1.1. Let $x \in (0,+\infty)$ then $\begin{array}{l}(0,+\infty) \longrightarrow \mathbb{R} \\ t \longmapsto t^{x-1}e^{-t}\end{array}$ is C^0 and >0

- **at 0**: $0 < t^{x-1}e^{-t} \leqslant t^{x-1}$

and $\int_a^1 t^{x-1}\,dt = \frac{1}{x} - \frac{a^x}{x} \xrightarrow[a\to 0]{} \frac{1}{x}$

so $\int_0^1 t^{x-1}e^{-t}\,dt < +\infty$

- **at $+\infty$**: $t^2(t^{x-1}e^{-t}) \xrightarrow[t\to+\infty]{} 0$

so $\int_1^{+\infty} t^{x-1}e^{-t}\,dt < +\infty$

Ccl: $\int_0^{+\infty} t^{x-1}e^{-t}\,dt < +\infty$

1.2.a. Let $x>0$

$$\int_a^b t^x e^{-t}\,dt \underset{\text{parts}}{=} \left[-e^{-t}t^x\right]_a^b + x\int_a^b t^{x-1}e^{-t}\,dt$$

parts:
$u = t^x \quad v' = e^{-t}$
$u' = xt^{x-1} \quad v = -e^{-t}$

By taking $a \to 0$ and $b \to +\infty$ we get:

$$\Gamma(x+1) = \int_0^{+\infty} t^x e^{-t}\,dt = x\int_0^{+\infty} t^{x-1}e^{-t}\,dt = x\Gamma(x)$$

1.2.b. Induction on n:

Base case: $n=0$ $\quad \Gamma(1) = \int_0^{+\infty} e^{-t}\,dt = \left[-e^{-t}\right]_0^{+\infty} = 1 = 0!$

Induction step: assume that $\Gamma(n+1) = n!$ for some $n \in \mathbb{N}_{\geq 0}$

then $\Gamma(n+2) = (n+1)\Gamma(n+1)$ by 1.2.a

$= (n+1)\,n! = (n+1)!$

1.3. a. Define $F(x,t) = t^{x-1} e^{-t}$ on $\mathbb{R}^2_{>0}$

We are going to prove by induction that for $n \in \mathbb{N}_{\geq 0}$

- $\dfrac{\partial^{n+1} F}{\partial x^{n+1}}(x,t) = \ln(t)^{n+1} t^{x-1} e^{-t}$
- $\Gamma^{(n)}(x)$ is C^1
- $\Gamma^{(n+1)}(x) = \displaystyle\int_0^{+\infty} \frac{\partial^{n+1} F}{\partial x^{n+1}}(x,t)\, dt$

<u>Base case</u> : <u>$n=0$</u>. $\dfrac{\partial F}{\partial x}(x,t) = \dfrac{\partial}{\partial x}(t^{x-1} e^{-t}) = \ln(t)\, t^{x-1} e^{-t}$

which is C^0 on $\mathbb{R}^2_{>0}$

- Take $K \subset (0,+\infty)$ compact then

  $\exists a,b$ s.t. $0 < a \leq 1 \leq b < +\infty$ and $K \subset [a,b]$

  and $\left|\dfrac{\partial F}{\partial x}(x,t)\right| \leq \underbrace{|\ln(t)| \cdot \max(t^{a-1}, t^{b-1})\, e^{-t}}_{= \phi(t)}$

  where ϕ is C^0, ≥ 0 and integrable on $(0,+\infty)$

  and we already know that $\int F$ is abscv (by 1.1)

  Hence by the theorem $\Gamma^{(0)} = \Gamma$ is C^1

  and $\Gamma'(x) = \displaystyle\int_0^{+\infty} \frac{\partial F}{\partial x}(x,t)\, dt$

- The inductive case is very similar as the base case.

1.3.b. We know that

- Γ is C^2 on $(0,+\infty)$
- $\Gamma''(x) = \int_0^{+\infty} (\ln t)^2\, t^{x-1} e^{-t}\, dt > 0$

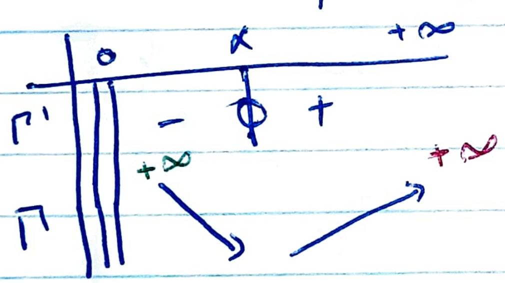
Hence Γ is convex on $(0,+\infty)$

1.3.c. $\Gamma(x)\big/_{1/x} = x\Gamma(x) \underset{\text{by } 1.2.a.}{=} \Gamma(x+1) \xrightarrow[x \to 0^+]{} \Gamma(1) = 0! = 1$

since Γ is C^0 by 1.2.b.

1.3.d. Since $\Gamma'' > 0$, Γ' is strictly increasing on $(0,+\infty)$
since $\Gamma(1) = \Gamma(2)$, by Rolle's theorem, $\exists \alpha \in (0,1)$
s.t. $\Gamma'(\alpha) = 0$

Hence we have the following monotonicity:

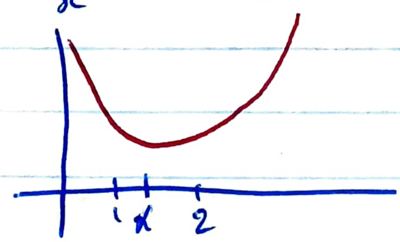


by 1.3.c : $\lim_{x \to 0^+} \Gamma(x) = +\infty$

We know that Γ is strictly increasing on $(\alpha,+\infty)$
and that $\Gamma(n+1) = n! \xrightarrow[\substack{n \to +\infty \\ n \in \mathbb{N}}]{} +\infty$ so $\lim_{x \to +\infty} \Gamma(x) = +\infty$

$$\frac{\Gamma(x)}{x} \underset{\text{by } 1.2.a}{=} \frac{(x-1)\Gamma(x-1)}{x} = \frac{x-1}{x}\Gamma(x-1) \xrightarrow[x \to +\infty]{} +\infty$$

Hence:



↳ Γ goes further and further away from the x-axis

1.4.a. $\displaystyle\int_a^b t^{x-1} e^{-t} dt \underset{\substack{t=u^2 \\ dt = 2u\,du}}{=} \int_{a^2}^{b^2} u^{2(x-1)} e^{-u^2} 2u\,du$

$$= \int_{a^2}^{b^2} 2u^{2x-1} e^{-u^2} du$$

$$\xrightarrow[\substack{a\to 0 \\ b \to +\infty}]{} \int_0^{+\infty} 2u^{2x-1} e^{-u^2} du$$

Hence $\Gamma(x) = \displaystyle\int_0^{+\infty} 2u^{2x-1} e^{-u^2} du$

1.4.b. $\Gamma(1/2) = 2\displaystyle\int_0^{+\infty} u^0 e^{-u^2} du = \int_{-\infty}^{+\infty} e^{-u^2} du$

1.4.c. Notice that $G(u,v) = 4e^{-u^2-v^2} u^{2r-1} v^{2s-1}$ is continuous and positive, so the value of the integral doesn't depend on the exhaustion (possibly $+\infty$)

Let $C_k = [0,k] \times [0,k]$.

$$\int_{C_k} G(u,v) = \int_0^k\int_0^k 4e^{-u^2-v^2} u^{2r-1} v^{2s-1} du\,dv$$

$$= \int_0^k 2e^{-u^2} u^{2r-1} du \int_0^k 2e^{-v^2} v^{2s-1} dv$$

$$\xrightarrow[k\to+\infty]{} \Gamma(r)\Gamma(s) < +\infty$$

by 1.4.a.

1.h.d: Let $D_k = B(\vec{0},k) \cap \{x>0,\ y>0\}$

then by 1.h.c.

$$\Gamma(r)\Gamma(s) = \lim_{k\to+\infty} \int_{D_k} G(u,v)$$

polar coordinates $\rightarrow$
$$= \lim_{k\to+\infty} \int_{[0,k]\times[0,\pi/2]} 4 e^{-\rho^2} \cos^{2r-1}\theta \sin^{2s-1}\theta\, \rho^{2r+2s-2+1}\, d\rho\, d\theta$$

$$= \lim_{k\to+\infty} 2\int_0^k 2\rho e^{-\rho^2} \rho^{2(r+s)-1}\, d\rho \int_0^{\pi/2} \cos^{2r-1}\theta \sin^{2s-1}\theta\, d\theta$$

$$= 2\Gamma(r+s) \int_0^{\pi/2} \cos^{2r-1}\theta \sin^{2s-1}\theta\, d\theta$$

by 1.h.a

1.h.e. By 1.h.d

$$\Gamma(1/2)^2 = 2\Gamma(1)\int_0^{\pi/2} 1\, d\theta = 2 \times \overset{1}{\overset{\shortparallel}{0!}} \times \frac{\pi}{2} = \pi$$

So
$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx \underset{\substack{\uparrow \\ 1.h.b}}{=} \Gamma(1/2) = \sqrt{\pi}$$

1.5.a. $c^x \Gamma(x) = \int_0^{+\infty} (ct)^x \frac{e^{-t}}{t} \, dt$

Following the hint we are first proving that $\varphi : x \mapsto (ct)^x e^{-t}/t$ is convex

$\varphi'(x) = \ln(ct)\,(ct)^x e^{-t}/t$

$\varphi''(x) = (\ln(ct))^2 (ct)^x e^{-t}/t$

So φ is C^2 and $\varphi'' > 0$ so φ is convex on $\mathbb{R}_{>0}$

Let $\lambda \in [0,1]$, then by convexity: $\forall x, y \in \mathbb{R}_{>0}^2$

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda \varphi(x) + (1-\lambda)\varphi(y)$$

$$(ct)^{\lambda x + (1-\lambda)y} e^{-t}/t \leq \lambda (ct)^x e^{-t}/t + (1-\lambda)(ct)^y e^{-t}/t$$

$$\Rightarrow c^{\lambda x + (1-\lambda)y} \Gamma(\lambda x + (1-\lambda)y) \leq \lambda c^x \Gamma(x) + (1-\lambda) c^y \Gamma(y)$$

by taking $\int_0^{+\infty}$

1.5.b. We divide by $c^{\lambda x + (1-\lambda)y}$

$$\Gamma(\lambda x + (1-\lambda)y) \leq \lambda c^{(1-\lambda)(x-y)} \Gamma(x) + (1-\lambda) c^{\lambda(y-x)} \Gamma(y)$$

Take $c = \left(\frac{\Gamma(y)}{\Gamma(x)}\right)^{\frac{1}{x-y}}$

1.5.c. We apply 1.5.b with $x = x$, $y = x+1$, $\lambda = 1-s$

$$\Gamma(x+s) \leq \Gamma(x)^{1-s} \Gamma(x+1)^s = x^{s-1} \Gamma(x+1) \text{ by 1.2.a}$$

so $x^{1-s} \leq \Gamma(x+1)/\Gamma(x+s)$

Again with "x" $= x+s$, $y = x+s+1$, $\lambda = s$

$$\Gamma(x+1) \leq \Gamma(x+s)^s \Gamma(x+s+1)^{1-s} = (x+s)^{1-s} \Gamma(x+s) \leq (x+1)^{1-s} \Gamma(x+s)$$

So $\Gamma(x+1)/\Gamma(x+s) \leq (x+1)^{1-s}$

2.1. Let $r, s > 0$ then $\begin{array}{l} (0,1) \longrightarrow \mathbb{R} \\ t \mapsto t^{r-1}(1-t)^{s-1} \end{array}$ is C^0 and >0

• at 0 : $0 < t^{r-1}(1-t)^{s-1} \leq C\, t^{r-1}$ on $(0, \frac{1}{2})$ for some C

and $\int_a^{1/2} t^{r-1}\, dt = \frac{1}{2^r r} - \frac{a^r}{r} \xrightarrow[a \to 0^+]{} \frac{1}{2^r r}$

so $\int_0^{1/2} t^{r-1}(1-t)^{s-1}\, dt < +\infty$

• at 1 : $0 < t^{r-1}(1-t)^{s-1} \leq D(1-t)^{s-1}$ on $(1/2, 1)$

and $\int_{1/2}^{b} (1-t)^{s-1}\, dt = -\frac{(1-b)^s}{s} + \frac{1}{2^s s} \xrightarrow[b \to 1^-]{} \frac{1}{2^s s}$

so $\int_{1/2}^{1} t^{r-1}(1-t)^{s-1}\, dt < +\infty$

CCL : $\int_0^1 t^{r-1}(1-t)^{s-1}\, dt < +\infty$

2.2. a $\int_a^b t^{r-1}(1-t)^{s-1}\, dt \underset{u=1-t}{=} -\int_{1-a}^{1-b} (1-u)^{r-1} u^{s-1}\, du = \int_{1-b}^{1-a} (1-u)^{r-1} u^{s-1}\, du$

$\downarrow$ $a \to 0^+$, $b \to 1^-$: $B(r,s)$ $\qquad$ $\downarrow$ $a \to 0^+$, $b \to 1^-$: $B(s,r)$

So $B(r,s) = B(s,r)$

$\int_a^b t^{s-1}(1-t)^{r-1}\, dt \underset{\uparrow}{=} 2\int_{\arcsin a}^{\arcsin b} \cos^{2r-1}\theta \sin^{2s-1}\theta\, d\theta$

$t = \sin^2\theta$
$dt = 2\cos\theta \sin\theta\, d\theta$

$\downarrow$ $a \to 0^+$, $b \to 1^-$: $B(s,r)$ $\qquad$ $\downarrow$ $2\int_0^{\pi/2} \cos^{2r-1}\theta \sin^{2s-1}\theta\, d\theta$

Hence $B(s,r) = 2\int_0^{\pi/2} \cos^{2r-1}\theta \sin^{2s-1}\theta\, d\theta$

2.2.b $B(r,s) = 2\int_0^{\pi/2} \cos^{2r-1}\theta \sin^{2s-1}\theta \, d\theta$ by 2.2.a

$$= \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} \text{ by 1.4.d.}$$

2.3.a. $B\left(\frac{m+1}{2}, \frac{1}{2}\right) = 2\int_0^{\pi/2} \cos^m\theta \sin^0\theta \, d\theta = 2W_m$

Hence $W_m = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{1}{2}\right)$

2.3.b $\sqrt{\frac{\pi}{2m}} / W_m \underset{2.3.a}{=} \sqrt{\frac{\pi}{2m}} / \frac{1}{2} B\left(\frac{m+1}{2}, \frac{1}{2}\right) \underset{\text{by } 2.2.b}{=} \frac{\sqrt{\pi}}{\sqrt{2m}} \cdot 2 \cdot \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)\Gamma(1/2)}$

$\Gamma(1/2) = \sqrt{\pi}$ by 1.4. $\rightarrow = \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)\left(\frac{m}{2}\right)^{1/2}}$

By Gautschi's inequality 1.5.c: $x = m/2$, $s = 1/2$

$$1 \leq \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{1}{2}\right)\left(\frac{m}{2}\right)^{1/2}} \leq \left(1+\frac{2}{m}\right)^{\frac{1}{2}} \xrightarrow[m\to+\infty]{} 1$$

By Squeeze theorem:

$$\lim_{m\to+\infty} \sqrt{\frac{\pi}{2m}} / W_m = 1, \text{ ie } W_m \underset{+\infty}{\sim} \sqrt{\frac{\pi}{2m}}$$

2.3.c. Let $x>0$ then

$$2^{-2x+1}B(1/2,x) = 2^{-2x+1}\int_0^1 t^{-1/2}(1-t)^{x-1}\,dt$$

$$= 4^{-1/2}\cdot(1/4)^{x-1}\int_0^1 t^{-1/2}(1-t)^{x-1}\,dt$$

$$= \int_0^1 (4t)^{-1/2}\left(\frac{1-t}{4}\right)^{x-1}dt$$

$$= \int_0^1 (4t)^{-1/2}\left(\frac{1-\sqrt{t}}{2}\right)^{x-1}\left(\frac{1+\sqrt{t}}{2}\right)^{x-1}dt$$

$$\left.\begin{array}{l} u = \dfrac{1+\sqrt{t}}{2} \\ du = \dfrac{1}{4\sqrt{t}} \\ \sqrt{t} = 2u-1 \end{array}\right\} \;=\; 2\int_{1/2}^1 (1-u)^{x-1}u^{x-1}\,du$$

$$= \int_0^1 (1-u)^{x-1}u^{x-1}\,du$$

$\varphi(u) = (1-u)^{x-1}u^{x-1}$

then $\varphi(1-u) = \varphi(u)$

but $[0,1/2] \xrightarrow{\sim} [1/2,1]$

$u \mapsto 1-u$

$$= B(x,x)$$

2.3.d Let $x>0$ then

$$B(x,x) = \Gamma(x)^2/\Gamma(2x) \text{ by } 2.2\,b$$

2.3.c $\to \|$

$\Gamma(1/2) = \sqrt{\pi}$ by 1.4.

$$2^{-2x+1}B(1/2,x) = \frac{1}{2^{2x-1}}\frac{\Gamma(x)\Gamma(1/2)}{\Gamma(x+1/2)} = \frac{1}{2^{2x-1}}\frac{\Gamma(x)\sqrt{\pi}}{\Gamma(x+1/2)}$$

$\Gamma(x)\neq 0$

$$\Rightarrow \quad \Gamma(x+1/2)\,\Gamma(x) = \frac{\sqrt{\pi}}{2^{x-1}}\,\Gamma(2x)$$

$$2.3.e\quad \Gamma\left(m+\frac{1}{2}\right) \overset{\overset{2.3.d}{\downarrow}}{=} \frac{2}{2^{2m}}\sqrt{\pi}\,\frac{\Gamma(2m)}{\Gamma(m)} \overset{1.2.b}{=} \frac{2}{2^{2m}}\sqrt{\pi}\,\frac{(2m-1)!}{(m-1)!}$$

$$= \frac{\sqrt{\pi}}{2^{2m}}\,\frac{2m}{m}\,\frac{(2m-1)!}{(m-1)!}$$

$$= \frac{\sqrt{\pi}}{2^{2m}}\,\frac{(2m)!}{m!}$$

$$2.3.f.\quad W_{2p} \overset{\overset{2.3.a}{\downarrow}}{=} \frac{1}{2}B\left(p+\frac{1}{2},\frac{1}{2}\right) \overset{\overset{2.2.b}{\downarrow}}{=} \frac{1}{2}\,\frac{\Gamma(p+1/2)\,\Gamma(1/2)}{\Gamma(p+1)}$$

$\Gamma(1/2)=\sqrt{\pi}$, 1.h
$\Gamma(p+1/2)$, 2.3.e
$\Gamma(p+1)=p!$, 1.2.b

$$\rightarrow = \frac{1}{2}\,\frac{\sqrt{\pi}}{2^{2p}}\cdot\frac{(2p)!}{p!}\cdot\frac{\sqrt{\pi}}{p!}$$

$$= \frac{\pi}{2}\,\frac{(2p)!}{(2^p p!)^2}$$

$$W_{2p+1} \overset{\overset{2.3.a}{\downarrow}}{=} \frac{1}{2}B(p+1,1/2) \overset{\overset{2.2b}{\downarrow}}{=} \frac{1}{2}\,\frac{\Gamma(p+1)\,\Gamma(1/2)}{\Gamma\left(p+1+\frac{1}{2}\right)}$$

$$= \frac{1}{2}\cdot\sqrt{\pi}\cdot p!\cdot\frac{2^{2p+2}\,(p+1)!}{\sqrt{\pi}\,(2p+2)!}$$

$$= \frac{(p!)^2\,2^{2p}}{(2p+1)!}\,\frac{2\cancel{(p+1)}}{\cancel{2p+2}}$$

$$= \frac{(2^p p!)^2}{(2p+1)!}$$

2.3.g. $W_{2p} \sim \sqrt{\dfrac{\pi}{4p}} = \dfrac{1}{2}\sqrt{\dfrac{\pi}{p}}$ by 2.3.a.

$$W_{2p} \sim \frac{\pi}{2} \cdot \frac{C\sqrt{2p}\left(\frac{2p}{e}\right)^{2p}}{2^{2p} C^2 p \left(\frac{p}{e}\right)^{2p}} = \frac{\pi}{2} \cdot \frac{1}{C} \cdot \sqrt{\frac{2}{p}}$$

$m! \sim C\sqrt{m}\left(\frac{m}{e}\right)^m$

Hence $\dfrac{1}{2}\sqrt{\pi} = \dfrac{\pi}{2} \cdot \dfrac{1}{C} \cdot \sqrt{2}$

$$\Rightarrow C = \sqrt{2\pi}$$

hence $m! \sim \sqrt{2m\pi}\left(\dfrac{m}{e}\right)^m$

2.3.h. $\dfrac{W_{2p}}{W_{2p+1}} \sim \dfrac{\sqrt{\frac{\pi}{4p}}}{\sqrt{\frac{\pi}{4p+2}}} = \sqrt{\dfrac{4p+2}{4p}} \xrightarrow[p\to+\infty]{} 1$

hence $\lim\limits_{p\to+\infty} \dfrac{W_{2p}}{W_{2p+1}} = 1$ $(*)$

$$\frac{\pi}{2}\frac{W_{2p+1}}{W_{2p}} = \frac{\prod_{k=1}^{p}\frac{2k}{2k+1}}{\prod_{k=1}^{p}\frac{2k-1}{2k}} = \prod_{k=1}^{p}\frac{4k^2}{(2k+1)(2k-1)} = \prod_{k=1}^{p}\frac{4k^2}{4k^2-1}$$

Hence $\displaystyle\prod_{k=1}^{\infty}\frac{4k^2}{4k^2-1} = \lim_{p\to+\infty}\prod_{k=1}^{p}\frac{4k^2}{4k^2-1} = \lim_{p\to+\infty}\frac{\pi}{2}\cdot\frac{W_{2p+1}}{W_{2p}} \overset{(*)}{=} \frac{\pi}{2}$

2.4.a. Let's prove by induction that $V_m(r) = r^m V_m(1)$

$\underline{m=1}$: $V_1(r) = \int_{-r}^{r} dx = 2r$

$V_1(1) = \int_{-1}^{1} dx = 2$ so $V_1(r) = r^1 V_1(1)$

<u>Induction step</u>:

$$V_{m+1}(r) = \int_{x_1^2+\dots+x_{m+1}^2 \leq r^2} 1 = \int_{-r}^{r} \left(\int_{x_1^2+\dots+x_m^2 \leq r^2 - x_{m+1}^2} 1 \right) dx_{m+1}$$

$$= \int_{-r}^{r} V_m\left(\sqrt{r^2 - x^2}\right) dx$$

$$= r \int_{-1}^{1} V_m\left(\sqrt{r^2 - (xr)^2}\right) dx$$

Induction assumption $\rightarrow$ $= r \int_{-1}^{1} r^m V_m\left(\sqrt{1-x^2}\right) dx$

$$= r^{m+1} \int_{-1}^{1} V_m\left(\sqrt{1-x^2}\right) dx = r^{m+1} V_{m+1}(1)$$

2.4.b. $V_{m+1}(1) = \int_{-1}^{1} V_m\left(\sqrt{1-x^2}\right) dx$

$$= \int_{-1}^{1} (1-x^2)^{\frac{m}{2}} V_m(1)\, dx$$

$$= V_m(1) \int_{-1}^{1} (1-x^2)^{\frac{m}{2}} dx$$

$$= 2 V_m(1) \int_{0}^{1} (1-x^2)^{m/2} dx$$

2.4.c. $B\left(\frac{1}{2}, \frac{m}{2}+1\right) = \int_0^1 t^{-\frac{1}{2}} (1-t)^{\frac{m}{2}} \, dt$

$x = \sqrt{t}$

$t = x^2 \quad \rightarrow \quad = 2\int_0^1 (1-x^2)^{\frac{m}{2}} dx$

$dt = 2x dx$

2.4.d. Hence $V_{m+1}(1) = V_m(1) \, B\left(\frac{1}{2}, \frac{m}{2}+1\right)$

$$2.2.b \rightarrow = V_m(1) \frac{\Gamma(1/2)\,\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{3}{2}\right)}$$

$$= V_m(1) \sqrt{\pi} \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}+\frac{3}{2}\right)}$$

$$\text{induction step} \rightarrow = \frac{\pi^{\frac{m}{2}+\frac{1}{2}}}{\Gamma\left(\frac{m}{2}+\frac{3}{2}\right)} = \frac{\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}+1\right)}$$

$$\text{So } V_m(1) = \frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2}+1\right)} = \frac{\pi^{m/2}}{\frac{m}{2}\Gamma\left(\frac{m}{2}\right)} \text{ by 1.2.a.}$$

2.4.e. $$V_m(r) = r^m V_m(1) = \frac{2(\sqrt{\pi}\, r)^m}{m\,\Gamma(m/2)} \text{ or } \frac{2\pi^{\frac{m}{2}}}{m\,\Gamma\left(\frac{m}{2}\right)} r^m$$

2.4.f. $$V_m(r) = \int_0^1 A_m(s)\, ds \Rightarrow V_m'(r) = A_m(r)$$

$$A_m(r) = m\, r^{m-1} V_m(1) = \frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)} r^{m-1}$$