Q2 : (1) By elementary operations on differentiable functions

(2) $\frac{\partial f}{\partial x}(x,y) = ye^{x} + xye^{x} = (1+x)ye^{x}$

$\frac{\partial f}{\partial y}(x,y) = xe^{x}$

Hence $\frac{\partial f}{\partial x}(1,1) = 2e$

$\frac{\partial f}{\partial y}(1,1) = e$

(3) $\nabla f(1,1) = (2e, e)$

(4) Since f is differentiable : $\partial_{(1,2)} f(1,1) = \nabla f(1,1) \cdot (1,2)$

$= (2e, e)\cdot(1,2)$

$= 4e$

(5) Set $g(t) = f((1,1) + t(1,2)) = f(1+t, 1+2t)$

$= (1+t)(1+2t)e^{1+t}$

$= (1+3t+2t^2)e^{1+t}$

then $g'(t) = (3+4t)e^{1+t} + (1+3t+2t^2)e^{1+t}$

and $\partial_{(1,2)} f(1,1) = g'(0) = 3e + e = 4e$

Sometimes it is faster to use the definition
sometimes to use ~~that~~ $\partial_u f(a) = \nabla f(a) \cdot u$
for f differentiable

Q3 ① The only point where it is not obvious that f is continuous is $(0,0)$

$$y^2 \leq \sqrt{x^2+y^4}$$

$$\Rightarrow 0 \leq \left|\frac{y^3}{\sqrt{x^2+y^4}}\right| \leq |y| \xrightarrow[(x,y)\to(0,0)]{} 0 = f(0,0)$$

Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ and f is C^0 at $(0,0)$

② $v = (a,b)$

if $a \neq 0$: $\left|\dfrac{f((0,0)+t(a,b)) - f(0,0)}{t}\right| = \left|\dfrac{f(ta,tb) - f(0,0)}{t}\right|$

$$= \left|\frac{b^3 t^2}{\sqrt{a^2t^2+b^4t^4}}\right| \leq \left|\frac{b^3}{a}\right| |t| \xrightarrow[t\to 0]{} 0$$

if $a=0$: $\dfrac{f((0,0)+t(a,b)) - f(0,0)}{t} = b \xrightarrow[t\to 0]{} b$

Hence $\partial_v f(0,0) = \begin{cases} 0 & \text{if } a \neq 0 \\ b & \text{if } a = 0 \end{cases}$

③ Assume that f is differentiable at $(0,0)$ then

$$\partial_{(1,1)} f(0,0) = d_{(0,0)} f(1,1) = d_{(0,0)} f(1,0) + d_{(0,0)} f(0,1)$$

$$\overset{\parallel}{0} \qquad = \partial_{(1,0)} f(0,0) + \partial_{(0,1)} f(0,0)$$

$$= 0 + 1$$

Contradiction

(4) Assume that f is differentiable at $(0,0)$ then

$$f(\vec{0}+h) = f(\vec{0}) + d_{\vec{0}}f(h) + E(h) \quad \text{where} \quad \frac{E(h)}{\|h\|} \xrightarrow[h\to 0]{} 0$$

$$f(h) = 0 + \frac{\partial f}{\partial x_1}(\vec{0})h_1 + \frac{\partial f}{\partial x_2}(\vec{0})h_2 + E(h)$$

$$= 0 + 0 + 1\times h_2$$

$$\Rightarrow \quad \frac{f(h)-h_2}{\|h\|} = \frac{E(h)}{\|h\|} \xrightarrow[h\to 0]{} 0$$

$$\frac{1}{\sqrt{h_1^2+h_2^2}}\left(\frac{h_2^3}{\sqrt{h_1^2+h_2^4}} - h_2\right)$$

But, for $h_1 = h^2$, $h_2 = h$, we get

$$\frac{1}{\sqrt{h^4+h^2}}\left(\frac{h^3}{\sqrt{h^4+h^4}} - h\right) = \frac{1}{|h|\sqrt{h^2+1}}\left(\frac{h^3}{\sqrt{2}\,h^2} - h\right)$$

$$= \frac{h}{|h|}\cdot\frac{1}{\sqrt{h^2+1}}\left(\frac{1}{\sqrt{2}} - 1\right) \xrightarrow[h\to 0^+]{} \frac{1}{\sqrt{2}} - 1 \neq 0$$

Contradiction

CQ4 ① By elementary operations on differentiable functions

②

$g(t) = f((2,0) + t(1,1)) = f(2+t, t) = (2+t)^2 e^t$

$g'(t) = 2(2+t)e^t + (2+t)^2 e^t$

$\partial_{(1,1)} f(2,0) = g'(0) = 4 + 4 = 8$

$h(t) = f((2,0) + t(1,-1)) = f(2+t, -t) = (2+t)^2 e^{-t}$

$h'(t) = 2(2+t)e^{-t} - (2+t)^2 e^{-t}$

$\partial_{(1,-1)} f(2,0) = h'(0) = 4 - 4 = 0$

③ Write $\nabla f(2,0) = (X, Y)$

Then, since f is differentiable,

$$\begin{cases} \partial_{(1,1)} f(2,0) = \nabla f(2,0) \cdot (1,1) \\ \partial_{(1,-1)} f(2,0) = \nabla f(2,0) \cdot (1,-1) \end{cases}$$

$$\Leftrightarrow \begin{cases} 8 = X + Y \\ 0 = X - Y \end{cases}$$

$$\Leftrightarrow \begin{cases} X = 4 \\ Y = 4 \end{cases}$$

Hence $\nabla f(2,0) = (4,4)$

Q5 : ① $f(a+h) = \|a+h\|^2 = (a+h)\cdot(a+h)$

$= a\cdot a + 2a\cdot h + h\cdot h$

$= \|a\|^2 + 2a\cdot h + \|h\|^2$

$= f(a) + 2a\cdot h + E(h)$

where $E(h) = \|h\|^2$ satisfies $\frac{E(h)}{\|h\|} = \|h\| \xrightarrow[h\to 0]{} 0$

and $h \mapsto 2a\cdot h$ is linear
$\mathbb{R}^n \to \mathbb{R}$

Hence f is differentiable at a and $d_af(h) = 2a\cdot h$

Moreover $\frac{\partial f}{\partial x_i}(a) = \partial_{e_i} f(a) = d_af(e_i) = 2a\cdot e_i = 2a_i$

Therefore $\nabla f(a) = 2(a_1, \dots, a_n)$

② $f(x) = \sum_{i=1}^{n} x_i^2$ ~~is differentiable~~

$\frac{\partial f}{\partial x}(x) = 2x_i$ are continuous

hence f is C^1 and hence differentiable

$\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a)\right) = (2a_1, \dots, 2a_n)$

③ $g(x) = \sqrt{x_1^2 + \dots + x_n^2}$ is not differentiable at $\vec{0}$

indeed $\lim_{t\to 0} \frac{g(0+te_1) - g(0)}{t} = \lim_{t\to 0} \frac{|t|}{t}$ DNE

hence $\partial_{e_1} g(0)$ DNE

Comment: when possible work with $\|x\|^2$ instead of $\|x\|$ to kill the $\sqrt{\ }$.

Recall that $x \mapsto \sqrt{x}$ ~~[illegible]~~ is increasing ⚠

Q6 : ① f is obviously C^1 on the open set $\mathbb{R}^2 \setminus \{(0,0)\}$
let's study at $(0,0)$

$$\frac{f(t,0) - f(0,0)}{t} = \frac{t^4}{t^3} = t \xrightarrow[t \to 0]{} 0$$

hence $\frac{\partial f}{\partial x}(0,0) = 0$

for $(x,y) \neq (0,0)$, $\frac{\partial f}{\partial x}(x,y) = \frac{-2x y^4}{(x^2+y^2)^2}$

$$\left|\frac{\partial f}{\partial x}(x,y)\right| = \left|\frac{2xy}{x^2+y^2}\right| \cdot \left|\frac{y^2}{x^2+y^2}\right| |y|$$

$$\leq \frac{2}{2} \cdot 1 \cdot |y| \xrightarrow[(x,y)\to(0,0)]{} 0$$

hence $\frac{\partial f}{\partial x}$ is continuous at 0 and $\frac{\partial f}{\partial x}(0,0) = 0$

Similarly we can prove ~~that~~ $\frac{\partial f}{\partial y}$ is C^0 and $\frac{\partial f}{\partial y}(0,0) = 0$
hence f is C^1

② $$\frac{\frac{\partial f}{\partial x}(0,t) - \frac{\partial f}{\partial x}(0,0)}{t} = 0 \xrightarrow[t \to 0]{} 0$$

hence $\frac{\partial^2 f}{\partial y \partial x}(0,0) = 0$

Similarly, we can ~~pro~~ prove ~~that~~ $\frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$

③ After a computation, for $(x,y) \neq (0,0)$, we get

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = - \frac{8x^3y^3}{(x^2+y^2)^3}$$

Hence $\frac{\partial^2 f}{\partial x \partial y}(t,t) = -1 \xrightarrow[t \to 0]{} -1 \neq 0 = \frac{\partial^2 f}{\partial x \partial y}(0,0)$

hence $\frac{\partial^2 f}{\partial x \partial y}$ is not continuous at $(0,0)$

Similarly for $\frac{\partial^2 f}{\partial y \partial x}$

⚠ In this example, $\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial^2 f}{\partial y \partial x}(0,0)$,

but $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are not continuous at $(0,0)$

Q7: Define $f : \mathbb{R}^3 \setminus \{a\} \longrightarrow \mathbb{R}$ by $f(x) = \ln(\|x-a\|)$

$$= \frac{1}{2} \ln(\|x-a\|^2)$$

↳ the advantage of this form is that the differential of $\|\cdot\|^2$ is easy to compute: kill the square root when you can

f is C^1 on $\mathbb{R}^3 \setminus \{a\}$ by elementary operations on C^1 functions

$$f(x+h) = \frac{1}{2} \ln(\|x+h-a\|^2)$$

$$= \frac{1}{2} \ln\left(\|x-a\|^2 + 2(x-a)\cdot h + \|h\|^2\right)$$

$$= \frac{1}{2} \ln\left(\|x-a\|^2 \left(1 + 2\frac{(x-a)\cdot h}{\|x-a\|^2} + \frac{\|h\|^2}{\|x-a\|^2}\right)\right)$$

$$= \ln(\|x-a\|) + \frac{1}{2} \ln\left(1 + \frac{2(x-a)\cdot h}{\|x-a\|^2} + \frac{\|h\|^2}{\|x-a\|^2}\right)$$

$$= f(x) + \left(\frac{x-a}{\|x-a\|^2}\right)\cdot h + E(h) \text{ where } \frac{E(h)}{\|h\|} \xrightarrow[h \to 0]{} 0$$

using the Taylor expansion of ln.

Hence $d_x f(h) = \dfrac{x-a}{\|x-a\|^2} \cdot h$

and $\nabla f(x) = \dfrac{x-a}{\|x-a\|^2}$

Q8 : Fix $x \in \mathbb{R}^2$ and define $g : (0, +\infty) \to \mathbb{R}$ by

$$g(t) = f(t^{w_1} x_1, \dots, t^{w_m} x_m)$$

thus $$g'(t) = \sum_{i=1}^{m} w_i x_i t^{w_i - 1} \frac{\partial f}{\partial x_i}(t^{w_1} x_1, \dots, t^{w_m} x_m)$$

by the chain rule

But by assumption $g(t) = t^r f(x_1, \dots, x_m)$

and $g'(t) = r t^{r-1} f(x_1, \dots, x_m)$

Hence $$\sum_{i=1}^{m} w_i x_i t^{w_i - 1} \frac{\partial f}{\partial x_i}(t^{w_1} x_1, \dots, t^{w_m} x_m) = r t^{r-1} f(x_1, \dots, x_m)$$

for any $t > 0$

By taking $t = 1$ we get :

$$\sum_{i=1}^{m} w_i x_i \frac{\partial f}{\partial x_i}(x_1, \dots, x_m) = r f(x_1, \dots, x_m)$$

$\frac{\partial^2 f}{\partial x^2}(r\cos\theta, r\sin\theta)$ $\frac{\partial^2 \varphi}{\partial r^2}(r,\theta)$ $\frac{\partial \varphi}{\partial r}(r,\theta)$

Q9

WTS: $\underbrace{\partial_x^2 f(r\cos\theta, r\sin\theta)}_{} + \partial_y^2 f(r\cos\theta, r\sin\theta) = \underbrace{\partial_r^2\varphi(r,\theta)}_{} + \frac{1}{r}\underbrace{\partial_r\varphi(r,\theta)}_{} + \frac{1}{r^2}\partial_\theta^2\varphi(r,\theta)$

⚠ We omit the variables for conciseness but they are there and they are important for the chain-rule, be careful

For instance

By the chain rule:

$\partial_r\varphi(r,\theta) = \cos\theta\,\partial_x f(r\cos\theta, r\sin\theta) + \sin\theta\,\partial_y f(r\cos\theta, r\sin\theta)$

$$\partial_r \varphi = \cos\theta\,\partial_x f + \sin\theta\,\partial_y f$$

$$\partial_r^2 \varphi = \cos^2\theta\,\partial_x^2 f + \cos\theta\sin\theta\,\partial_{yx}^2 f + \cos\theta\sin\theta\,\partial_{xy}^2 f + \sin^2\theta\,\partial_y^2 f$$

$$= \cos^2\theta\,\partial_x^2 f + 2\cos\theta\sin\theta\,\partial_{xy}^2 f + \sin^2\theta\,\partial_y^2 f \quad \text{by Clairaut's theorem}$$

$$\partial_\theta \varphi = -r\sin\theta\,\partial_x f + r\cos\theta\,\partial_y f$$

$$\partial_\theta^2 \varphi = -r\cos\theta\,\partial_x f + r^2\sin^2\theta\,\partial_x^2 f - r^2\sin\theta\cos\theta\,\partial_{yx}^2 f$$

$$- r\sin\theta\,\partial_y f + r^2\cos^2\theta\,\partial_y^2 f - r^2\sin\theta\cos\theta\,\partial_{xy}^2 f$$

$\partial_r\varphi =$

$$= -r\,\partial_r\varphi + r^2\sin^2\theta\,\partial_x^2 f + r^2\cos^2\theta\,\partial_y^2 f - \underbrace{2r^2\cos\theta\sin\theta\,\partial_{xy}^2 f}_{\text{by Clairaut's theorem}}$$

Hence: $\partial_r^2\varphi + \frac{1}{r}\partial_r\varphi + \frac{1}{r^2}\partial_\theta^2\varphi$

$$= (\cos^2\theta + \sin^2\theta)\,\partial_x^2 f + (\cos^2\theta + \sin^2\theta)\,\partial_y^2 f$$

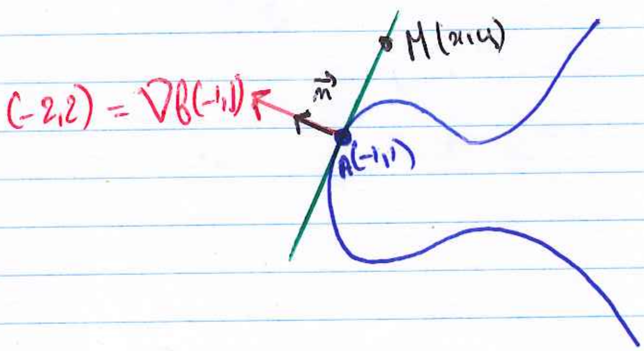
$$= \partial_x^2 f + \partial_y^2 f$$

Q10.

1)a) Let $f(x,y) = x + y^2 - x^3$

We want to find the line tangent to the level set $f(x,y) = 1$ at $(-1,1)$.

By the course we know that it is orthogonal to $\nabla f(-1,1)$



$$\frac{\partial f}{\partial x}(x,y) = 1 - 3x^2 \qquad \frac{\partial f}{\partial y} = 2y \quad \rightsquigarrow \quad \nabla f(-1,1) = (-2,2) = 2(-1,1)$$

Let $\vec{n} = (-1,1)$, $A = (-1,1)$

$\underset{(x,y)}{M} \in$ Tangent line $\Leftrightarrow \vec{AM} \perp \vec{n}$

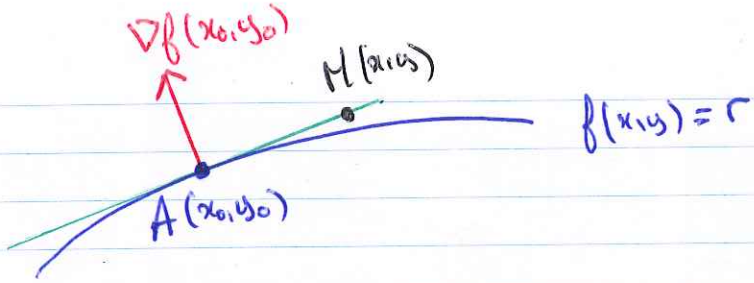
$\Leftrightarrow (x+1, y-1) \perp (-1,1)$

$\Leftrightarrow (x+1, y-1) \cdot (-1,1) = 0$

$\Leftrightarrow -x-1+y-1 = 0$

$\Leftrightarrow \boxed{x - y + 2 = 0}$

1) b)



$M(x,y) \in$ Tangent line

$\Leftrightarrow \overrightarrow{AM} \perp \nabla f(x_0,y_0)$

$\Leftrightarrow \left(x-x_0,\ y-y_0\right)\cdot\left(\frac{\partial f}{\partial x}(x_0,y_0),\ \frac{\partial f}{\partial y}(x_0,y_0)\right)=0$

$\Leftrightarrow \boxed{(x-x_0)\frac{\partial f}{\partial x}(x_0,y_0)+(y-y_0)\frac{\partial f}{\partial y}(x_0,y_0)=0}$

$\Leftrightarrow \frac{\partial f}{\partial x}(x_0,y_0)\,x+\frac{\partial f}{\partial y}(x_0,y_0)\,y-\frac{\partial f}{\partial x}(x_0,y_0)\,x_0-\frac{\partial f}{\partial y}(x_0,y_0)\,y_0=0$

2) a) Similarly, we know that the tangent plane of $f(x,y,z)=0$ at $(1,2,3)$ is orthogonal to $\nabla f(1,2,3)$, here $f(x,y,z)=x^3+zx^2-y^2$

$\nabla f(1,2,3)=\cancel{(9i-4j+k)}\ (9,-4,1)$

Let $A(1,2,3)$, $M(x,y,z)$

$M\in$ Tangent plane $\Leftrightarrow \overrightarrow{AM}\perp\nabla f(1,2,3)$

$\Leftrightarrow (x-1,y-2,z-3)\cdot(9,-4,1)=0$

$\Leftrightarrow \boxed{9x-4y+z-4=0}$

At $(0,0,1)$, $\nabla f(0,0,1)=(0,0,0) \rightsquigarrow$ there is no tangent plane
(there is a singularity) (draw $x^3+zx^2-y^2=0$ online)

2) b) As in 1) b) $(x-x_0)\frac{\partial f}{\partial x}(x_0,y_0,z_0)+(y-y_0)\frac{\partial f}{\partial y}(x_0,y_0,z_0)+(z-z_0)\frac{\partial f}{\partial z}(x_0,y_0,z_0)=0$

Method 1

3)a) We know that the tangent plane to the graph of f at $(1,1,2)$ is the translation of the graph of the differential at $(1,1)$

Hence it contains:

$$\vec{u} = (1,0,\ d_{(1,1)}f(1,0)) = \left(1,0,\ \frac{\partial f}{\partial x}(1,1)\right) = (1,0,2)$$

$$\vec{v} = (0,1,\ d_{(1,1)}f(0,1)) = \left(0,1,\ \frac{\partial f}{\partial y}(1,1)\right) = (0,1,3)$$

Then it is orthogonal to

$$\vec{n} = \vec{u} \times \vec{v} = \begin{pmatrix}1\\0\\2\end{pmatrix} \times \begin{pmatrix}0\\1\\3\end{pmatrix} = \begin{pmatrix}-2\\-3\\1\end{pmatrix}$$

And $A(1,1,2)$ is in the plane hence $M(x,y,z)$ is in it if and only if $\overrightarrow{AM} \perp \vec{n}$

$$\iff (x-1,\ y-1,\ z-2)\cdot(-2,-3,1) = 0$$

$$\iff -2x - 3y + z + 2 + 3 - 2 = 0$$

$$\iff \boxed{2x + 3y - z - 3 = 0}$$

3) a) Method 2 :

Notice that the graph of f is

$$\Gamma = \{(x,y,z) \in \mathbb{R}^3, \quad z = f(x,y)\}$$

$$f(x,y) = z \iff f(x,y) - z = 0 \iff F(x,y,z) = 0$$

where $F(x,y,z) = f(x,y) - z$

hence Γ is the 0-level set of F

hence the tangent plane is orthogonal to $\nabla F(1,1,2)$:

$$\nabla F(1,1,2) = \left(\frac{\partial f}{\partial x}(1,1,2), \frac{\partial f}{\partial y}(1,1,2), -1\right)$$

$$= (2, 3, -1)$$

And $A(1,1,2)$ is in the tangent plane, hence

$M(x,y,z) \in$ tangent plane

$$\iff \overrightarrow{AM} \perp \nabla F(1,1,2)$$

$$\iff (x-1, y-1, z-2) \cdot (2,3,-1) = 0$$

$$\iff 2x + 3y - z - 3 = 0$$

Comment: Method 1 is closer to the intuition, but Method 2 is faster ~~since the~~ and safer: there is no cross-product to compute

3)b) : Method 1

The tangent plane supports the vectors

$$\left(1, 0, \frac{\partial f}{\partial x}(x_0,y_0)\right) \quad \text{and} \quad \left(0, 1, \frac{\partial f}{\partial y}(x_0,y_0)\right)$$

hence it is normal to

$$\vec{n} = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(x_0,y_0) \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y}(x_0,y_0) \end{pmatrix} = \begin{pmatrix} -\frac{\partial f}{\partial x}(x_0,y_0) \\ -\frac{\partial f}{\partial y}(x_0,y_0) \\ 1 \end{pmatrix}$$

and it contains $A(x_0, y_0, f(x_0,y_0))$

hence $M(x,y,z) \in$ Tangent plane

$$\Leftrightarrow \vec{AM} \perp \vec{n}$$

$$\Leftrightarrow \left(x-x_0, y-y_0, z-f(x_0,y_0)\right) \cdot \left(-\frac{\partial f}{\partial x}(x_0,y_0), -\frac{\partial f}{\partial y}(x_0,y_0), 1\right) = 0$$

$$\Leftrightarrow -(x-x_0)\frac{\partial f}{\partial x}(x_0,y_0) - (y-y_0)\frac{\partial f}{\partial y}(x_0,y_0) + z - f(x_0,y_0) = 0$$

$$\Leftrightarrow (x-x_0)\frac{\partial f}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(x_0,y_0) - z + f(x_0,y_0) = 0$$

3) b) Method 2

$\Gamma_f = \{(x,y,z) \in \mathbb{R}^2,\ f(x,y) - z = 0\}$ the graph of f

Set $F : \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x,y,z) = f(x,y) - z$

then Γ_f is the level set $F(x,y,z) = 0$

hence the tangent plane of Γ_f at $(x_0, y_0, f(x_0,y_0))$

is orthogonal to $\nabla F(x_0, y_0, f(x_0,y_0)) = \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0), -1\right)$

and it contains $A(x_0, y_0, f(x_0,y_0))$

hence $M(x,y,z) \in$ tangent plane $\Leftrightarrow \vec{AM} \perp$

$$\Leftrightarrow \left(x - x_0, y - y_0, z - f(x_0,y_0)\right) \cdot \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0), -1\right) = 0$$

$$\Leftrightarrow (x - x_0)\frac{\partial f}{\partial x}(x_0,y_0) + (y - y_0)\frac{\partial f}{\partial y}(x_0,y_0) - z + f(x_0,y_0) = 0$$

Q11:

1) $P_{a,0}(h) = f(a)$

$$P_{a,1}(h) = f(a) + \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(a) h_i$$

$$P_{a,2}(h) = f(a) + \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(a) h_i + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j}(a) h_i h_j$$

don't forget this ~~one~~ coefficient $\frac{1}{2}$!

2) $P_{a,0}(h) = f(a)$

$$P_{a,1}(h) = f(a) + \nabla f(a) \cdot h$$

$$P_{a,2}(h) = f(a) + \nabla f(a)\cdot h + \frac{1}{2} h^t H_f(a) h$$

$$= f(a) + \nabla f(a) \cdot h + \frac{1}{2} h \cdot (H_f(a) h)$$

Recall that $\nabla f(a) = \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_m}(a)\right)$

$$H_f(a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(a) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(a) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(a) & \cdots & & \frac{\partial^2 f}{\partial x_m \partial x_m}(a) \end{pmatrix}$$

We can recover the coefficient of $P_{a,2}(h)$ from these data

Q12

i) Method 1:

$A\begin{pmatrix}1\\1\end{pmatrix} = 0\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow 0$ is an eigenvalue

hence A can't be positive definite or negative definite

Since A has only two eigenvalues, we can't have

a positive and a negative one so A is not indefinite

Method 2: Write $A = \begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}$

$$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Leftrightarrow \begin{pmatrix}\alpha+\beta\\ \beta+\gamma\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Leftrightarrow \begin{cases}\alpha = -\beta\\ \gamma = -\beta\end{cases}$$

hence $A = \begin{pmatrix}-\beta & \beta\\ \beta & -\beta\end{pmatrix}$

$\det A = \beta^2 - \beta^2 = 0$

~~so one eigen~~ so A is not $\begin{cases}\text{positive definite}\\ \text{negative definite}\\ \text{indefinite}\end{cases}$

In this situation, if A is an Hessian matrix at a critical point, the second derivative test doesn't allow to conclude

2. $A\begin{pmatrix}1\\1\\1\end{pmatrix} = 0\begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow 0$ is an eigenvalue

- A is not positive definite
- A is not negative definite
- A may ~~be~~ be or not be indefinite $\rightsquigarrow$ we don't have enough information

the two remaining eigenvalues have two ≠ sign
$\Rightarrow$ A is indefinite

otherwise, A not indefinite

In this case, we can't even apply the second ~~deriv~~ derivative test since we don't have enough information about A

3. A is negative definite by the course (local max)

4. $A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix} = (-1)\begin{pmatrix}1\\1\end{pmatrix} \rightsquigarrow -1$ eigenvalue <0
$A\begin{pmatrix}1\\2\end{pmatrix} = 2\begin{pmatrix}1\\2\end{pmatrix} \rightsquigarrow 2$ eigenvalue >0
A is indefinite (saddle point)

5. $\det A = 3-4 < 0$, A is indefinite (saddle point)

6. $\det A = 2-1 = 1 > 0$, A is either positive or negative definite

et $Ae_1 = 1 > 0 \Rightarrow A$ is positive definite (local min)

$\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}$

Q13: $f(x,y) = x^4 + y^4 - 2(x-y)^2$

$$\frac{\partial f}{\partial x}(x,y) = 4x^3 - 4(x-y) = 4(x^3 - x + y)$$

$$\frac{\partial f}{\partial y}(x,y) = 4y^3 + 4(x-y) = 4(y^3 + x - y)$$

hence $\nabla f(x,y) = 4(x^3 - x + y,\ y^3 + x - y)$

$$\nabla f(x,y) = (0,0) \iff \begin{cases} x^3 - x + y = 0 \\ y^3 + x - y = 0 \end{cases}$$

$$\iff \begin{cases} x^3 + y^3 = 0 & \leftarrow L_1 + L_2 \\ x^3 - x + y = 0 & \leftarrow L_1 \end{cases}$$

$$\iff \begin{cases} (x+y)(x^2 - xy + y^2) = 0 \\ x^3 - x + y = 0 \end{cases}$$

- $(x,y) = (0,0)$ is a trivial solution
- if $(x,y) \neq (0,0)$, $x^2 - xy + y^2 = \frac{(x-y)^2}{2} + \frac{x^2}{2} + \frac{y^2}{2} > 0$

hence $\begin{cases} x + y = 0 \\ x^3 - x + y = 0 \end{cases} \iff \begin{cases} x + y = 0 \\ x^3 - 2x = 0 \end{cases}$

Solution: $(0,0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$

There are 3 critical points : $(0,0)$, $(\sqrt{2},-\sqrt{2})$, $(-\sqrt{2},\sqrt{2})$

$\frac{\partial^2 f}{\partial x^2}(x,y) = 12x^2 - 4$ $\frac{\partial^2 f}{\partial y^2}(x,y) = 12y^2 - 4$

$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = 4$

↳ (f is C^2 so by Clairaut's thm)

and $Hf(x,y) = \begin{pmatrix} 12x^2-4 & 4 \\ 4 & 12y^2-4 \end{pmatrix}$

At $(0,0)$: $Hf(0,0) = \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}$ and $(-4)^2 - 4^2 = 0$.

So we can't conclude from the second derivative test

but $f(x,-x) = 2x^4 - 8x^2 = -2x^2(4-x^2) < 0$ for x small

$f(x,x) = 2x^4 > 0$

$(0,0)$ is $\boxed{\text{not}}$ a local extremum

At $(\sqrt{2},-\sqrt{2})$: $Hf(\sqrt{2},-\sqrt{2}) = \begin{pmatrix} 20 & 4 \\ 4 & 20 \end{pmatrix}$,

$\begin{cases} 20 \times 20 - 4 \times 4 > 0 \\ 20 > 0 \end{cases} \Rightarrow Hf(\sqrt{2},-\sqrt{2})$ is positive definite

$\Rightarrow (\sqrt{2},-\sqrt{2})$ local min

At $(-\sqrt{2},\sqrt{2})$: local min ~~by~~ (same computations)

Q14:

1) $f'(t) = (f_1'(t), f_2'(t), f_3'(t)) \in \mathbb{R}^3$

$Df(t) = \begin{pmatrix} f_1'(t) \\ f_2'(t) \\ f_3'(t) \end{pmatrix} \in M_{3,1}(\mathbb{R}) \simeq \mathbb{R}^3$

So it is safe to identify $f'(t)$ with $Df(t)$

2) $d_tf : \mathbb{R} \longrightarrow \mathbb{R}^3$ is a linear map given

$$d_t f(h) = \begin{pmatrix} d_t f_1(h) \\ d_t f_2(h) \\ d_t f_3(h) \end{pmatrix} = \begin{pmatrix} f_1'(t) h \\ f_2'(t) h \\ f_3'(t) h \end{pmatrix}$$

hence $f'(t) = d_t f(1)$