University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus

Change of variables: usual coordinate systems

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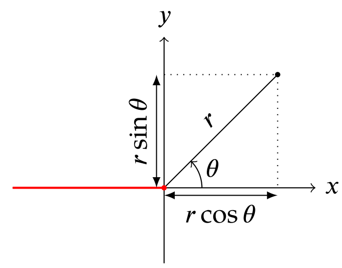
1 Polar coordinates

$$\Phi : \begin{array}{ccc} (0,+\infty)\times(-\pi,\pi) & \to & \mathbb{R}^2 \setminus \{(x,0)\in\mathbb{R}^2 \ :\ x\le 0\} \\ (r,\theta) & \mapsto & (r\cos\theta, r\sin\theta) \end{array}$$



- Φ is C^1.
- Φ is bijective.
- $\det D\Phi(r,\theta) = \det\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix} = r\left(\cos^2\theta+\sin^2\theta\right) = r > 0.$
- Hence Φ is a C^1-diffeomorphism.
- And $|\det D\Phi(r,\theta)| = r$.

Example 1. Let $\Delta = \left\{(x,y)\in\mathbb{R}^2 \ :\ 1\le x^2+y^2\le 9,\ x\ge 0\right\}$.
We want to compute $\displaystyle\int_\Delta e^{x^2+y^2}$.

First, notice that $\Delta = \Phi\left([1,3]\times[-\pi/2,\pi/2]\right)$.
Hence,

$$\begin{aligned}\int_\Delta e^{x^2+y^2}\,dx\,dy &= \int_{[1,3]\times[-\pi/2,\pi/2]} e^{r^2} r\,dr\,d\theta && \text{by the CoV formula} \\ &= \int_{-\pi/2}^{\pi/2}\int_1^3 re^{r^2}\,dr\,d\theta && \text{by the iterated integrals theorem} \\ &= \int_{-\pi/2}^{\pi/2}\frac{e^9-e}{2}\,d\theta \\ &= \frac{\pi}{2}\left(e^9-e\right)\end{aligned}$$

Example 2. We want to compute $\displaystyle\int_{\overline{B}((1,1),1)} x^2 + y^2 - 2y dxdy$.

First notice that $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\Psi(x, y) = (x + 1, y + 1)$ is a C^1-diffeomorphism and that $|\det D\Psi(x, y)| = 1$.
Moreover $\overline{B}((1, 1), 1) = \Psi\left(\overline{B}(\mathbf{0}, 1)\right)$. Hence

$$\begin{aligned}\int_{\overline{B}((1,1),1)} x^2 + y^2 - 2y dxdy &= \int_{\overline{B}(\mathbf{0},1)} (x+1)^2 + (y+1)^2 - 2(y+1) dxdy \qquad \text{by the CoV formula}\\ &= \int_{\overline{B}(\mathbf{0},1)} x^2 + y^2 - 2x dxdy\end{aligned}$$

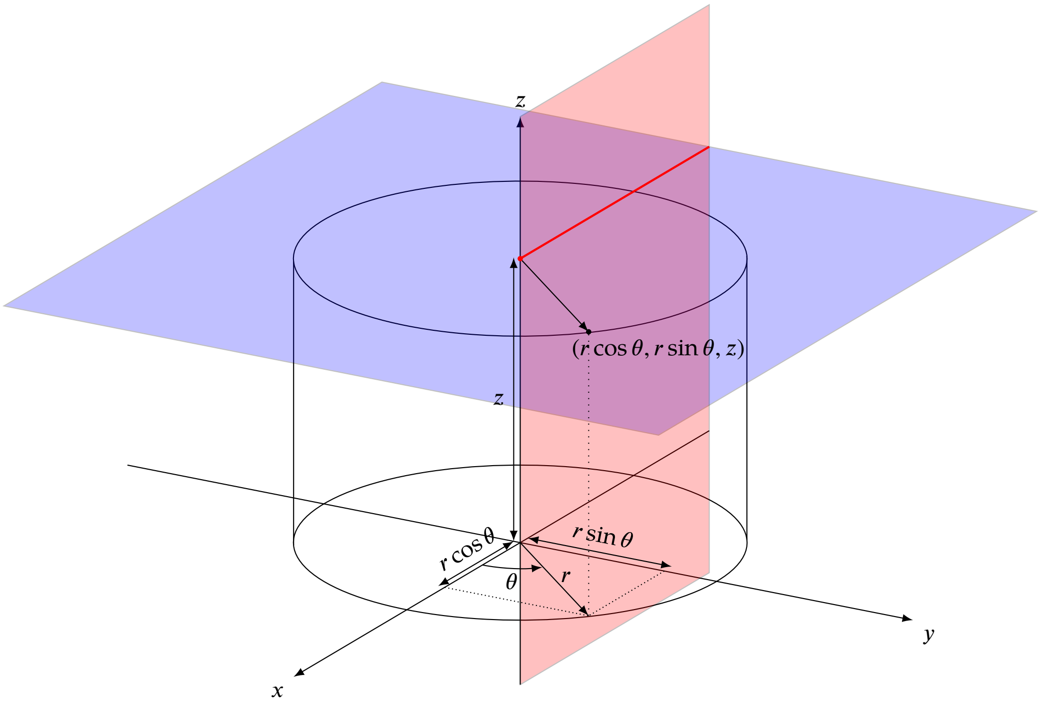
Next, we have $\overline{B}(\mathbf{0}, 1) = \Phi([0, 1] \times [-\pi, \pi])$.
Notice that there is an issue for $r = 0$ or $\theta = \pm\pi$ (i.e. $\{(x, 0) : x \in [-1, 0]\}$) but these sets have zero content.
Hence

$$\int_{\overline{B}(\mathbf{0},1)} x^2 + y^2 - 2x dxdy = \int_{[0,1]\times[-\pi,\pi]} (r^2 - 2r\cos\theta) r dr d\theta \qquad \text{by the CoV formula}$$

$$\begin{aligned}&= \int_{-\pi}^{\pi}\int_0^1 r^3 - 2r^2\cos\theta dr d\theta \qquad \text{by the iterated integrals theorem}\\ &= \int_{-\pi}^{\pi} \frac{1}{4} - \frac{2}{3}\cos\theta d\theta\\ &= \frac{\pi}{2}\end{aligned}$$

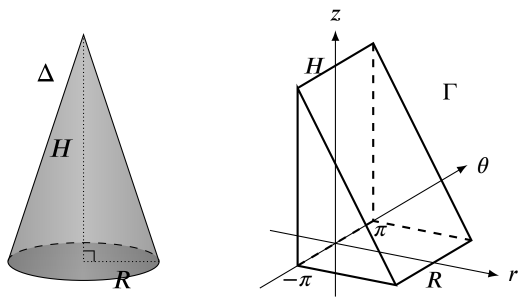
2 Cylindrical coordinates

$$\Phi : \begin{array}{ccc} (0,+\infty)\times(-\pi,\pi)\times\mathbb{R} & \to & \mathbb{R}^3\setminus((-\infty,0]\times\{0\}\times\mathbb{R}) \\ (r,\theta,z) & \mapsto & (r\cos\theta, r\sin\theta, z) \end{array}$$



- Φ is C^1.
- Φ is bijective.
- $\det D\Phi(r,\theta,z) = \det\begin{pmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} = \det\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix} = r > 0.$
- Hence Φ is a C^1-diffeomorphism.
- And $|\det D\Phi(r,\theta,z)| = r$.

Example 3. We want to compute $\int_\Delta z dx dy dx$
where $\Delta = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 \leq \frac{R^2}{H^2}(H-z)^2,\ 0 \leq z \leq H\}$.



Notice that $\Delta = \Phi(\Gamma)$ where $\Gamma = \left\{(r, \theta, z) \ : \ 0 \leq r \leq \frac{R}{H}(H-z),\ \theta \in [-\pi, \pi],\ z \in [0, H]\right\}$.
Again, Γ goes outside the domain of Φ but the involved sets have zero content.
Hence

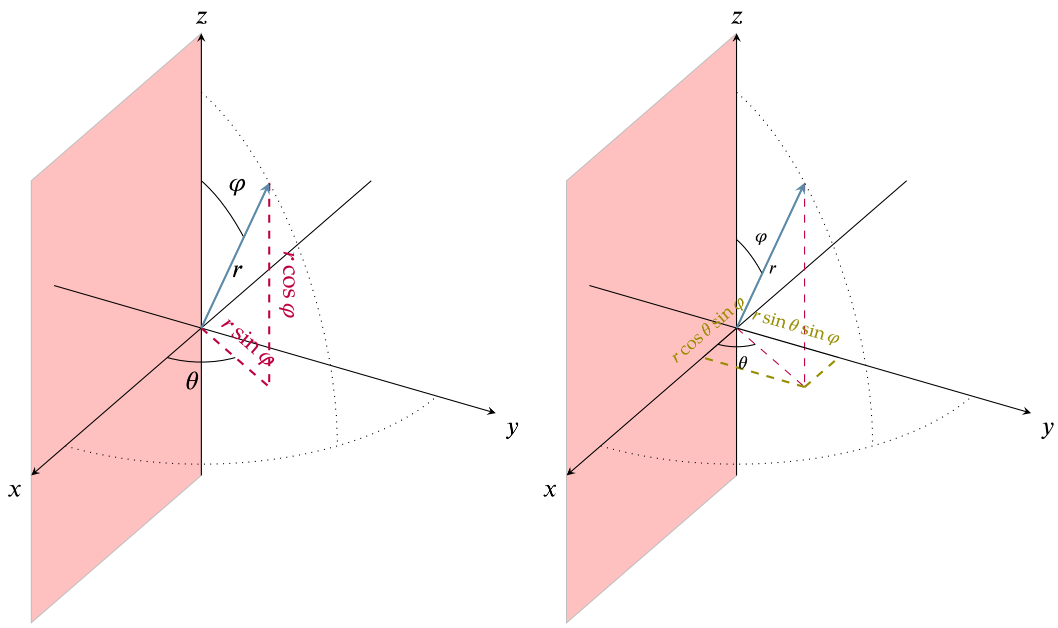
$$\begin{aligned}
\int_\Delta z dx dy dx &= \int_\Gamma z r dr d\theta dz \qquad \text{by the CoV formula} \\
&= \int_0^H \int_{-\pi}^{\pi} \int_0^{\frac{R}{H}(H-z)} r z dr d\theta dz \\
&= \int_0^H \int_{-\pi}^{\pi} \frac{R^2}{2H^2}(H-z)^2 z d\theta dz \\
&= \frac{\pi R^2}{H^2} \int_0^H (H-z)^2 z dz \\
&= \frac{\pi R^2 H^2}{12}
\end{aligned}$$

3 Spherical coordinates

$$\Phi : \begin{array}{ccc} (0,+\infty)\times(0,2\pi)\times(0,\pi) & \to & \mathbb{R}^3 \setminus ([0,+\infty)\times\{0\}\times\mathbb{R}) \\ (r,\theta,\varphi) & \mapsto & (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi) \end{array}$$

In this course, we use the following convention ⋆ :

$$(r,\theta,\varphi) = (\text{radius/distance to the origin}, \text{longitude}, \text{colatitude})$$



- Φ is C^1.
- Φ is bijective.
- The Jacobian determinant is

$$\begin{aligned}
\det D\Phi(r,\theta,\varphi) &= \det\begin{pmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\varphi & 0 & -r\sin\varphi \end{pmatrix} \\
&= \cos\varphi\det\begin{pmatrix} -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{pmatrix} - r\sin\varphi\det\begin{pmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi \end{pmatrix} \\
&= r^2\cos^2\varphi\sin\varphi\det\begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} - r^2\sin^3\varphi\det\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
&= -r^2\cos^2\varphi\sin\varphi - r^2\sin^3\varphi \\
&= -r^2\sin\varphi < 0 \text{ since } \varphi\in(0,\pi)
\end{aligned}$$

- Hence Φ is a C^1-diffeomorphism.
- And $|\det D\Phi(r,\theta,\varphi)| = r^2\sin\varphi$.

⋆ This convention may differ from the one used in other courses in math or in physics (the meaning of θ and φ may be swapped, some people use the latitude and not the colatitude...).
I believe that the usual convention in physics is $(r,\theta,\varphi) = $ (radius, colatitude, longitude) as in ISO 80000-2, i.e. the meaning of θ and φ are swapped from our convention in MAT237.

Example 4. We want to compute $\int_\Delta z dx dy dz$ where $\Delta = \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2+y^2+z^2 \leq 1,\ z \geq 0\right\}$. Notice that $\Delta = \Phi\left([0,1]\times[0,2\pi]\times[0,\pi/2]\right)$.
Again, there is an issue with the domain of Φ but the involved sets have zero content.
Hence

$$\begin{aligned}
\int_\Delta z dx dy dz &= \int\limits_{[0,1]\times[0,2\pi]\times[0,\pi/2]} r^3 \cos\varphi \sin\varphi dr d\theta d\varphi && \text{by the CoV formula} \\
&= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 r^3 \frac{\sin(2\varphi)}{2} dr d\theta d\varphi && \text{by the iterated integrals theorem} \\
&= 2\pi \frac{1}{4}\left(\frac{\cos 0}{4} - \frac{\cos\pi}{4}\right) \\
&= \frac{\pi}{4}
\end{aligned}$$