University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus

Darboux's construction of Riemann's integral in one variable

Jean-Baptiste Campesato

January 21st, 2020

Contents

1 Historical comments **2**

2 Darboux's construction of Riemann's integral **3**

3 The ε-criterion for integrability **5**

4 Properties of Riemann's integral **6**

5 Some sufficient conditions for integrability **9**
5.1 Monotonicity . . . 9
5.2 Continuity . . . 10
5.3 Discontinuity set has zero content . . . 10

6 The MVT for Riemann's integral **12**

7 The Fundamental Theorem of Calculus **12**

8 Riemann sums **14**

A Supremum and Infimum **15**

B Uniform continuity **17**

C Riemann's integrability criterion in terms of oscillation (Addendum from Feb 3) **20**

D Lebesgue criterion for Riemann integrability (Addendum from Feb 3, extra-curricular, not part of MAT237) **23**

1 Historical comments

- Several methods allowing us to compute areas by finer approximations were already known in ancient Greece (Eudoxus' method of exhaustion, Archimedes' triangles...).

- During the 17th century Gregory, Barrow, Newton on one side and Leibniz on the other side independently proved the FTC (where the integral was defined as the area under the curve for a continuous function on a segment line).

- Cauchy ("Résumé des Leçons sur le calcul infinitésimal", 1823) gave a first constructive definition of "an integral" in terms of Cauchy sums (which are left-Riemann sums using today's terminology) but he restricted himself to continuous functions.

- Riemann ("La possibilité de représenter une fonction par une série trigonométrique" 1854 but published in 1873) generalized Cauchy's definition and removed the continuous assumption. The question at this time was to determine the largest class of functions for which we can compute integrals. He gave a non-constructible characterization of integrability in terms of "oscillation of a function" (this criterion has been superseded by Lebesgue's criterion for Riemann integrability, in terms of discontinuity set, proved in the below cited paper).
Riemann's paper is very important because it allowed mathematicians to construct examples of continuous functions which are not differentiable. Indeed, if $f : [a, b] \to \mathbb{R}$ is Riemann-integrable then $F(x) = \int_a^x f(t)dt$ is uniformly continuous on $[a, b]$, but if we start with a non-continuous f, we may obtain a non-differentiable F; hence such a F is continuous but not differentiable.
Before that most mathematicians believed that continuity implies differentiability.

- Darboux ("Mémoire sur les fonctions discontinues" 1875) gave a new definition equivalent to Riemann's integral. This is the construction we are going to present in these notes.
In this paper (§IX, pp. 109–110), among other results, Darboux also proved that a derivative may not be continuous (e.g. starting with $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and 0 otherwise) but, nevertheless, the conclusion of the IVT always holds for a derivative (even if it is not continuous!). The latter result is usually called "Darboux's theorem" and a function satisfying the IVT property while not being continuous (e.g. f' for the above f) is usually called a "Darboux function".
He also constructed a function everywhere differentiable but whose derivative is not continuous on the rational numbers: by Darboux's theorem, such a function satisfies the IVT property while being discontinuous on any interval.

- Lebesgue ("Leçons sur l'intégration et la recherche des fonctions primitives" 1904 after a CRAS note in 1901) gave an axiomatic definition of "an" integral operator. This led him to introduce *measure theory* and Lebesgue's integral (which is probably the common point of view of mathematicians nowadays).

- There are other integration theories:

    - The Riemann–Stieltjes integral. It may be seen as a *weighted* version of Riemann's integral. It was introduced by Stieltjes in order to modelize mass distributions on the real line in 1894.

    - The Henstock–Kurzweil integral. It was first defined by Denjoy in 1912, but there are several equivalent constructions (the nowadays common construction is due to Kurweil (1957) and was then developed by Henstock, it is similar to Riemann's integral construction with tagged partitions but it involves the notion of *gauge*). This integral is a little bit more difficult to construct than Riemann's integral, but it admits some powerful results available in Lebesgue's integral (e.g. monotone convergence theorem, dominated convergence theorem). Besides, the HK integral behaves well w.r.t. "improper integrals".

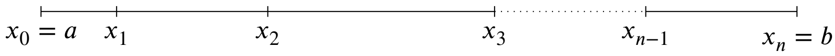
    - ...

2 Darboux's construction of Riemann's integral

Definition 1. A **partition** P of the segment line $[a, b]$ consists in breaking $[a, b]$ into finitely many closed subintervals. We simply describe it by giving the boundaries of the subintervals:

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

Hence P is a finite set of points of $[a, b]$ containing the endpoints a and b.



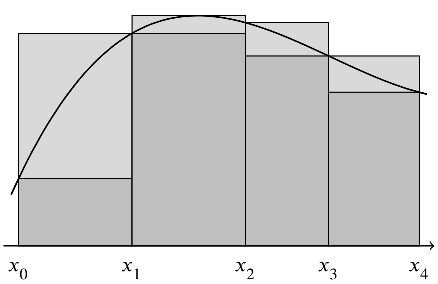
Definition 2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$.
We define the **upper Darboux sum** of f with respect to P by

$$U_P(f) = \sum_{k=1}^{n} \left((x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} f \right)$$

and the **lower Darboux sum** of f with respect to P by

$$L_P(f) = \sum_{k=1}^{n} \left((x_k - x_{k-1}) \inf_{[x_{k-1}, x_k]} f \right)$$

In the following figure, the upper Darboux sum is the area of the light grey and the dark grey rectangles together whereas the lower Darboux sum is the area of the dark grey rectangles only.



Remark 3. Notice that the assumption "f is bounded" ensures that the Darboux sums are well-defined. Indeed, then the infimum and the supremum on the subintervals exist thanks to the LUB and GLB principles.

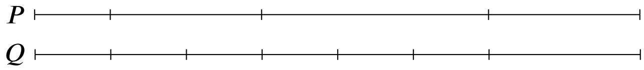
Proposition 4. *For any partition P of $[a, b]$, we have $U_P(f) \geq L_P(f)$.*

Proof.

$$\begin{aligned} U_P(f) &= \sum_{k=1}^{n} \left((x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} f \right) \\ &\geq \sum_{k=1}^{n} \left((x_k - x_{k-1}) \inf_{[x_{k-1}, x_k]} f \right) \quad \text{since } x_k > x_{k-1} \text{ and } \sup f \geq \inf f \\ &= L_P(f) \end{aligned}$$

■

Definition 5. Let P and Q be two partitions of $[a,b]$. We say that Q is **finer** than P if $P \subset Q$.



Proposition 6. *If Q is finer than P then*

$$U_Q(f) \leq U_P(f)$$

and

$$L_Q(f) \geq L_P(f)$$

Proof. By induction, it is enough to see what happens if we break one subinterval into two subintervals. I am just doing it for the upper sum.
Let $c \in (x_{k-1}, x_k)$. Then

$$\begin{aligned}
(x_k - x_{k-1}) \sup_{[x_{k-1},x_k]} f &= (x_k - c + c - x_{k-1}) \sup_{[x_{k-1},x_k]} f \\
&= (c - x_{k-1}) \sup_{[x_{k-1},x_k]} f + (x_k - c) \sup_{[x_{k-1},x_k]} f \\
&\geq (c - x_{k-1}) \sup_{[x_{k-1},c]} f + (x_k - c) \sup_{[c,x_k]} f
\end{aligned}$$

■

Proposition 7. *For any partitions P and Q of $[a,b]$, we have $L_P(f) \leq U_Q(f)$.*

Proof. Indeed, set $R = P \cup Q$ then R is finer than P and finer than Q, so

$$L_P(f) \leq L_R(f) \leq U_R(f) \leq U_Q(f)$$

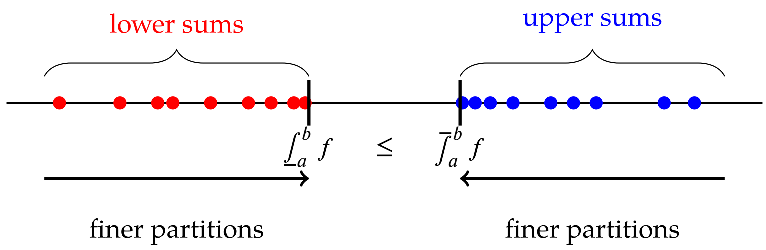
■

Definition 8. Let $f : [a,b] \to \mathbb{R}$ be a bounded function.
We define the **lower (Darboux) integral** of f by

$$\underline{\int_a^b} f = \sup \left\{ L_P(f), \forall P \text{ partition of [a,b]} \right\}$$

and the **upper (Darboux) integral** of f by

$$\overline{\int_a^b} f = \inf \left\{ U_P(f), \forall P \text{ partition of [a,b]} \right\}$$



Definition 9. Let $f : [a,b] \to \mathbb{R}$ be a bounded function.
We say that f is **integrable** on $[a,b]$ if $\underline{\int_a^b} f = \overline{\int_a^b} f$.
Then we denote this quantity by

$$\int_a^b f(x)dx$$

3 The ε-criterion for integrability

Theorem 10. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function.*
Then f is integrable on $[a,b]$ if and only if

$$\forall \varepsilon > 0,\ \exists \text{ a partition } P \text{ of } [a,b],\ U_P(f) - L_P(f) < \varepsilon$$

Remark 11. This criterion is not constructible: it doesn't give the value of the integral!

Proof.
$\Rightarrow$: We know that f is integrable on $[a,b]$, i.e.

(1) $\underline{\int_a^b} f = \overline{\int_a^b} f$

where $\underline{\int_a^b} f = \sup\left\{ L_P(f),\ \forall P \text{ partition of [a,b]} \right\}$ and $\overline{\int_a^b} f = \inf\left\{ U_P(f),\ \forall P \text{ partition of [a,b]} \right\}$.
We want to prove:

$$\forall \varepsilon > 0,\ \exists \text{ a partition } P \text{ of } [a,b],\ U_P(f) - L_P(f) < \varepsilon$$

Let $\varepsilon > 0$.

Then $\overline{\int_a^b} f + \frac{\varepsilon}{2}$ is greater than $\overline{\int_a^b} f$ which is the greatest lower bound of the upper Darboux sums.
Hence $\overline{\int_a^b} f + \frac{\varepsilon}{2}$ is not an lower bound of the upper Darboux sums.
That means that there exists a partition P_1 of $[a,b]$ such that

$$U_{P_1}(f) < \overline{\int_a^b} f + \frac{\varepsilon}{2}$$

Similarly $\underline{\int_a^b} f - \frac{\varepsilon}{2}$ is less than $\underline{\int_a^b} f$ which is the least upper bound of the lower Darboux sums.
Hence $\underline{\int_a^b} f - \frac{\varepsilon}{2}$ is not an upper bound of the lower Darboux sums.
That means that there exists a partition P_2 of $[a,b]$ such that

$$L_{P_2}(f) > \underline{\int_a^b} f - \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$.
Then P is finer than P_1, hence

(2) $U_P(f) \leq U_{P_1}(f) < \overline{\int_a^b} f + \frac{\varepsilon}{2}$

and similarly P is finer than P_2, hence

(3) $L_P(f) \geq L_{P_2}(f) > \underline{\int_a^b} f - \frac{\varepsilon}{2}$

We derive from (2) and (3) that

$$U_P(f) - L_P(f) < \overline{\int_a^b} f + \frac{\varepsilon}{2} - \underline{\int_a^b} f + \frac{\varepsilon}{2}$$

Using (1), we obtain that the RHS of the above inequality is ε.
Therefore we have well obtained a partition P of $[a,b]$ such that

$$U_P(f) - L_P(f) < \varepsilon$$

$\Leftarrow$: We know that

$$\forall \varepsilon > 0, \exists \text{ a partition } P \text{ of } [a,b],\ U_P(f) - L_P(f) < \varepsilon$$

and we want to prove that f is integrable, i.e. that

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

It is enough to prove that

$$\forall \varepsilon > 0,\ 0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon$$

Let $\varepsilon > 0$.
By our assumption, there exists a partition P of $[a,b]$ such that $U_P(f) - L_P(f) < \varepsilon$.
Then, we have

$$L_P(f) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U_P(f)$$

Hence

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq U_P(f) - L_P(f) < \varepsilon$$

We have well obtained

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon$$

■

4 Properties of Riemann's integral

Theorem 12. *Let $f, g : [a,b] \to \mathbb{R}$ be integrable and $c \in \mathbb{R}$. Then*

1. $(f+g) : [a,b] \to \mathbb{R}$ *is integrable too and* $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$
2. $(cf) : [a,b] \to \mathbb{R}$ *is integrable and* $\int_a^b cf(x)dx = c\int_a^b f(x)dx.$
3. $(fg) : [a,b] \to \mathbb{R}$ *is integrable and we have the following Cauchy–Schwarz inequality*

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b f(x)^2 dx \int_a^b g(x)^2 dx$$

4. *If* $\forall x \in [a,b],\ f(x) \leq g(x)$ *then* $\int_a^b f(x)dx \leq \int_a^b g(x)dx.$
5. $|f| : [a,b] \to \mathbb{R}$ *is integrable and* $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$

Remark 13. There is no equality formula for the product! You can have a look at the following example: $f, g : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x > \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Remark 14. The converse of 5 does **not** hold. Indeed, if we define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{otherwise} \end{cases}$$

then $|f|$ is integrable whereas f is not.

Proof of Theorem 12.

1.

$$\begin{aligned}
\int_a^b f + \int_a^b g &= \underline{\int_a^b} f + \underline{\int_a^b} g && \text{since } f \text{ and } g \text{ are integrable} \\
&\leq \underline{\int_a^b} (f+g) && \text{check it using lower Darboux sums and the definition} \\
&\leq \overline{\int_a^b} (f+g) \\
&\leq \overline{\int_a^b} f + \overline{\int_a^b} g && \text{check it using upper Darboux sums and the definition} \\
&= \int_a^b f + \int_a^b g && \text{since } f \text{ and } g \text{ are integrable}
\end{aligned}$$

Hence

$$\overline{\int_a^b} (f+g) = \underline{\int_a^b} (f+g) = \int_a^b f + \int_a^b g$$

2. If $c \geq 0$ then:

$$\begin{aligned}
\underline{\int_a^b} (cf) &= c \underline{\int_a^b} f && \text{check if using Darboux sums} \\
&= c \overline{\int_a^b} f && \text{since } f \text{ is integrable} \\
&= \overline{\int_a^b} (cf) && \text{check it using Darboux sums}
\end{aligned}$$

Hence

$$\overline{\int_a^b} (cf) = \underline{\int_a^b} (cf) = c \int_a^b f$$

Then, in order to get the case $c < 0$, it is enough to study $c = -1$:

$$\begin{aligned}
\underline{\int_a^b} (-f) &= -\overline{\int_a^b} (f) && \text{since } \inf(-f) = -\sup(f) \\
&= -\underline{\int_a^b} (f) && \text{since } f \text{ is integrable} \\
&= \overline{\int_a^b} (-f) && \text{since } \sup(-f) = -\inf(f)
\end{aligned}$$

Hence

$$\underline{\int_a^b} (-f) = \overline{\int_a^b} (-f) = -\int_a^b f$$

3. We are first going to prove that if f is integrable then so is f^2.
Since f is bounded, there exists $M > 0$ such that $\forall x \in [a,b]$, $|f(x)| \leq M$. Then
$$\begin{aligned} \left|f(x)^2 - f(y)^2\right| &= |f(x)+f(y)|\ |f(x)-f(y)| \\ &\leq (|f(x)|+|f(y)|)\,|f(x)-f(y)| \quad \text{by the Triangle Inequality} \\ &\leq 2M\ |f(x)-f(y)| \end{aligned}$$
Hence, for a segment line I,
$$\sup_I \left(f^2\right) - \inf_I \left(f^2\right) \leq 2M \left(\sup_I(f) - \inf_I(f) \right)$$
and hence for a partition P,
$$U_P\left(f^2\right) - L_P\left(f^2\right) \leq 2M \left(U_P(f) - L_P(f)\right)$$
Then we may conclude that f^2 is integrable using the ε-criterion (Theorem 10).

Next, $fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$ is integrable from the previous points.

The Cauchy–Schwarz inequality can be proved using the proof of the usual Cauchy–Schwarz inequality seen on September 5: study the discriminant of the quadratic polynomial $\theta(t) = \int_a^b (f(x)+tg(x))^2 dx$.

4. $\int_a^b g - \int_a^b f = \int_a^b (g-f) = \overline{\int_a^b} (g-f) \geq 0$
For the last inequality, use that $\forall x \in [a,b]$, $g(x) - f(x) \geq 0$ together with the definition involving the upper Darboux sums.

5. From the reverse triangle inequality we have
$$\begin{aligned} &\forall x,y \in [a,b],\ |f(x)| - |f(y)| \leq |f(x) - f(y)| \\ \implies &\sup_I(|f|) - \inf_I(|f|) \leq \sup_I(f) - \inf_I(f) \quad \text{for a segment line } I \\ \implies &U_P(|f|) - L_P(|f|) \leq U_P(f) - L_P(f) \quad \text{for a partition } P \end{aligned}$$
Then we may conclude that $|f|$ is integrable using the ε-criterion (Theorem 10).
For the remaining inequality, notice that
$$|f| \geq f \implies \int_a^b |f| \geq \left(\int_a^b f\right) \quad \text{and} \quad |f| \geq -f \implies \int_a^b |f| \geq -\left(\int_a^b f\right)$$
■

Exercise 15. *Prove that if $f : [a,b] \to \mathbb{R}$ is integrable then $f_+, f_- : [a,b] \to \mathbb{R}$ defined by $f_+ = \sup(f,0)$ and $f_- = -\inf(f,0)$ are integrable too.*

Theorem 16. *Let $f : [a,b] \to \mathbb{R}$ be integrable and $[c,d] \subset [a,b]$. Then f is integrable on $[c,d]$.*

Theorem 17 (Chasles' relation)**.** *Let $f : [a,b] \to \mathbb{R}$ and $c \in (a,b)$. If f is integrable on $[a,c]$ and on $[c,b]$ then f is integrable on $[a,b]$ and*
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
Therefore it is natural to introduce the following notation:

Notation 18. If $f : [a,b] \to \mathbb{R}$ is integrable then we set
$$\int_b^a f(x)dx := -\int_a^b f(x)dx$$

5 Some sufficient conditions for integrability

5.1 Monotonicity

Theorem 19. *If* $f : [a, b] \to \mathbb{R}$ *is non-decreasing or non-increasing then* f *is integrable on* $[a, b]$.

Remark 20. Notice that we don't assume that f is continuous, only that f is monotonic! Notice also that such a function is necessarily bounded!

Proof. Let's assume that f is non-decreasing (then, for the other case, replace f by $-f$).
First, notice that f is bounded.
Indeed, for any $x \in [a, b]$ we have $a \leq x \leq b$ and hence, since f is non-decreasing, we have

$$f(a) \leq f(x) \leq f(b)$$

Hence, f is bounded from above by $f(b)$ and from below by $f(a)$.

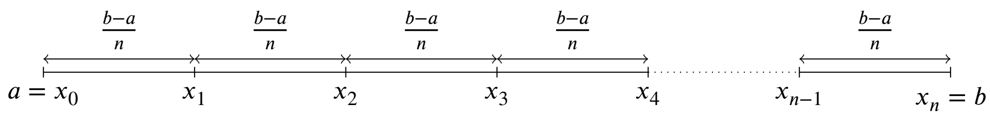
Then, according to the ε-criterion (Theorem 10), it is enough to prove that

$$\forall \varepsilon > 0, \exists \text{ a partition } P \text{ of } [a, b], U_P(f) - L_P(f) < \varepsilon$$

Let $\varepsilon > 0$.
Set $n = \left\lfloor \frac{(f(b)-f(a))(b-a)}{\varepsilon} \right\rfloor + 1$. Then

$$\frac{(f(b) - f(a))(b - a)}{n} < \varepsilon \tag{4}$$

Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be the partition of $[a, b]$ consisting in n subintervals of the same length, i.e. $x_k = a + k\frac{b-a}{n}$.



Since f is non-decreasing, we easily check (*do it!*) that

$$\sup_{[x_{k-1}, x_k]} f = f(x_k) \qquad \text{and} \qquad \inf_{[x_{k-1}, x_k]} f = f(x_{k-1})$$

Then

$$U_P(f) = \sum_{k=1}^{n} \left((x_k - x_{k-1}) \sup_{[x_{k-1}, x_k]} f \right) = \sum_{k=1}^{n} \left(\frac{b-a}{n} f(x_k) \right) = \frac{b-a}{n} \sum_{k=1}^{n} f(x_k)$$

and

$$L_P(f) = \sum_{k=1}^{n} \left((x_k - x_{k-1}) \inf_{[x_{k-1}, x_k]} f \right) = \sum_{k=1}^{n} \left(\frac{b-a}{n} f(x_{k-1}) \right) = \frac{b-a}{n} \sum_{k=1}^{n} f(x_{k-1})$$

Therefore

$$\begin{aligned} U_P(f) - L_P(f) &= \frac{b-a}{n} \sum_{k=1}^{n} \big(f(x_k) - f(x_{k-1}) \big) \\ &= \frac{b-a}{n} \big(f(x_1) - f(x_0) + f(x_2) - f(x_1) + f(x_3) - f(x_2) + \cdots + f(x_n) - f(x_{n-1}) \big) \\ &= \frac{b-a}{n} (f(x_n) - f(x_0)) = \frac{b-a}{n} (f(b) - f(a)) \end{aligned}$$

We deduce from (4) that

$$U_P(f) - L_P(f) < \varepsilon$$

which is what we wanted to prove! ■

5.2 Continuity

Theorem 21. *If $f : [a,b] \to \mathbb{R}$ is continuous then f is integrable on $[a,b]$.*

Remark 22. Notice that such a function is necessarily bounded.

Proof. First, notice that f is bounded since its image is compact as the continuous image of a compact subset.

We are going to prove that f is integrable using the ε-criterion (Theorem 10).

Let's $\varepsilon > 0$.
Since f is continuous on a segment line, it is uniformly continuous (Theorem 66).
So there exists $\delta > 0$ such that

$$\forall x_1, x_2 \in [a,b],\ |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \frac{\varepsilon}{2(b-a)}$$

Set $n = \left\lceil \frac{2(b-a)}{\delta} \right\rceil$ and define a partition $P = \{a = x_0 < \cdots < x_n = b\}$ by $x_k = a + k\frac{b-a}{n}$ so that the lengths of the intervals are less than δ. Then

$$\begin{aligned} U_P(f) - L_P(f) &= \sum_{k=1}^{n}(x_k - x_{k-1})\left(\sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f\right) \\ &\leq \sum_{k=1}^{n}(x_k - x_{k-1})\frac{\varepsilon}{2(b-a)} \\ &= \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

■

5.3 Discontinuity set has zero content

Definition 23. A set $S \subset \mathbb{R}$ has *zero content* if for every $\varepsilon > 0$ there exists finitely many segment lines $[a_1,b_1],[a_2,b_2],\dots,[a_r,b_r]$ such that

(i) $S \subset \displaystyle\bigcup_{i=1}^{r}[a_i,b_i]$

(ii) $\displaystyle\sum_{i=1}^{r}(b_i - a_i) < \varepsilon$

Proposition 24.

1. *If S has zero content then S is bounded.*
2. $\left.\begin{array}{l} S \text{ has zero content} \\ \tilde{S} \subset S \end{array}\right\} \implies \tilde{S}$ *has zero content*
3. *S has zero content if and only if its closure $\overline{S}$ has zero content.*
4. *If $S_1,\dots,S_r$ have zero content then $S = \displaystyle\bigcup_{i=1}^{r} S_i$ has zero content.*
5. *A finite set has zero content.*

Proof. Exercise! ■

Exercise 25. *Prove that $\left\{\frac{1}{n} : n \in \mathbb{N}_{>0}\right\}$ has zero content (even if it is not finite).*

Exercise 26. *Prove that* $[0,1]$ *doesn't have zero content.*

Theorem 27. *Let* $f : [a,b] \to \mathbb{R}$ *be a bounded a function.*
If the set $\{x \in [a,b] : f \text{ is not continuous at } x\}$ *has zero content then* f *is integrable.*

Proof. We are going to use the ε-criterion (Theorem 10).
Notice that if f is constant then there is nothing to prove, so we may assume that f is not constant.
Let $\varepsilon > 0$.
By assumption, we may find a partition $\left\{a = x_0 < \dots < x_n = b\right\}$ of $[a,b]$ such that either $[x_k, x_{k+1}]$ has no discontinuity point or

$$\sum_{k \text{ s.t. } [x_k,x_{k+1}] \text{ has a discontinuity}} (x_{k+1} - x_k) < \frac{\varepsilon}{2\left(\sup_{[a,b]} f - \inf_{[a,b]} f\right)}$$

Notice that the denominator is not zero since we assumed that f is not constant.
Hence

$$\sum_{k \text{ s.t. } [x_k,x_{k+1}] \text{ has a discontinuity}} (x_{k+1} - x_k)\left(\sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f\right) < \frac{\varepsilon}{2}$$

Following the proof of Theorem 21, we may refine the partition by breaking the intervals $[x_k, x_{k+1}]$ with no discontinuity point in order to obtain

$$\sum_{k \text{ s.t. } [x_k,x_{k+1}] \text{ has no discontinuity}} (x_{k+1} - x_k)\left(\sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f\right) < \frac{\varepsilon}{2}$$

Then, if we denote $P = \left\{a = x_0 < \dots < x_n = b\right\}$ the partition obtained, $U_P - L_P < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
Hence f is integrable. ∎

Exercise 28 (Thomae's function)**.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q},\ p \in \mathbb{Z} \setminus \{0\},\ q \in \mathbb{N}_{>0},\ \gcd(p,q) = 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

1. *Prove that* f *is discontinuous at all the rational points but is continuous at all irrational numbers.*
2. *Prove that* f *is integrable on* $[0,1]$ *using the definition of integrability.*
3. *Prove that* $[0,1] \cap \mathbb{Q}$ *doesn't have zero content (hint: compute* $\overline{[0,1] \cap \mathbb{Q}}$*).*
4. *Conclude that the converse of Theorem 27 is false.*

Remark 29 (Lebesgue's criterion for Riemann integrability, **extra-curricular**)**.**
If we replace "finitely many" by "countably many" in Definition 79, we obtain the notion of "set of measure zero". Then we have the following characterization (**not part of MAT237**).
Let $f : [a,b] \to \mathbb{R}$ *be a bounded function.*
Then f *is integrable if and only if* $\{x \in [a,b] : f \text{ is not continuous at } x\}$ *is of measure* 0.
In this case, the converse is also true!

Exercise 30 (Dirichlet's function)**.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

1. *Prove that* f *is nowhere continuous.*
2. *Prove that* f *is not integrable on* $[0,1]$ *using the definition of integrability.*

6 The MVT for Riemann's integral

Theorem 31. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a,b]$ such that*

$$\int_a^b f(x)dx = (b-a)f(c)$$

Proof. Since $f : [a,b] \to \mathbb{R}$ is continuous on a segment line, it has a min $m = f(s)$ and a max $M = f(S)$. It is also integrable as a continuous function, hence we deduce from

$$\forall x \in [a,b],\ m \leq f(x) \leq M$$

that (by taking the integral between a and b)

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

and therefore that

$$f(s) = m \leq \frac{\int_a^b f(x)dx}{b-a} \leq M = f(S)$$

Since f is continuous, by the IVT, there exists $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

■

7 The Fundamental Theorem of Calculus

Theorem 32 (FTC – Part 1)**.** *Let $f : I \to \mathbb{R}$ be a continuous function defined on an interval I and $a \in I$. Define $F : I \to \mathbb{R}$ by $F(x) = \int_a^x f(t)dt$. Then F is differentiable and $F' = f$.*

Proof. Let $x_0, x \in I$ with $x \neq x_0$, then

$$F(x) - F(x_0) = \int_a^x f(t)dt - \int_a^{x_0} f(t)dt = \int_{x_0}^x f(t)dt$$

Then, by Theorem 31, there exists $\xi \in [x, x_0]$ if $x_0 > x$ or $\xi \in [x_0, x]$ otherwise such that

$$F(x) - F(x_0) = (x - x_0)f(c)$$

i.e.

$$\frac{F(x) - F(x_0)}{x - x_0} = f(c)$$

Notice that c tends to x_0 when x tends to x_0, hence, by continuity of f,

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$$

Therefore F is differentiable at x_0 and $F'(x_0) = f(x_0)$. ■

Definition 33. Let I be an interval and $f : I \to \mathbb{R}$ be a function. We say that $F : I \to \mathbb{R}$ is an *antiderivative* (or a *primitive*) of f if F is differentiable and $F' = f$.

Remark 34. Hence, the FTC Part 1 (Theorem 32) states that function which is continuous on an interval has an antiderivative.

Corollary 35. *Let $f : I \to \mathbb{R}$ be a continuous function defined on* ***an interval*** *I and let $a \in I$. If $F : I \to \mathbb{R}$ is an antideriative of f then there exists $C \in \mathbb{R}$ such that*

$$\forall x \in I,\ F(x) = \int_a^x f(t)dt + C$$

Proof. Define $G : I \to \mathbb{R}$ by $G(x) = F(x) - \int_a^x f(t)dt$.
Then, by the FTC Part 1 (Theorem 32) G is differentiable on I and $G' = f - f = 0$.
Hence, by the MVT, G is constant on I (since it is an interval!), i.e. $\exists C \in \mathbb{R}, \forall x \in I, G(x) = C$. Otherwise stated

$$\forall x \in I,\ F(x) = \int_a^c f(t)dt + C$$

∎

Remark 36. Particularly, **on an interval**, two antiderivatives differ by a constant.

Remark 37. The assumption that the domain is an interval is very important in the above result: otherwise we may find two antiderivatives which don't differ by a constant.
For instance define $F_1, F_2 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$F_1(x) = \ln(|x|) \qquad \text{and} \qquad F_2(x) = \begin{cases} \ln(|x|) + 42 & \text{if } x > 0 \\ \ln(|x|) - \pi & \text{if } x < 0 \end{cases}$$

then they are both antiderivatives of $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ but $F_1 - F_2$ is not constant!

Theorem 38 (FTC – Part 2)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $F : [a, b] \to \mathbb{R}$ be an antiderivative of f. Then*

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof. By Corollary 35, there exists $C \in \mathbb{R}$ such that $\forall x \in [a, b],\ F(x) = \int_a^x f(t)dt + C$.

Then $F(b) - F(a) = \int_a^b f(t)dt + C - \int_a^a f(t)dt - C = \int_a^b f(t)dt.$ ∎

Actually, the continuity assumption is superfluous in the above theorem. We can weaken the result by replacing the *continuity* assumption by *integrability*.

Theorem 39. *Assume that $F : [a, b] \to \mathbb{R}$ is an antiderivative of $f : [a, b] \to \mathbb{R}$.*
If f is integrable then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof. Let $\varepsilon > 0$. Since f is integrable, by the ε-criterion (Theorem 10), there exists a partition $P = \{a = x_0 < \cdots < x_n = b\}$ such that $U_P(f) - L_P(f) < \varepsilon$.
Next, notice that,

$$\begin{aligned} F(b) - F(a) &= \sum_{k=1}^n F(x_k) - F(x_{k-1}) \\ &= \sum_{k=1}^n (x_k - x_{k-1}) F'(c_k) \text{ for some } c_k \in [x_{k-1}, x_k] \text{ by the MVT} \end{aligned}$$

Since $\inf_{[x_{k-1},x_k]} f \leq F'(c_k) \leq \sup_{[x_{k-1},x_k]} f$, we get that

$$\sum_{k=1}^n (x_k - x_{k-1}) \inf_{[x_{k-1},x_k]} f \leq F(b) - F(a) \leq \sum_{k=1}^n (x_k - x_{k-1}) \sup_{[x_{k-1},x_k]} f$$

i.e.

$$L_P(f) \leq F(b) - F(a) \leq U_P(f)$$

But we also know that

$$L_P(f) \leq \int_a^b f(x)dx \leq U_P(f)$$

Hence

$$\left| F(b) - F(a) - \int_a^b f(x)dx \right| \leq U_P(f) - L_P(f) < \varepsilon$$

Hence we proved that

$$\forall \varepsilon > 0, \left| F(b) - F(a) - \int_a^b f(x)dx \right| < \varepsilon$$

■

Remark 40. The integrability assumption is necessary. It is not enough to have an antiderivative to be integrable!!!
Indeed, define for instance $F : [0,1] \to \mathbb{R}$ by

$$F(x) = \begin{cases} x^2 \sin\left(\pi/x^2\right) & \text{if x}\neq 0 \\ 0 & \text{otherwise} \end{cases}$$

then F is differentiable but $f = F'$ is not integrable (whereas it has obviously an antiderivative).

8 Riemann sums

Definition 41. Let $f : [a,b] \to \mathbb{R}$ be a bounded function.
Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$.
For any $k = 1, 2, \ldots, n$, pick a point $x_k^* \in [x_{k-1}, x_k]$ (*then we say that P is a tagged partition*).
Then the following sum

$$S_P^*(f) = \sum_{k=1}^{n} \left((x_k - x_{k-1})f(x_k^*)\right)$$

is called *a* **Riemann sum** of f with respect to P.

Remark 42. Quite often (but not always!), we pick an endpoint of the subinterval $[x_{k-1}, x_k]$:

1. If for all k, we fix $x_k^* = x_k$, then we talk about the right Riemann sum of f with respect to P.
2. If for all k, we fix $x_k^* = x_{k-1}$, then we talk about the left Riemann sum of f with respect to P.

Definition 43. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$.
The **norm** of P is the length of the longest subinterval of P:

$$\|P\| = \max\left\{x_1 - x_0,\, x_2 - x_1,\, \ldots, x_n - x_{n-1}\right\}$$

Theorem 44. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function.*
Let $S_{P_1}^*(f), S_{P_2}^*(f), \ldots, S_{P_n}^*(f), \ldots$ be a sequence of Riemann sums such that $\lim_{n\to+\infty} \|P_n\| = 0$.
If f is integrable on $[a,b]$ then

$$\lim_{n\to+\infty} S_{P_n}^*(f) = \int_a^b f(x)dx$$

Remark 45. Quite often (but not always!), we define P_n as the partition breaking $[a,b]$ into n closed subintervals of the same length.

A Supremum and Infimum

Definition 46. Let $A \subseteq \mathbb{R}$ and $U \in \mathbb{R}$. We say that U is an **upper bound** of A if

$$\forall x \in A,\ x \leq U$$

Definition 47. Let $A \subseteq \mathbb{R}$ and $L \in \mathbb{R}$. We say that L is a **lower bound** of A if

$$\forall x \in A,\ L \leq x$$

Definition 48. We say that a subset $A \subseteq \mathbb{R}$ is **bounded from above** if it admits an upper bound.

Definition 49. We say that a subset $A \subseteq \mathbb{R}$ is **bounded from below** if it admits a lower bound.

Definition 50. Let $A \subseteq \mathbb{R}$ and $S \in \mathbb{R}$.
We say that S is the **supremum** (or **least upper bound**) of A if
1. S is an upper bound of A, and,
2. for all upper bounds T of A, $S \leq T$.

Then we use the notation $S = \sup(A)$.

Definition 51. Let $A \subseteq \mathbb{R}$ and $I \in \mathbb{R}$.
We say that I is the **infimum** (or **greatest lower bound**) of A if
1. I is a lower bound of A, and,
2. for all lower bounds J of A, $J \leq I$.

Then we use the notation $I = \inf(A)$.

Remark 52. Notice that we talk about **the** supremum of a set but about **an** upper bound of a set.
Indeed, if a set admits a supremum then it is unique (prove it!).
Beware, it is possible for a set to not have a supremum.

As already discussed (see the slides from September 24), the real line $\mathbb{R}$ satisfies the following very fundamental property: we say that $\mathbb{R}$ is Dedekind-complete.

Theorem 53 (The least upper bound property)**.**
If a non-empty subset of $\mathbb{R}$ is bounded from above then it admits a least upper bound (supremum).

The following result is a direct corollary of the previous theorem.

Theorem 54 (The greatest lower bound property)**.**
If a non-empty subset of $\mathbb{R}$ is bounded from below then it admits a greatest lower bound (infimum).

Remark 55. The "non-empty" assumption is essential here!

The following characterizations may be very useful when writing proofs!

Proposition 56. *Let $A \subseteq \mathbb{R}$ and $S \in \mathbb{R}$. Then*

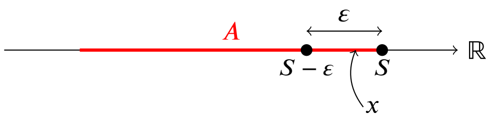
$$S = \sup(A) \Leftrightarrow \begin{cases} \forall x \in A,\ x \leq S \\ \forall \varepsilon > 0,\ \exists x \in A,\ S - \varepsilon < x \end{cases}$$

Proposition 57. *Let $A \subseteq \mathbb{R}$ and $I \in \mathbb{R}$. Then*

$$I = \inf(A) \Leftrightarrow \begin{cases} \forall x \in A,\ I \leq x \\ \forall \varepsilon > 0,\ \exists x \in A,\ x < I + \varepsilon \end{cases}$$

We will only focus on the characterization of the supremum (that's similar for the infimum).

Notice that the first line simply means that S is an upper bound.
Then the second line of the characterization means that S is the smallest one!
Indeed, for any $\varepsilon > 0$, even a very very very small one, $S - \varepsilon < S$. So the fact that S is the least upper bound means exactly that $S - \varepsilon$ isn't an upper bound, or, equivalently, that there is at least one $x \in A$ such that $S - \varepsilon < x$.



Beware, for simplicity I represented A as an interval in the above figure, but A may not be an interval!

Proof of proposition 11. Let $A \subseteq \mathbb{R}$ and $S \in \mathbb{R}$.

1. Proof of $\Rightarrow$.
   Assume that $S = \sup(A)$.

   Then S is an upper bound of A so $\forall x \in A,\ x \leq S$.

   We know that if T is an upper bound of A then $S \leq T$.
   So, by taking the contrapositive, if $T < S$ then T isn't an upper bound of A. We are going to use this fact to prove the second part of the characterization.
   Let $\varepsilon > 0$. Since $S - \varepsilon < S$, we know that $S - \varepsilon$ is not an upper bound of A, meaning that there exists $x \in A$ such that $S - \varepsilon < x$.

2. Proof of $\Leftarrow$.
   We assume that
   $$\begin{cases} \forall x \in A,\ x \leq S \\ \forall \varepsilon > 0,\ \exists x \in A,\ S - \varepsilon < x \end{cases}$$
   The first part of the characterization ensures that S is an upper bound of A.

   We still have to prove that if T is an upper bound of A then $S \leq T$.
   We will show the contrapositive: if $T < S$ then T isn't an upper bound.
   Let $T \in \mathbb{R}$. Assume that $T < S$. Let $\varepsilon = S - T > 0$. Then there exists $x \in A$ such that $S - \varepsilon < x$, i.e. $T < x$.
   Hence T isn't an upper bound.

■

B Uniform continuity

We first recall the definition of continuity:

Definition 58. Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$.
We say that f is *continuous at* x_0 if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$$

Definition 59. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$.
We say that f is *continuous* if it is continuous everywhere, i.e.

$$\forall x_0 \in I, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in I, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$$

Definition 60. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$.
We say that f is *uniformly continuous* if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in I, |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon$$

Let's compare carefully these two definitions:
f is continuous when

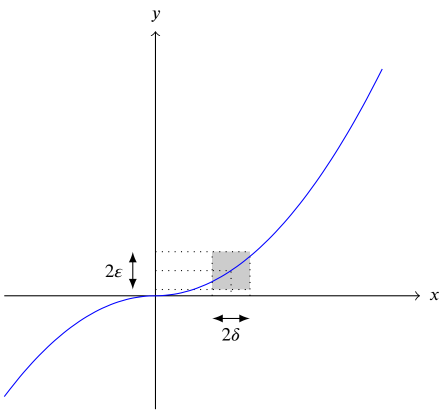
$$\forall \varepsilon > 0, \forall x_1 \in I, \exists \delta > 0, \forall x_2 \in I, |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon$$

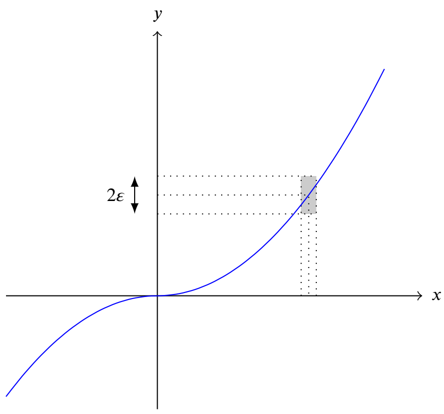
f is uniformly continuous when

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x_1 \in I, \forall x_2 \in I, |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon$$

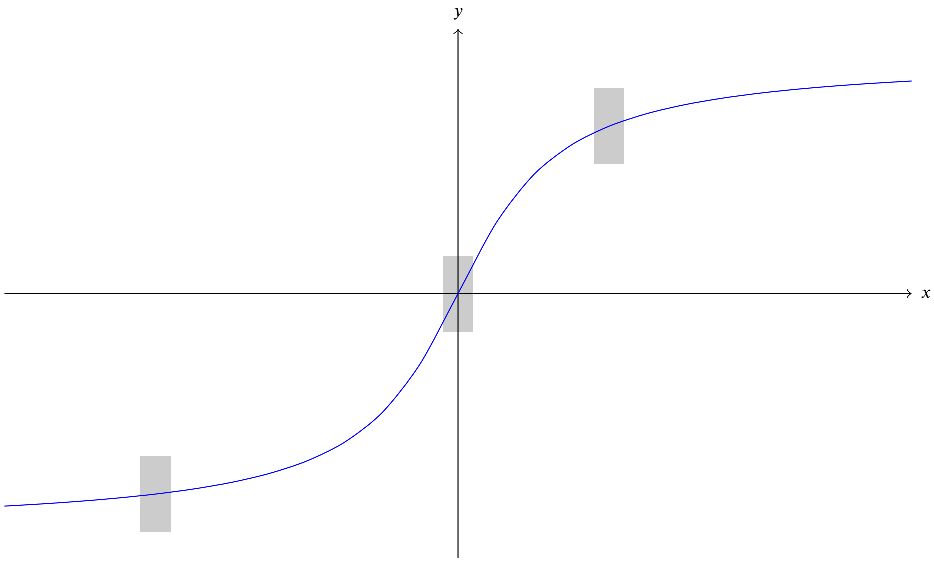
Hence the only difference is that we permuted two quantifiers: the universal quantifier for x_1 and the existential quantifier for δ.
Therefore δ may depend on x_1 for continuity whereas for uniform continuity we need to find a δ suitable for any x_1 in the domain.

Example 61. The following function is continuous but not uniformly continuous: for a given ε, the more we look to the right the smaller δ should be for the graph to not leave the square from the top or the bottom.





Example 62. The following function is uniformly continuous.



Remark 63. Notice that continuity is a *local* notion whereas uniform continuity is a *global* notion.

The following proposition is obvious:

Proposition 64. *A uniformly continuous function is continuous.*

Notice that the converse is false.

Example 65. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous but not uniformly continuous.

However we have the following result for a continuous function defined on a **segment line**.

Theorem 66 (Heine–Cantor theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous then it is uniformly continuous.*

Proof. We are going to prove the contrapositive: if f is not uniformly continuous then f is not continuous.

Let's assume that f is not uniformly continuous. Then there exists $\varepsilon > 0$ such that

$$\forall \delta > 0,\ \exists x_1, x_2 \in [a, b],\ |x_1 - x_2| < \delta \text{ and } |f(x_1) - f(x_2)| \geq \varepsilon$$

Hence, for any $n \in \mathbb{N}_{>0}$ there exists $x_{1,n}, x_{2,n} \in [a, b]$ such that

$$|x_{1,n} - x_{2,n}| < \frac{1}{n} \text{ and } |f(x_{1,n}) - f(x_{2,n})| \geq \varepsilon$$

Since the sequence $(x_{1,n})$ lies in the compact set $[a, b]$, it admits a subsequence $(x_{1,\varphi(n)})$ convergent to $\ell \in [a, b]$.
Notice that

$$\begin{aligned} |x_{2,\varphi(n)} - \ell| &= |(x_{2,\varphi(n)} - x_{1,\varphi(n)}) + (x_{1,\varphi(n)} - \ell)| \\ &\leq |x_{2,\varphi(n)} - x_{1,\varphi(n)}| + |x_{1,\varphi(n)} - \ell| \\ &\leq \frac{1}{\varphi(n)} + |x_{1,\varphi(n)} - \ell| \\ &\leq \frac{1}{n} + |x_{1,\varphi(n)} - \ell| \xrightarrow[n \to +\infty]{} 0 \end{aligned}$$

Hence $(x_{2,\varphi(n)})$ is also convergent to ℓ.

Assume by contradiction that f is continuous at $\ell \in [a,b]$ then from

$$\forall n,\ \left|f\left(x_{1,\varphi(n)}\right) - f\left(x_{2,\varphi(n)}\right)\right| \geq \varepsilon$$

we derive by taking the limit that

$$0 = |f(\ell) - f(\ell)| \geq \varepsilon > 0$$

which is impossible.
So f is not continuous at ℓ. ■

The following exercises are useful to check whether a function is uniformly continuous or not!

Exercise 67. *Let $f : I \to \mathbb{R}$ be a function defined on an interval I.*
Prove that if f is Lipschitz then f is uniformly continuous.

Exercise 68. *Let $f : I \to \mathbb{R}$ be a differentiable function defined on an interval I.*
Prove that if f' is bounded then f is uniformly continuous.

Exercise 69. *Let $f : I \to \mathbb{R}$ where $I = (a,b)$ with $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$.*

1. *Prove that if f is uniformly continuous then $\lim\limits_{x\to a^+} f(x)$ exists (and is finite!).*
2. *Prove that if $\lim\limits_{x\to a^+} f(x)$ DNE then f is not uniformly continuous.*

Exercise 70. *Let $f : [0,+\infty) \to \mathbb{R}$.*

1. *Prove that if f is uniformly continuous then $\exists a,b \in \mathbb{R}, \forall x \in [0,+\infty),\ f(x) \leq ax+b$.*

   Remark: the above question remains true if the domain is $(-\infty,0]$ but not if the domain is the entire real line $\mathbb{R}$, for instance $f(x) = |x|$ is uniformly continuous but not upper bounded by an affine function.

2. *Prove that if $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = +\infty$ then f is not uniformly continuous.*
3. *Prove that if $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = -\infty$ then f is not uniformly continuous.*

Exercise 71. *Let $f : [a,+\infty) \to \mathbb{R}$.*
Prove that if f is continuous and $\lim\limits_{x\to+\infty} f(x) = \ell \in \mathbb{R}$ then f is uniformly continuous.

Exercise 72.

1. *Prove that $x^2 : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous.*
2. *Prove that $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is not uniformly continuous.*
3. *Prove that $\frac{1}{x} : (0,+\infty) \to \mathbb{R}$ is not uniformly continuous.*
4. *Prove that $\exp : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous.*
5. *Prove that $\exp : \left[-\pi, \sqrt{42}\right] \to \mathbb{R}$ is uniformly continuous.*
6. *Prove that $\sqrt{\cdot} : [0,+\infty) \to \mathbb{R}$ is uniformly continuous.*
7. *Prove that $\sqrt[3]{\cdot} : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.*
8. *Prove that $\sin : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.*
9. *Prove that $\sin(1/x) : (0,1) \to \mathbb{R}$ is not uniformly continuous.*
10. *Prove that $\sin(x^2) : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous (hint: $\lim\limits_{n\to+\infty}\left(\sqrt{n\pi + \frac{\pi}{2}} - \sqrt{n\pi}\right) = 0$).*

C Riemann's integrability criterion in terms of oscillation (Addendum from Feb 3)

In this section we introduce the original integrability criterion due to Riemann, which is now superseded by the Lebesgue criterion (i.e. *a function defined on a segment line is Riemann integrable if and only if its discontinuity set has measure 0*).

Definition 73. Let $f : A \to \mathbb{R}$ be a bounded function defined on a subset $A \subset \mathbb{R}$ and $a \in A$.
The *oscillation of f at a* is defined by

$$o(f,a) = \lim_{\delta\to 0^+}\left(\sup_{A\cap(a-\delta,a+\delta)} f - \inf_{A\cap(a-\delta,a+\delta)} f\right)$$

Remark 74. Notice that $o(f,a)$ is always well-defined (as soon as f is bounded):

- For $\delta > 0$, $a \in A \cap (a-\delta, a+\delta)$, hence $f(a) \in \{f(x) : x \in A \cap (a-\delta, a+\delta)\}$ and the latter set is not empty. Moreover it is bounded by assumption. Hence the supremum and the infimum are well-defined.
- And $g(\delta) = \left(\sup_{A\cap(a-\delta,a+\delta)} f - \inf_{A\cap(a-\delta,a+\delta)} f\right)$ decreases when δ decreases on $(0,+\infty)$ and is bounded from below by 0. Hence $\lim_{\delta\to 0^+} g(\delta)$ is well defined by the monotone convergence theorem.

Proposition 75. *Let $f : A \to \mathbb{R}$ be a bounded function defined on a subset $A \subset \mathbb{R}$ and $a \in A$. Then f is continuous at a if and only if $o(f,a) = 0$.*

Proof.
$\Rightarrow$: assume that f is continuous at a.
Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for all $x \in A$, if $|x-a| < \delta$ then $|f(x) - f(a)| < \frac{\varepsilon}{2}$.
Hence $o(f,a) \le \sup_{A\cap(a-\delta,a+\delta)} f - \inf_{A\cap(a-\delta,a+\delta)} f \le \varepsilon$ (for the first inequality, use the monotonicity of g).
Therefore we proved that $\forall \varepsilon > 0$, $o(f,a) \le \varepsilon$. So $o(f,a) = 0$.

$\Leftarrow$: assume that $o(f,a) = 0$.
Let $\varepsilon > 0$.
Then there exists $\delta > 0$ such that $\sup_{A\cap(a-\delta,a+\delta)} f - \inf_{A\cap(a-\delta,a+\delta)} f < \varepsilon$ (definition of $\lim_{\delta\to 0^+} g(\delta) = 0$).
If $x \in (a-\delta, a+\delta) \cap A$ then $|f(x) - f(a)| \le \sup_{A\cap(a-\delta,a+\delta)} f - \inf_{A\cap(a-\delta,a+\delta)} f < \varepsilon$.
Hence

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in A,\ |x-a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

i.e. f is continuous at a. ■

Remark 76. When $o(f,a) > 0$ then this number quantify "how far" is f to be continuous at a.

The following proposition will be useful to prove Lebesgue's criterion (Theorem 80).

Proposition 77. *Assume that $A \subset \mathbb{R}$ is closed and that $f : A \to \mathbb{R}$ is bounded. Let $\varepsilon > 0$. Then $\{x \in A : o(f,x) \ge \varepsilon\}$ is closed.*

Proof. Notice that $B = \mathbb{R} \setminus \{x \in A : o(f,x) \ge \varepsilon\} = (\mathbb{R} \setminus A) \cup \{x \in A : o(f,x) < \varepsilon\}$.
It is enough to prove that B is open.

Let $x_0 \in B$.

- First case: $x_0 \in \mathbb{R} \setminus A$.

Since $\mathbb{R} \setminus A$ is open by assumption, there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (\mathbb{R} \setminus A) \subset B$.

- Second case: $x_0 \in A$ and $o(f, x_0) < \varepsilon$.

Then there exists[⋆] $\delta > 0$ such that $\sup\limits_{A\cap(x_0-\delta,x_0+\delta)} f - \inf\limits_{A\cap(x_0-\delta,x_0+\delta)} f < \varepsilon$.
Let $y \in \left(x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}\right) \cap A$.
If $z \in \left(y - \frac{\delta}{2}, y + \frac{\delta}{2}\right)$ then $|x_0 - z| = |x_0 - y + y - z| \leq |x_0 - y| + |y - z| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$.
Hence $\left(y - \frac{\delta}{2}, y + \frac{\delta}{2}\right) \subset \left(x_0 - \delta, x_0 + \delta\right)$.
So $\sup\limits_{A\cap(y-\delta/2,a+\delta/2)} f - \inf\limits_{A\cap(y-\delta/2,a+\delta/2)} f \leq \sup\limits_{A\cap(x_0-\delta,x_0+\delta)} f - \inf\limits_{A\cap(x_0-\delta,x_0+\delta)} f < \varepsilon$.
Then $o(f, y) < \varepsilon$ (again, use that $g(\delta)$ is monotonic).
Therefore $\left(x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}\right) \cap A \subset \{x \in A \ : \ o(f, x) < \varepsilon\}$.
Hence $\left(x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}\right) \subset (\mathbb{R} \setminus A) \cup \{x \in A \ : \ o(f, x) < \varepsilon\} = B$.

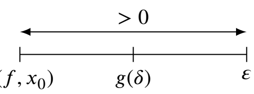
■

Theorem 78 (Riemann's criterion for integrability)**.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\varepsilon > 0$, $\{x \in [a, b] \ : \ o(f, x) \geq \varepsilon\}$ has zero content, i.e. for every $\varepsilon > 0$, for very $\alpha > 0$ there exists finitely many segment lines $I_1 = \left[a_1, b_1\right], \dots, I_q = \left[a_q, b_q\right]$ such that $\sum_{i=1}^{q}(b_i - a_i) < \alpha$ and $\{x \in A \ : \ o(f, x) \geq \varepsilon\} \subset \bigcup_{i=1}^{q}[a_i, b_i]$.*

Proof.
$\Rightarrow$: assume that f is integrable.
Let $\varepsilon > 0$ and $\alpha > 0$.
By the ε-criterion (Theorem 10) there exists a partition $P = \left\{a = x_0 < \cdots < x_n = b\right\}$ of $[a, b]$ such that $U_P(f) - L_P(f) < \frac{\varepsilon\alpha}{2}$.
We define $\mathcal{K} = \left\{k = 0, \dots, n \ : \ (x_k, x_{k+1}) \cap \{x \in [a, b] \ : \ o(f, x) \geq \varepsilon\} \neq \varnothing\right\}$. Then[†]

$$\begin{aligned}
\varepsilon \sum_{k\in\mathcal{K}} (x_{k+1} - x_k) &= \sum_{k\in\mathcal{K}} (x_{k+1} - x_k)\varepsilon \\
&\leq \sum_{k\in\mathcal{K}} (x_{k+1} - x_k) \left(\sup_{[x_k,x_{k+1}]} f - \inf_{[x_k,x_{k+1}]} f \right) \\
&\leq \sum_{k=1}^{n-1} (x_{k+1} - x_k) \left(\sup_{[x_k,x_{k+1}]} f - \inf_{[x_k,x_{k+1}]} f \right) \\
&= U_P(f) - L_P(f) \\
&< \frac{\varepsilon\alpha}{2}
\end{aligned}$$

⋆ Use the definition of limit: $o(f, x_0) = \lim\limits_{\delta\to 0^+} g(\delta)$, so there exits $\delta > 0$ such that:



Here it is important that $x_0 \in A$ so that $g(\delta)$ is well-defined (to avoir taking the supremum/infimum of an empty set).

† Notice that we excluded the endpoints here: to compute $o(f, x_k)$ in order to obtain the first inequality we need to work on a small interval around x_k and hence to leave $[x_k, x_{k+1}]$ or $[x_{k-1}, x_k]$.
Indeed define $f : [0, 2] \to \mathbb{R}$ by $f(x) = 0$ on $[0, 1)$ and $f(x) = \pi$ on $[1, 2]$, then $o(f, 1) = \pi$ and $1 \in [1, 2]$ but $\sup\limits_{[1,2]} f - \inf\limits_{[1,2]} f = 0$, so it is false that $\sup\limits_{[1,2]} f - \inf\limits_{[1,2]} f \geq \pi$.
But there are only finitely many endpoints so we will take care of them later.

Hence $\sum_{k\in\mathcal{K}}(x_{k+1}-x_k)<\frac{\alpha}{2}$ and $\{x\in[a,b]\ :\ o(f,x)\geq\varepsilon\}\setminus\{x_0,\dots,x_n\}\subset\bigcup_{k\in\mathcal{K}}[x_k,x_{k+1}]$.
Since $\{x_0,\dots,x_n\}$ is finite, there exists finitely many segment lines $[a_1,b_1]\dots[a_m,b_m]$ such that $\sum_{i=1}^{m}(b_i-a_i)<\frac{\alpha}{2}$ and $\{x_0,\dots,x_n\}\subset\bigcup_{i=1}^{m}[a_i,b_i]$.
Then $\{x\in[a,b]\ :\ o(f,x)\geq\varepsilon\}\subset\left(\bigcup_{k\in\mathcal{K}}[x_k,x_{k+1}]\right)\cup\left(\bigcup_{i=1}^{m}[a_i,b_i]\right)$ and $\sum_{k\in\mathcal{K}}(x_{k+1}-x_k)+\sum_{i=1}^{m}(b_i-a_i)<\alpha$.

$\Leftarrow$: assume that for any $\varepsilon>0$, $\{x\in[a,b]\ :\ o(f,x)\geq\varepsilon\}$ has zero content.
Let $\varepsilon>0$. Then we may find a partition $P=\{a=x_0<\cdots<x_n=b\}=\mathcal{K}\sqcup\mathcal{L}$ of $[a,b]$ such that

$$\left\{x\in[a,b]\ :\ o(f,x)\geq\frac{\varepsilon}{2(b-a)}\right\}\subset\bigcup_{k\in\mathcal{K}}(x_k,x_{k+1})\quad\text{and}\quad\sum_{k\in\mathcal{K}}(x_{k+1}-x_k)<\frac{\varepsilon}{2\left(\sup_{[a,b]}f-\inf_{[a,b]}f\right)}$$

Moreover, by Heine–Borel theorem ⋆, we may refine P so that for $k\in\mathcal{L}$,

$$\sup_{[x_k,x_{k+1}]}f-\inf_{[x_k,x_{k+1}]}f<\frac{\varepsilon}{2(b-a)}$$

Then

$$\begin{aligned}
U_P(f)-L_P(f)&=\sum_{k=0}^{n-1}\left(x_{k+1}-x_k\right)\left(\sup_{[x_k,x_{k+1}]}f-\inf_{[x_k,x_{k+1}]}f\right)\\
&=\sum_{k\in\mathcal{K}}\left(x_{k+1}-x_k\right)\left(\sup_{[x_k,x_{k+1}]}f-\inf_{[x_k,x_{k+1}]}f\right)+\sum_{k\in\mathcal{L}}\left(x_{k+1}-x_k\right)\left(\sup_{[x_k,x_{k+1}]}f-\inf_{[x_k,x_{k+1}]}f\right)\\
&<\sum_{k\in\mathcal{K}}\left(x_{k+1}-x_k\right)\left(\sup_{[a,b]}f-\inf_{[a,b]}f\right)+\sum_{k\in\mathcal{L}}\left(x_{k+1}-x_k\right)\frac{\varepsilon}{2(b-a)}\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\
&=\varepsilon
\end{aligned}$$

Hence, according to the ε-criterion (Theorem 10), f is integrable. ■

⋆ If $k\in\mathcal{L}$ then for any $x\in[x_k,x_{k+1}]$ we have $o(f,x)<\frac{\varepsilon}{2(b-a)}$ so we can find an interval $[x-\delta,x+\delta]$ having the expected property. By Heine-Borel theorem, we may take a finite subcover to refine $[x_k,x_{k+1}]$ in finitely many interval with the wanted property.

D Lebesgue criterion for Riemann integrability (Addendum from Feb 3, extra-curricular, not part of MAT237)

Definition 79. A set $S \subset \mathbb{R}$ has *measure zero* if for every $\varepsilon > 0$ there exists countably many (possibly empty) segment lines $\left([a_n, b_n]\right)_{n\in\mathbb{N}}$ such that

(i) $S \subset \bigcup_{n\in\mathbb{N}} [a_n, b_n]$

(ii) $\sum_{n\geq 0} (b_n - a_n) < \varepsilon$

Theorem 80 (Lebesgue's Criterion). *Let $f : [a,b] \to \mathbb{R}$ be a bounded a function.*
Then f is integrable if and only if the set $\{x \in [a,b] : f \text{ is not continuous at } x\}$ has measure zero.

Proof.
$\Rightarrow$: assume that f is integrable.
Then, by Theorem 78, for any $n \in \mathbb{N}_{>0}$, $\left\{x \in [a,b] \ : \ o(f,x) \geq \frac{1}{n}\right\}$ has measure 0.
Hence $\{x \in [a,b] : f \text{ is not continuous at } x\} = \bigcup_{n\in\mathbb{N}_{>0}} \left\{x \in [a,b] \ : \ o(f,x) \geq \frac{1}{n}\right\}$ has measure 0.

$\Leftarrow$: assume that $D = \{x \in [a,b] : f \text{ is not continuous at } x\}$ has measure zero.
Then, for any $\varepsilon > 0$, $E = \{x \in [a,b] \ : \ o(f,x) \geq \varepsilon\}$ has measure zero as a subset of D.
But E is also compact by Proposition 77 and hence has content zero by Heine–Borel theorem.
Hence f is integrable by Theorem 78. ■