{1}{r}\nabla f}_{(2)} = -\underbrace{\iiint_{R\setminus B(a,\varepsilon)} \frac{1}{r}\nabla^2 f}_{(1)} + \iint_{\partial R} \frac{1}{r}\nabla f - f\nabla(1/r)$$

• (1): Since f is C^2, $\nabla^2 f$ is C^0 on R compact hence

$$\left|\frac{1}{r}\nabla^2 f\right| \leqslant \frac{M}{\|x-a\|^1}, \quad 1 < 3$$

so $\iiint \frac{1}{\|x-a\|}$ is (absolutely cv)

and $\lim_{\varepsilon\to 0} \iiint_{R\setminus B(a,\varepsilon)} \frac{\nabla^2 f}{r} = \iiint_R \frac{\nabla^2 f}{r}$

② $\lim\limits_{\varepsilon \to 0} \iint\limits_{\partial B(a,\varepsilon)} \left(f \nabla(1/r) - \frac{\nabla f}{r} \right) \cdot \vec{n} = ?$

Cauchy-Schwarz

$\cdot \left| \iint\limits_{\partial B(a,\varepsilon)} \frac{\nabla f}{r} \cdot \vec{n} \right| \leq \iint\limits_{\partial B(a,\varepsilon)} \left| \frac{\nabla f}{r} \cdot \vec{n} \right| \leq \iint\limits_{\partial B(a,\varepsilon)} \left\| \frac{\nabla f}{r} \right\| \underbrace{\| \vec{n} \|}_{=1}$

∇f is C^0 on ∂B compact hence bounded $\rightarrow \leq M \iint\limits_{\partial B(a,\varepsilon)} \frac{1}{r}$

area sphere

$r = \varepsilon$ on $\partial B(a,\varepsilon) \longrightarrow = \frac{M}{\varepsilon} \iint\limits_{\partial B(a,\varepsilon)} 1 = \frac{M}{\varepsilon} 4\pi\varepsilon^2 = 4\pi M\varepsilon \xrightarrow[\varepsilon \to 0]{} 0$

So $\lim\limits_{\varepsilon \to 0} \iint\limits_{\partial B(a,\varepsilon)} \frac{\nabla f}{r} \cdot \vec{n} = 0$

$\cdot \iint\limits_{\partial B(a,\varepsilon)} f \nabla\left(\frac{1}{r}\right) \cdot \vec{n} = \iint\limits_{\partial B(a,\varepsilon)} f \underbrace{\frac{1}{r^3} \begin{pmatrix} a_x - x \\ a_y - y \\ a_z - z \end{pmatrix}}_{\nabla(1/r)} \cdot \underbrace{\left(-\frac{1}{r} \begin{pmatrix} x - a_x \\ y - a_y \\ z - a_z \end{pmatrix} \right)}_{\vec{n}}$

$\vec{n}$: remember that $\vec{n}$ points to a since outward for R'

$= \iint\limits_{\partial B(a,\varepsilon)} f \frac{1}{r^3} r^2 \frac{1}{r}$

$= \iint\limits_{\partial B(a,\varepsilon)} f/r^2 = \iint\limits_{\partial B(a,\varepsilon)} \frac{f}{\varepsilon^2}$ since $r = \varepsilon$ on $\partial B(a,\varepsilon)$

Let $\delta > 0$, since f is C^0, $\exists \varepsilon > 0$ s.t. $x \in B(a, 2\varepsilon) \Rightarrow |f(a) - f(x)| \leq \delta$

$\Rightarrow \underbrace{\iint\limits_{\partial B(a,\varepsilon)} \frac{f(a) - \delta}{\varepsilon^2}}_{= 4\pi(f(a) - \delta)} \leq \iint\limits_{\partial B(a,\varepsilon)} \frac{f}{\varepsilon^2} \leq \underbrace{\iint\limits_{\partial B(a,\varepsilon)} \frac{f(a) + \delta}{\varepsilon^2}}_{= 4\pi(f(a) + \delta)}$

hence $\lim\limits_{\varepsilon \to 0} \iint\limits_{\partial B(a,\varepsilon)} \frac{f}{\varepsilon^2} = 4\pi f(a)$

and $\lim \iint \left(f \nabla(1/r) - \frac{\nabla f}{r} \right) \cdot \vec{n} = 0 + 4\pi f(a)$

CCL: By ① & ②: $4\pi f(a) = - \iiint\limits_{R} \frac{\nabla^2 f}{r} + \iint\limits_{\partial R} \left(\frac{\nabla f}{r} - f \nabla(1/r) \right) \cdot \vec{n}$

Ex8: (1) By Green's third identity

$$f(a) = -\frac{1}{4\pi} \iiint_{B(a,\varepsilon)} \overset{=0}{\cancel{\frac{\nabla^2 f}{r}}} + \frac{1}{4\pi} \iint_{\partial B(a,\varepsilon)} \left(\frac{\nabla f}{r} - f \nabla (1/r) \right) \cdot \vec{m}$$

↳ outward pointing

$$\bullet \iint_{\partial B(a,\varepsilon)} \frac{\nabla f}{r} \cdot \vec{m} \underset{r=\varepsilon \text{ on } \partial B(a,\varepsilon)}{=} \frac{1}{\varepsilon} \iint_{\partial B(a,\varepsilon)} \nabla f \cdot \vec{m} \underset{\text{Divergence thm}}{=} \frac{1}{\varepsilon} \iiint_{B(a,\varepsilon)} \operatorname{div}(\nabla f) = \frac{1}{\varepsilon} \iiint_{B(a,\varepsilon)} \overset{=0}{\nabla^2 f} = 0$$

$$\bullet \iint_{\partial B(a,\varepsilon)} f \nabla (1/r) \cdot \vec{m} = -\iint_{\partial B(a,\varepsilon)} f \frac{1}{r^3} \vec{r} \cdot \vec{m}$$

$$= -\iint_{\partial B(a,\varepsilon)} f \frac{1}{r^3} \vec{r} \cdot \frac{\vec{r}}{r} \qquad m = \frac{\vec{r}}{r}$$

$$= -\iint_{\partial B(a,\varepsilon)} \frac{f}{r^2}$$

$$= -\frac{1}{\varepsilon^2} \iint_{\partial B(a,\varepsilon)} f$$

Hence $\boxed{f(a) = \frac{1}{4\pi\varepsilon^2} \iint_{\partial B(a,\varepsilon)} f}$

Then
$$\iiint_{B(a,\varepsilon)} f = \int_0^\varepsilon \left(\iint_{\partial B(a,t)} f \right) dt = \int_0^\varepsilon f(a) 4\pi t^2 \, dt$$
$$= f(a) \frac{4\pi\varepsilon^3}{3}$$

Hence $\boxed{f(a) = \frac{3}{4\pi\varepsilon^3} \iiint_{B(a,\varepsilon)} f}$

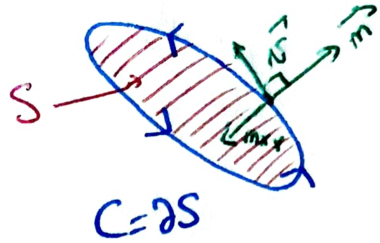
(2) We apply Green's first identity to $g = f$: $\iint_{\partial R} \overset{=0 \text{ on } \partial R}{\cancel{f}} \nabla f \cdot \vec{m} = \iiint_R \|\nabla f\|^2 + \cancel{f \overset{=0}{\nabla^2 f}}$

Hence $\iiint_R \|\nabla f\|^2 = 0$. Since ∇f is C^0 we get that $\nabla f = \vec{0}$ on R.

Hence f is constant on each connected component of R.

Since $f = 0$ on ∂R, by C^0, $f \equiv 0$ on R.

Ex 9



S has to be oriented by $\vec{n}$ pointing upward for the given orientation on $C=\partial S$ to be the positive one

By Stokes theorem:

$$\int_C y\,dx + y^2\,dy + (x+2z)\,dz = \iint_S \text{curl}(y, y^2, x+2z)\cdot\vec{n} = \iint_S (0,-1,-1)\cdot\vec{n}$$

$$(1)\begin{cases} x^2+y^2+z^2=a^2 \\ y+z=a \end{cases} \Rightarrow (\sqrt{2}\,x)^2 + (2y-a)^2 = a^2$$

↳ $z=a-y$ in (1)

Hence we set $(\sqrt{2}\,x,\ 2y-a) = (r\cos\theta, r\sin\theta)$, $r\in[0,a]$, $\theta\in[0,2\pi]$

and S is parametrized by

$$\sigma(r,\theta) = \left(\frac{r}{\sqrt{2}}\cos\theta,\ \frac{r}{2}\sin\theta + \frac{a}{2},\ -\frac{r}{2}\sin\theta + \frac{a}{2}\right),\ \begin{matrix} r\in[0,a] \\ \theta\in[0,2\pi] \end{matrix}$$

$$\partial_r\sigma\times\partial_\theta\sigma = \begin{pmatrix} \frac{\cos\theta}{\sqrt{2}} \\ \frac{\sin\theta}{2} \\ -\frac{\sin\theta}{2} \end{pmatrix} \times \begin{pmatrix} -\frac{r}{\sqrt{2}}\sin\theta \\ \frac{r}{2}\cos\theta \\ -\frac{r}{2}\cos\theta \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{r}{2\sqrt{2}} \\ \frac{r}{2\sqrt{2}} \end{pmatrix}$$

$\frac{r}{2\sqrt{2}} > 0$ So we have the good orientation

and
$$\iint_S (0,-1,-1)\cdot\vec{n} = \int_0^{2\pi}\!\!\int_0^a \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ r/2\sqrt{2} \\ r/2\sqrt{2} \end{pmatrix} dr\,d\theta$$

$$= \int_0^{2\pi}\!\!\int_0^a -\frac{r}{\sqrt{2}}\,dr\,d\theta$$

$$= -\pi\frac{\sqrt{2}}{2}\left[r^2\right]_0^a$$

$$= -\pi\frac{\sqrt{2}}{2}a^2$$

and $\int_C y\,dx + y^2\,dy + (x+2z)\,dz = -\pi\frac{\sqrt{2}}{2}a^2$

Ex10 : Let $(x,y,z) \in S$ then

(1) $z \leq 2 \Leftrightarrow \dfrac{x^2+y^2}{2} \leq 2 \Leftrightarrow x^2+y^2 \leq 2^2$

Hence $S = \{ \underbrace{(x, y, \tfrac{x^2+y^2}{2})}_{\sigma(x,y)} : x^2+y^2 \leq 2^2 \}$

$$\partial_x\sigma \times \partial_y\sigma = \begin{pmatrix}1\\0\\x\end{pmatrix} \times \begin{pmatrix}0\\1\\y\end{pmatrix} = \begin{pmatrix}-x\\-y\\1\end{pmatrix} \quad 1 \to < 0$$

we have the bad orientation

$$\iint_S \vec{F}\cdot\vec{n} = \iint_{x^2+y^2\leq 2^2} \begin{pmatrix}3y\\-4xz\\-4yx\end{pmatrix} \cdot \begin{pmatrix}x\\y\\-1\end{pmatrix}$$

$$= \iint_{x^2+y^2\leq 2^2} 3xy - 4xy\cdot\frac{x^2+y^2}{2} + 4yx$$

polar coord $\to$ $$= \int_0^{2\pi}\int_0^2 3r^3\cos\theta\sin\theta - 2r^5\cos\theta\sin\theta + 4r^3\cos\theta\sin\theta \, dr\, d\theta$$

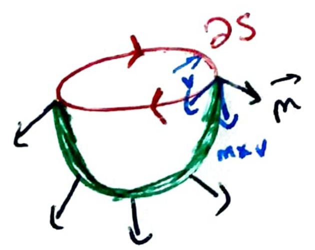
$$= \int_0^{2\pi} \left(12 - \frac{64}{3} + 16\right) \underbrace{\cos\theta\sin\theta}_{=\frac{\sin(2\theta)}{2}} d\theta$$

$$= 0$$

(2) $\operatorname{div} \vec{F} = 0$ on $\mathbb{R}^3$ star-shaped.
since $\forall t \in [0,1]$ $\forall x \in \mathbb{R}^3$, $(1-t)0 + t\,x \in \mathbb{R}^3$,

$F = \operatorname{curl} G$ where $G = \displaystyle\int_0^1 F(tr)\times tr \, dt = \int_0^1 \begin{pmatrix}3ty\\-4t^2xz\\-4t^2xy\end{pmatrix} \times \begin{pmatrix}tx\\ty\\tz\end{pmatrix} dt$

$$= \int_0^1 \begin{pmatrix}4t^3xy^2 - 4t^3xz^2\\ -4t^3x^2y - 3t^2yz\\ 3t^2y^2 + 4t^3x^2z\end{pmatrix} dt = \begin{pmatrix}xy^2 - xz^2\\ -x^2y - yz\\ y^2 + x^2z\end{pmatrix}$$



By Stokes : $\displaystyle\iint_S \vec{F}\cdot\vec{n} = \int_{\partial S} \vec{G}\cdot d\vec{x}$

positive orientation

$\partial S = \{ (2\cos\theta, -2\sin\theta, 2) : \theta \in [0, 2\pi] \}$

$$\int_{\partial S} \vec{G}\cdot d\vec{x} = \int_0^{2\pi} \vec{G}(\sigma(\theta)) \cdot (-2\sin\theta, -2\cos\theta, 0)\, d\theta$$

$$= 16\int_0^{2\pi} 2\cos\theta\sin\theta + \cos\theta\sin\theta(\cos^2 - \sin^2)\, d\theta$$

$$= 16\int_0^{2\pi} \sin(2\theta) + \frac{\sin(2\theta)}{2}\cos(2\theta)\, d\theta$$

$$= 16\int_0^{2\pi} \sin(2\theta) + \frac{\sin(4\theta)}{4} d\theta = 0$$

Ex 11 :

(1) F is defined on $\mathbb{R}^3$ and S is closed hence by Stokes theorem : $\iint_S \operatorname{curl}\vec{F}\cdot\vec{n} = 0$

(2) By the divergence theorem (outward orientation ✓)

$$\iint_S \operatorname{curl}\vec{F}\cdot\vec{n} = \iiint_R \operatorname{div}(\operatorname{curl}\vec{F}) \underset{\substack{\uparrow \\ F \text{ is } C^2 \text{ on } \mathbb{R}^3}}{=} \iiint_R 0 = 0$$

(3) You have to work separately on the 6 sides...

Ex 12 :

$$\int_{\partial S} f\nabla g\cdot d\vec{x} \overset{\text{Stokes}}{=} \iint_S \operatorname{curl}(f\nabla g)\cdot\vec{n}$$

$\operatorname{curl}(fG) = f\operatorname{curl}G + \nabla f\times G$ $\to$

$$= \iint_S \Big(f\,\underbrace{\operatorname{curl}(\nabla g)}_{\substack{= 0 \\ \text{since } g \text{ is } C^2}} + \nabla f\times\nabla g\Big)\cdot\vec{n}$$

$$= \iint_S (\nabla f\times\nabla g)\cdot\vec{n}$$

$$\int_{\partial S}(f\nabla g + g\nabla f)\cdot dx \overset{\text{above}}{=} \iint_S (\nabla f\times\nabla g + \nabla g\times\nabla f)\cdot\vec{n}$$

$$= \iint_S (\nabla f\times\nabla g - \nabla f\times\nabla g)\cdot\vec{n}$$

$$= \iint_S \vec{0}\cdot\vec{n} = 0$$

Ex 13 :

$n = 2$:

Prop : $F : \mathcal{U} \to \mathbb{R}^2$ C^1, $\mathcal{U} \subset \mathbb{R}^2$ open

F conservative $\Rightarrow \dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$

- **Contrapositive** : $F : \mathcal{U} \to \mathbb{R}^2$ C^1, $\mathcal{U} \subset \mathbb{R}^2$ open

  $\dfrac{\partial F_2}{\partial x} \neq \dfrac{\partial F_1}{\partial y} \Rightarrow F$ **not** conservative

- **"Converse"** (Poincaré lemma)

  $F : \mathcal{U} \to \mathbb{R}^2$ C^1, $\mathcal{U} \subset \mathbb{R}^2$ open and **star-shaped**

  $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y} \Rightarrow F$ conservative

- **How to compute** : $F : \mathcal{U} \to \mathbb{R}^2$ C^1, $\mathcal{U} \subset \mathbb{R}^2$ open **rectangle**

  If F is conservative then $F = \nabla f$ where

  $$f(x,y) = \int_a^x F_1(t,b)\,dt + \int_b^y F_2(x,t)\,dt \ , \quad (a,b) \in \mathcal{U}$$

$n=3$:

Prop : $F : \mathcal{U} \to \mathbb{R}^3$ C^1, $\mathcal{U} \subset \mathbb{R}^3$ open

F conservative $\Rightarrow \operatorname{curl} F = \vec{0}$

- **Contrapositive** : $F : \mathcal{U} \to \mathbb{R}^3$ C^1, $\mathcal{U} \subset \mathbb{R}^3$ open

  $\operatorname{curl} \vec{F} \neq \vec{0} \Rightarrow F$ is **not** conservative

- **"Converse"** (Poincaré lemma)

  $F : \mathcal{U} \to \mathbb{R}^3$ C^1, $\mathcal{U} \subset \mathbb{R}^3$ open and **star-shaped**

  $\operatorname{curl} \vec{F} = \vec{0} = F$ is conservative

- **How to compute** : $F : \mathcal{U} \to \mathbb{R}^3$ C^1, $\mathcal{U} \subset \mathbb{R}^3$ open rectangle

  If F is conservative then $F = \nabla f$ where

  $$f(x,y,z) = \int_a^x F_1(t,b,c)\,dt + \int_b^y F_2(x,t,c)\,dt + \int_c^z F_2(x,y,t)\,dt \ , \quad (a,b,c) \in \mathcal{U}$$

Ex 13

1. $\frac{\partial F_2}{\partial x}(x,y) = 2y$

$\frac{\partial F_1}{\partial y}(x,y) = 2y$

$U = \mathbb{R}^2$ is star shaped and F is C^1 so it is conservative

$U = \mathbb{R}^2$ is a rectangle (infinite rectangle, not as in Darboux)

$$f(x,y) = \int_0^x F_1(t,0)\,dt + \int_0^y F_2(x,t)\,dt$$
$$= \int_0^x t^2\,dt + \int_0^y 2xt\,dt$$
$$= \frac{x^3}{3} + xy^2$$

$F = \nabla f$

2. $\frac{\partial F_2}{\partial x} = y$ hence F is not conservative

$\frac{\partial F_1}{\partial y} = x$

3. $\frac{\partial F_2}{\partial x} = -2x\sin y$ $\quad \frac{\partial F_1}{\partial y} = -2x\sin y$

$U = \mathbb{R}^2$ is star-shaped hence F C^1 is conservative

$$f(x,y) = \int_0^x 2t\underbrace{\cos(\pi/2)}_{=0}\,dt + \int_{\pi/2}^y -x^2\sin(t) - \sin(t)\,dt$$

$(x_0, y_0) = (0, \pi/2)$

$$= \left[(x^2+1)\cos(t)\right]_{\pi/2}^y = (x^2+1)\cos(y)$$

$F = \nabla f$

4. Here $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ but we can't conclude using Poincaré Lemma since the domain $U = \mathbb{R}^2 \setminus \{(0,0)\}$ is not star-shaped

If $C = \{(x,y) : x^2+y^2=1\}$ is oriented clockwise

then $\int_C \vec{F} \cdot d\vec{x} \neq 0$. but C is closed

Hence $\vec{F}$ is not conservative

($\int_C Df \cdot d\vec{x} = 0$ by the Gradient thm)

5. $\operatorname{curl} \vec{F} = (0,0,-x) \neq \vec{0}$

Hence $\vec{F}$ is <u>not</u> conservative

6. $\operatorname{curl} \vec{F} = \vec{0}$, $\vec{F}$ is C^1, $U = \mathbb{R}^3$ is star-shaped

Hence by Poincaré Lemma $\vec{F}$ is conservative

$$f(x,y,z) = \int_0^x \underbrace{F_1(t,0,0)}_{=0} dt + \int_0^y \underbrace{F_2(x,t,0)}_{=0} dt + \int_0^z F_3(x,y,t) dt$$

$$= \int_0^z x^2 y \, dt = x^2 y z$$

$\vec{F} = Df$

7. Same as above but now $f(x,y,z) = \int_0^x \sin t \, dt + \int_0^z x^2 y \, dt$

$$= -\cos x + x^2 y z$$

8. $\operatorname{curl} F = \vec{0}$ but $U = \mathbb{R}^3 \setminus \{z=0\}$ is not star-shaped hence we can't conclude using Poincaré Lemma

Set $C = \{(\cos\theta, \sin\theta, 0)\}$ then

$$\int_C \vec{F} \cdot d\vec{x} = \int_0^{2\pi} \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} d\theta = \int_0^{2\pi} -1 \, d\theta = -2\pi$$

but C is closed

so $\vec{F}$ is not conservative

($\int_C Df \cdot d\vec{x} = 0$ by the Gradient thm)

Ex14:

Prop: $U \subset \mathbb{R}^3$ open, $F: U \to \mathbb{R}^3$ C^1

If $F = \operatorname{curl} G$ for $G: U \to \mathbb{R}^3$ C^2 then $\operatorname{div} F = 0$ on U

Contrapositive: $U \subset \mathbb{R}^3$ open, $F: U \to \mathbb{R}^3$ C^1

$\operatorname{div} F \neq 0$ on $U \Rightarrow \nexists\, G: U \to \mathbb{R}^3$ C^2 s.t. $F = \operatorname{curl} G$

"Converse" (Poincaré lemma)

$U \subset \mathbb{R}^3$ open, star-shaped, $F: U \to \mathbb{R}^3$ C^1

If $\operatorname{div} F = 0$ then $\exists\, G: U \to \mathbb{R}^3$ C^2 s.t. $F = \operatorname{curl} G$

How to compute G:

Take $P_0 \in U$ s.t. $\forall q \in U$, $\forall t \in [0,1]$, $(1-t)P_0 + tq \in U$

then you can take

$$G(x,y,z) = \int_0^1 F\begin{pmatrix} x_0 + t(x-x_0) \\ y_0 + t(y-y_0) \\ z_0 + t(z-z_0) \end{pmatrix} \times \begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix} t\, dt$$

where $P_0 = (x_0, y_0, z_0)$

For $P_0 = (0,0,0)$, if it works,

$$G(x,y,z) = \int_0^1 F(tr) \times (tr)\, dt$$

where $r(x,y,z) = (x,y,z)$

Ex14

① F is C^1 on $\mathcal{U} = \mathbb{R}^3$ star-shaped and $\operatorname{div} F = 0$ on $\mathcal{U}$

Hence by Poincaré Lemma, $\exists G : \mathcal{U} \to \mathbb{R}^3$ C^2 s.t. $F = \operatorname{curl} G$

Since $\mathcal{U} = \mathbb{R}^3$ we can take

$$G(x,y,z) = \int_0^1 F\begin{pmatrix} tx \\ ty \\ tz \end{pmatrix} \times \begin{pmatrix} tx \\ ty \\ tz \end{pmatrix} dt$$

$$= \int_0^1 \begin{pmatrix} -t^3xy^2 - t^3yz^2 \\ t^3x^2y - t^3xz^2 \\ 2t^3xyz \end{pmatrix} dt$$

$$= \frac{1}{4}\begin{pmatrix} -xy^2 - yz^2 \\ x^2y - xz^2 \\ 2xyz \end{pmatrix}$$

② $\operatorname{div} \vec{F} = \nabla^2(x^3 + xyz - z^2) = 6x - 2 \neq 0$

Hence there is no such G

③ $\operatorname{div} \vec{F} = 1 \neq 0$ hence there is no such G

④ $\operatorname{div} \vec{F} = 0$ on $\mathcal{U} = \mathbb{R}^3$ star shaped and F is C^1 hence, ~~..~~ $\exists G \dots$

and we can take since $\mathcal{U} = \mathbb{R}^3$:

$$G(x,y,z) = \int_0^1 \begin{pmatrix} t^2x^2 + 1 \\ tz - 2t^2xy \\ t^2x^2 \end{pmatrix} \times \begin{pmatrix} tx \\ ty \\ tz \end{pmatrix} dt$$

$$= \int_0^1 \begin{pmatrix} -t^3x^2y - t^3xyz + t^2z^2 \\ t^3x^3 - tz - t^3x^2z \\ ty + 3t^3x^2y - t^2xz \end{pmatrix} dt = \begin{pmatrix} \frac{1}{12}(\cancel{6} - 3x^2y - 6xyz + 4z^2) \\ \frac{1}{4}(x^3 - x^2z - 2z) \\ \frac{1}{12}((19x^2+6)y - 4xz \end{pmatrix}$$

⑤ Here $\operatorname{div} \vec{F} = 0$ but $\mathcal{U} = \mathbb{R}^3 \setminus \{0\}$ is not star-shaped

Hence we can't conclude with Poincaré Lemme.

However by Gauss-Law, $\displaystyle\iint_{x^2+y^2+z^2=1} \vec{F} \cdot \vec{n} = \pm 4\pi \neq 0$

But if $F = \operatorname{curl} G$ then $\displaystyle\iint_{x^2+y^2+z^2=1} \vec{F} \cdot \vec{n} = 0$ by Stokes theorem.

since the unit sphere is closed