The Implicit Function Theorem and the Inverse Function Theorem

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Exercise 1. *Let $B = \overline{B}(a,r) \subset \mathbb{R}^n$. Let $f : B \to B$ be a contraction mapping (i.e. a Lipschitz mapping with constant $q \in [0,1)$, or equivalently $\exists q \in [0,1),\ \forall x,y \in B,\ \|f(x)-f(y)\| \le q\|x-y\|$).*

1. *Prove that f is continuous.*

We define a sequence inductively by picking $x_0 \in B$ and then setting $x_{n+1} = f(x_n)$.

2. *Prove that $\forall n \in \mathbb{N}_{\ge 0},\ \|x_{n+1}-x_n\| \le q^n\|x_1-x_0\|$.*
3. *Prove that $\forall m,n \in \mathbb{N}_{\ge 0},\ m > n \implies \|x_m - x_n\| \le \frac{q^n}{1-q}\|x_1-x_0\|$.*
4. *Prove that $(x_n)_{n\in\mathbb{N}_{\ge 0}}$ is a Cauchy sequence.*
5. *Prove that $(x_n)_{n\in\mathbb{N}_{\ge 0}}$ is convergent in B.*
6. *Prove that f admits a fixed point, i.e. $\exists x \in B,\ f(x) = x$.*
7. *Prove that f admits only one fixed point (i.e. if $y \in B$ is another fixed of f point then $x = y$).*

You just proved the

Theorem 2 (Banach fixed point theorem)**.** *A contraction mapping $f : \overline{B}(a,r) \to \overline{B}(a,r)$ admits a unique fixed point.*

Exercise 3. *Let $U \subset \mathbb{R}^n$ be an open subset containing $\mathbf{0}$ and $f : U \to \mathbb{R}^n$ be a C^1 function satisfying $f(\mathbf{0}) = \mathbf{0}$ and $Df(\mathbf{0}) = I_{n,n}$.*

1. *Prove that there exists $t > 0$ such that $B(\mathbf{0},t) \subset U$ and $\forall x \in B(\mathbf{0},t),\ \det(Df(x)) \neq 0$.*
2. *Define $F : U \to \mathbb{R}^n$ by $F(x) = f(x) - x$.*
   (a) *Compute $DF(\mathbf{0})$.*
   (b) *Prove that there exists $r \in (0,t)$ such that $\forall x \in B(\mathbf{0},r),\ \|DF(x)\| \le \frac{1}{2}$.*
   (c) *Prove that there exists $s \in (0,r)$ such that*
   $$\forall x,y \in \overline{B}(\mathbf{0},s),\ \|F(x)-F(y)\| \le \left(\sup_{z\in\overline{B}(\mathbf{0},s)} \|DF(z)\|\right)\|x-y\|$$
   Comment: we use the Frobenius norm for matrices as in PS4 (recall that $\|AB\| \le \|A\|\|B\|$).
   (d) *Prove that $\forall x,y \in \overline{B}(\mathbf{0},s),\ \|F(x)-F(y)\| \le \frac{1}{2}\|x-y\|$.*
3. *Let $y \in B\left(\mathbf{0},\frac{s}{2}\right)$. Define $\theta_y : U \to \mathbb{R}^n$ by $\theta_y(x) = y - F(x)$.*
   (a) *Prove that $\theta_y\left(\overline{B}(\mathbf{0},s)\right) \subset B(\mathbf{0},s)$.*
   (b) *Prove that $\theta_y : \overline{B}(\mathbf{0},s) \to \overline{B}(\mathbf{0},s)$ is a contraction mapping with constant $\frac{1}{2}$.*
   (c) *We set $V = B(\mathbf{0},s) \cap f^{-1}\left(B\left(\mathbf{0},\frac{s}{2}\right)\right)$ and $W = B\left(\mathbf{0},\frac{s}{2}\right)$.*
   Prove that $f : V \to W$ is a well-defined bijection between two open subsets of $\mathbb{R}^n$ containing $\mathbf{0}$.
4. (a) *Prove that $f^{-1} : W \to V$ is Lipschitz with constant 2 and then that it is continuous.*
   (b) *Prove that $f^{-1} : W \to V$ is differentiable.*
   (Hint: use the very definition of differentiability together with Question 1.)
   (c) *Prove that $f^{-1} : W \to V$ is C^1.*
   (Hint: study $D\left(f^{-1}\right)$ using the Chain Rule.)

You just proved that

Claim 4. *Let $U \subset \mathbb{R}^n$ be an open subset containing $\mathbf{0}$ and $f : U \to \mathbb{R}^n$ be of class C^1 such that $f(\mathbf{0}) = \mathbf{0}$ and $Df(\mathbf{0}) = I_{n,n}$. Then there exist $V, W \subset \mathbb{R}^n$ two open subsets such that $\mathbf{0} \in V \subset U$, $\mathbf{0} \in W$ and $f : V \to W$ is a C^1-diffeomorphism (i.e. f is C^1, bijective and f^{-1} is C^1).*

Exercise 5. *Prove the Inverse Function Theorem (the statement is below).*
(Hint: reduce to the case considered in the above claim.)

Theorem 6 (The Inverse Function Theorem)**.** *Let $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ of class C^1 and $a \in U$. Assume that $Df(a)$ is invertible.*
Then there exist $V, W \subset \mathbb{R}^n$ two open subsets such that $a \in V \subset U$, $f(a) \in W$ and $f : V \to W$ is a C^1-diffeomorphism (i.e. f is C^1, bijective and f^{-1} is C^1).

Exercise 7. *Derive the Implicit Function Theorem from the Inverse Function Theorem (the statement is below).*

Theorem 8 (The Implicit Function Theorem)**.**
Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^p$ be two open subsets. Let $(x_0, y_0) \in U \times V$.
Let $F : \begin{array}{ccc} U \times V & \to & \mathbb{R}^p \\ (x, y) & \mapsto & F(x, y) \end{array}$ be of class C^1.
If $D_y F(x_0, y_0)$ is invertible then there exist $r, s > 0$ such that $B(x_0, r) \subset U$, $B(y_0, s) \subset V$ and there exists $\varphi : B(x_0, r) \to B(y_0, s)$ of class C^1 such that

$$\forall (x, y) \in B(x_0, r) \times B(y_0, s),\ F(x, y) = F(x_0, y_0) \Leftrightarrow y = \varphi(x)$$