University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus!

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Question 1. *(Re)solve the questions 13 to 18 from the Test 1 Reviews sheet (Oct 10) and read the review slides.*

Question 2. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by* $f(x,y) = xye^x$.

1. *Justify that* f *is differentiable on* $\mathbb{R}^2$.
2. *Compute the partial derivatives at* $(1,1)$.
3. *Determine* $\nabla f(1,1)$.
4. *Compute* $\partial_{(1,2)} f(1,1)$ *the directional derivative of* f *at* $(1,1)$ *along* $(1,2)$ *using the previous question.*
5. *Compute* $\partial_{(1,2)} f(1,1)$ *the directional derivative of* f *at* $(1,1)$ *along* $(1,2)$ *from the very definition.*

Question 3. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by* $f(x,y) = \begin{cases} \frac{y^3}{\sqrt{x^2+y^4}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$

1. *Prove that* f *is continuous on* $\mathbb{R}^2$.
2. *Prove that the directional derivative* $\partial_v f(0,0)$ *exists for every* $v \in \mathbb{R}^2$.
3. *Prove that* f *is not differentiable at* $(0,0)$ *using the results from Slide 3 of the reviews.*
4. *Prove that* f *is not differentiable at* $(0,0)$ *using the definition of the differentiability.*

Question 4. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by* $f(x,y) = x^2e^y$.

1. *Justify that* f *is differentiable on* $\mathbb{R}^2$.
2. *Compute directly from the definition* $\partial_{(1,1)} f(2,0)$ *and* $\partial_{(1,-1)} f(2,0)$
3. *Determine* $\nabla f(2,0)$.

Question 5. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be defined by* $f(\mathbf{x}) = \|\mathbf{x}\|^2$.

1. *Prove that* f *is differentiable on* $\mathbb{R}^n$ *and determine its differential and its gradient at* $\mathbf{a} \in \mathbb{R}^n$ *using the definition of differentiability.*
2. *Same question using the fact that* f *is* C^1 *and the results from Slide 3 of the reviews.*
3. *Let* $g : \mathbb{R}^n \to \mathbb{R}$ *be defined by* $g(\mathbf{x}) = \|\mathbf{x}\|$. *Is* g *differentiable on* $\mathbb{R}^n$?

Question 6. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by* $f(x,y) = \begin{cases} \frac{y^4}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$

1. *Prove that* f *is* C^1 *on* $\mathbb{R}^2$.
2. *Compute* $\frac{\partial^2 f}{\partial x \partial y}(0,0)$ *and* $\frac{\partial^2 f}{\partial y \partial x}(0,0)$.
3. *Prove that* $\frac{\partial^2 f}{\partial x \partial y}$ *and* $\frac{\partial^2 f}{\partial y \partial x}$ *are not continuous.*

Question 7. *Let* $\mathbf{a} \in \mathbb{R}^3$ *and define* $\mathbf{F} : \mathbb{R}^3 \setminus \{\mathbf{a}\} \to \mathbb{R}^3$ *by* $\mathbf{F}(\mathbf{x}) = \frac{\mathbf{x}-\mathbf{a}}{\|\mathbf{x}-\mathbf{a}\|^2}$.
Prove that there exists a C^1 *function* $f : \mathbb{R}^3 \setminus \{\mathbf{a}\} \to \mathbb{R}$ *such that* $\forall \mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{a}\}$, $\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$.

Question 8 (Euler's identity)**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a differentiable weighted homogeneous function of degree r with weights* $(w_1, \dots, w_n)$*, i.e.* $\forall \mathbf{x} \in \mathbb{R}^n$, $\forall t > 0$, $f(t^{w_1}x_1, \dots, t^{w_n}x_n) = t^r f(x_1, \dots, x_n)$.
Prove that $\forall \mathbf{x} \in \mathbb{R}^n$, $\sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i}(\mathbf{x}) = r f(\mathbf{x})$

Question 9. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be of class* C^2*. Let* $U = \left\{(r,\theta) \in \mathbb{R}^2 \ : \ r > 0\right\}$.
Define $\varphi : U \to \mathbb{R}$ *by* $\varphi(r,\theta) = f(r\cos(\theta), r\sin(\theta))$*. Prove that* $\partial_x^2 f + \partial_y^2 f = \partial_r^2 \varphi + \frac{1}{r}\partial_r \varphi + \frac{1}{r^2}\partial_\theta^2 \varphi$.

Question 10 (Tangent lines/planes)**.**

1. *Tangent line of an implicit curve.*
   (a) *Give an equation of the tangent line of* $x + y^2 - x^3 = 1$ *at* $(-1, 1)$.
   (b) *Give an equation of the tangent line of* $f(x,y) = r$ *at* (x_0, y_0) *such that* $f(x_0, y_0) = r$.
2. *Tangent plane of an implicit surface.*
   (a) *Give an equation of the tangent plane of* $x^3 + zx^2 - y^2 = 0$ *at* $(1, 2, 3)$.
   Try to do the same at $(0, 0, 1)$*. What do you notice? Can you see why?*
   (b) *Give an equation of the tangent line of* $f(x,y,z) = r$ *at* (x_0, y_0, z_0) *such that* $f(x_0, y_0, z_0) = r$.
3. *Tangent plane of a graph.*
   (a) *Give an equation of the plane tangent to the graph of* $f(x,y) = x^2 + y^3$ *at* $(1, 1, 2)$.
   Method 1: *find two linear independant vectors supported in the tangent plane and use their cross-product to deduce a normal vector.*
   Method 2: *reduce to the study of a level set in order to deduce a normal vector without computing a cross-product.*
   (b) *Give an equation of the plane tangent to the graph* $z = f(x,y)$ *at* $(x_0, y_0, f(x_0, y_0))$.

Question 11. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function of class* C^2.

1. *Write the Taylor polynomials* $P_{\mathbf{a},0}(\mathbf{h})$, $P_{\mathbf{a},1}(\mathbf{h})$ *and* $P_{\mathbf{a},2}(\mathbf{h})$ *of* f *at* $\mathbf{a}$ *in terms of the partial derivatives.*
2. *Same question but using the gradient and the Hessian matrix of* f.

Question 12. *Determine whether the following* ***symmetric*** *matrices are positive definite? negative definite? indefinite?*

1. $A \in M_{2,2}(\mathbb{R})$ *such that* $A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$
2. $A \in M_{3,3}(\mathbb{R})$ *such that* $A\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$
3. $A \in M_{n,n}(\mathbb{R})$ *such that* $\forall h \in M_{n,1}(\mathbb{R}) \simeq \mathbb{R}^n$, $h^t A h \leq -\pi\|h\|^2$
4. $A \in M_{2,2}(\mathbb{R})$ *such that* $A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}$ *and* $A\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$
5. $A = \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}$
6. $A = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}$

Question 13. *Define* $f : \mathbb{R}^2 \to \mathbb{R}$ *by* $f(x,y) = x^4 + y^4 - 2(x-y)^2$*. Study the critical points of* f.

Question 14. *Let* $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$ *be a differentiable parametric curve.*
We define $\mathbf{f}' : \mathbb{R} \to \mathbb{R}^3$ *by* $\mathbf{f}'(t) = \left(f_1'(t), f_2'(t), f_3'(t)\right)$.

1. *Compare* $\mathbf{f}'(t)$ *and the Jacobian matrix* $D\mathbf{f}(t)$.
2. *Compare* $\mathbf{f}'(t)$ *and the differential* $d_t\mathbf{f}$.